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On the tree cover number and the positive semidefinite maximum nullity of a graph

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Abstract

Let $G = (V, E)$ be a simple graph. A tree cover of G is a collection of vertex-disjoint simple trees occurring as induced subgraphs of G that together cover all the vertices of G. The tree cover number of G, denoted $T(G)$, is the minimum cardinality of a tree cover. We show that if G is connected with $n \geq 2$ vertices, then $T(G) \leq \lceil \frac{n}{2} \rceil$, and we give a characterization of connected outerplanar graphs achieving this bound. We also show that for connected graphs on $n \geq 6$ vertices with no 3- or 4-cycles, $T(G) \leq \frac{n}{3}$. In 2011, Barioli et al. [*Elec. J. Lin. Alg.* 22 (2011), 10–21] introduced the tree cover number as a tool for studying the maximum nullity of a family of i matrices associated with a graph: Let $\mathcal{S}_+(G)$ denote the set of real positive semidefinite matrices $A = (a_{ij})$ such that for $i \neq j$, $a_{ij} \neq 0$ if $\{i, j\} \in E$ and $a_{ij} = 0$ if $\{i, j\} \notin E$. The positive semidefinite maximum nullity of G, denoted $\mathrm{M}_+(G)$, is $\max\{\mathrm{null}(A) \mid A \in \mathcal{S}_+(G)\}$. It was conjectured in 2011 that $T(G) \leq \mathrm{M}_+(G)$ holds for all graphs, and shown that equality holds when G is outerplanar. Therefore our bounds on $T(G)$ give bounds on $\mathrm{M}_+(G)$ for outerplanar graphs. We show that the conjecture $T(G) \leq \mathrm{M}_+(G)$ is true for certain graph families.

1 Introduction

A *graph* is a pair $G = (V, E)$ where V is the vertex set and E is the set of edges (2-element subsets of V). All graphs discussed here are simple (no loops or multiple edges) and finite. When G has n vertices, we use $\mathcal{S}_+(G)$ to denote the set of real $n \times n$ positive semidefinite matrices $A = (a_{ij})$ satisfying $a_{ij} \neq 0$ if and only if $\{i, j\} \in E$, for $i \neq j$, and a_{ii} is any nonnegative real number. The positive semidefinite maximum nullity of G, denoted $\mathrm{M}_+(G)$, is defined as $\max\{\mathrm{null}(A) \mid A \in \mathcal{S}_+(G)\}$. The minimum positive semidefinite rank of G, denoted $\mathrm{mr}_+(G)$, is defined as $\min\{\mathrm{rank}(A) \mid A \in \mathcal{S}_+(G)\}$, and it follows from the Rank-Nullity Theorem that $\mathrm{M}_+(G) + \mathrm{mr}_+(G) = n$.

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Barioli et al. [2] define a *tree cover* of G to be a collection of vertex-disjoint simple trees occurring as induced subgraphs of G that together cover all the vertices of G. The *tree cover number* of G, denoted $T(G)$, is the minimum cardinality of a tree cover, and it is used as a tool for studying the positive semidefinite maximum nullity of G. (In [2], G is allowed to be a multigraph, but we restrict ourselves to simple graphs.) It was conjectured in [2] that $T(G) \leq \mathrm{M}_+(G)$ for all graphs, and it is shown there that $T(G) = \mathrm{M}_+(G)$ for outerplanar graphs.

We show that $T(G) \leq \mathrm{M}_+(G)$ for certain families of graphs in Section 2. In Section 3, we show that $T(G) \leq \lceil \frac{n}{2} \rceil$ if G is a connected graph on $n \geq 2$ vertices. In Section 3, we also study $T(G)$ for connected graphs with girth at least 5 and deduce bounds on $\mathrm{M}_+(G)$ for such graphs that are outerplanar. We characterize connected outerplanar graphs on n vertices having positive semidefinite maximum nullity and tree cover number equal to the upper bound of $\lceil \frac{n}{2} \rceil$ in Section 4.

1.1 Graph theory terminology

For a graph $G = (V, E)$ and $v \in V$, the *neighborhood* of v, denoted $N(v)$, is the set of vertices adjacent to v. The *degree* of v is the cardinality of $N(v)$ and is denoted by $\deg(v)$. A vertex of degree one is called a *pendant vertex*. A set $S \subseteq V$ is *independent* if no two of the vertices of S are adjacent. The *independence number* of G, denoted $\alpha(G)$, is the maximum cardinality of an independent set in G.

The *path* P_n is the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{\{v_i, v_{i+1}\} \mid i \in \{1, \ldots, n-1\}\}$. The *cycle* C_n is formed by adding the edge $\{v_n, v_1\}$ to P_n. The *girth* of a graph is the size of a smallest cycle in the graph. If the graph contains no cycle, the girth is defined to be infinite. We denote the graph on n vertices containing every edge possible by K_n, and we use $K_{s,t}$ to denote the *complete bipartite graph*, the graph whose vertex set may be partitioned into two independent sets X and Y with $|X| = s$ and $|Y| = t$, such that for each $x \in X$ and $y \in Y$, $\{x, y\}$ is an edge and there are no additional edges. The graph $K_{1,t}$ is called a *star*.

For a graph $G = (V, E)$, a graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. A subgraph G' is an *induced subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') = \{\{u, v\} \mid \{u, v\} \in E(G) \text{ and } u, v \in V(G')\}$. For a subset $S \subseteq V(G)$, the *subgraph induced by* S, denoted $G[S]$, is the induced subgraph of G whose vertex set is the set S. For $S \subseteq V(G)$, we use $G - S$ to denote $G[V(G) \setminus S]$, and for $e \in E$, $G - e$ denotes the graph obtained by deleting e. For a graph G and an induced subgraph H, $G - H$ denotes the graph that results from G by deleting $V(H)$. A subgraph H of a graph G is a *clique* if for each $u, v \in V(H)$, $\{u, v\} \in E(H)$. The *clique number* of G, denoted $\omega(G)$, is $\omega(G) = \max\{|V(H)| : H \text{ is a subgraph of } G \text{ and } H \text{ is a clique}\}$.

A graph is *connected* if there is a path from each vertex to each other vertex. For a connected graph $G = (V, E)$, an edge $e \in E$ is called a *bridge* if $G - e$ is disconnected. We *subdivide* an edge $e = \{u, w\} \in E$ by removing e and adding a new vertex v_e such that $N(v_e) = \{u, w\}$.

A graph $G = (V, E)$ is *outerplanar* if it has a crossing-free embedding in the plane with every vertex on the boundary of the unbounded face. A *cut-vertex* of a connected graph is a vertex $v \in V$ such that $G - v$ is disconnected. A graph is *nonseparable* if it is connected and does not have a cut-vertex. A *block* is a maximal nonseparable induced subgraph, and G is a *block-clique graph* if every block is a clique.

Throughout this paper, given a graph $G = (V, E)$ and a tree cover $\mathcal{T}$ of G, we use $T_v \in \mathcal{T}$ to denote the unique tree in $\mathcal{T}$ containing $v \in V$.

1.2 Preliminaries

The following two propositions will be used repeatedly throughout this paper.

Proposition 1.1 ([2, Proposition 3.3]). *Let $G = (V, E)$ be a graph, let $v \in V$ be a pendant vertex, and let $e \in E$. Then $T(G - v) = T(G)$ and $T(G_e) = T(G)$, where G_e is the graph obtained from G by subdividing the edge e.*

Proposition 1.2 ([8, Proposition 2.5]). *Suppose $G_i, i = 1, \ldots, h$, are graphs, there is a vertex v for all $i \neq j$, $G_i \cap G_j = \{v\}$ and $G = \cup_{i=1}^{h} G_i$. Then*

$$T(G) = \left(\sum_{i=1}^{h} T(G_i) \right) - h + 1.$$

In the case that $h = 2$ in Proposition 1.2, we say G is the *vertex-sum* of G_1 and G_2 and write $G = G_1 \oplus_v G_2$.

Bozeman et al. [5] give the following bound on the tree cover number in terms of the independence number.

Proposition 1.3 ([5, Proposition 2]). *Let G be a connected graph, and let S be an independent set of G. Then $T(G) \leq |G| - |S|$. In particular, $T(G) \leq |G| - \alpha(G)$, where $\alpha(G)$ is the independence number of G. Furthermore, this bound is tight.*

It is also shown in Proposition 6 of [5] that for a graph $G = (V, E)$ and a bridge $e \in E$, every minimum tree cover of G includes some tree containing e. Embedded in the proof of this proposition is the following lemma.

Lemma 1.4. *Let $G = (V, E)$ be a connected graph and e a bridge in E. Let G_1 and G_2 be the connected components of $G - e$. Then $T(G) = T(G_1) + T(G_2) - 1 = T(G - e) - 1$.*

The following theorem will be used throughout the paper.

Theorem 1.5 ([3, Theorem 4.2]). *For a graph $G = (V, E)$ and an edge $e \in E$, $\mathrm{M}_+(G) - 1 \leq \mathrm{M}_+(G - e) \leq \mathrm{M}_+(G) + 1$.*

It is shown in [5] that an analogous bound holds for the tree cover number. That is, $T(G) - 1 \leq T(G - e) \leq T(G) + 1$. Furthermore, it is known that for $v \in V$, $\mathrm{M}_+(G) - 1 \leq \mathrm{M}_+(G - v) \leq \mathrm{M}_+(G) + \deg(v) - 1$ (see Fact 11 of page 46-11 of [12]). We show that an analogous bound holds for $T(G)$.

Proposition 1.6. *For a graph $G = (V, E)$ and vertex $v \in V$,*

$$T(G) - 1 \leq T(G - v) \leq T(G) + \deg(v) - 1.$$

Proof. Any tree cover of $G - v$ together with the tree consisting of the single vertex v is a tree cover for G, so $T(G) \leq T(G - v) + 1$, which gives the lower bound. To see the upper bound, let E_v denote the set of edges incident to v, and let $G - E_v$ denote the graph resulting from deleting the edges in E_v. Note that $|E_v| = \deg(v)$, and that $T(G - E_v) = T(G - v) + 1$. By [5, Theorem 3], the deletion of an edge can raise the tree cover number by at most 1, so $T(G - v) + 1 = T(G - E_v) \leq T(G) + \deg(v)$, and the upper bound follows. □

Section 4 studies tree covers of outerplanar graphs; the following facts are used.

Theorem 1.7 ([7]). *A graph is outerplanar if and only if it does not contain K_4 or $K_{2,3}$ as a minor, a graph obtained from it by deleting vertices, deleting edges, and contracting edges.*

Corollary 1.8. *Let G be an outerplanar graph and suppose that u and v are adjacent vertices in G that have common neighbors x and y with $x \neq y$. Then there is no path from x to y in $G' = G[V(G) \setminus \{u, v\}]$.*

Proof. If there is a path between x and y in $G' = G[V(G) \setminus \{u, v\}]$, then the edges of this path can be contracted to a single edge, forming a K_4 subgraph. By Theorem 1.7, this contradicts the fact that G is outerplanar. □

2 Some graphs with $T(G) \leq \mathrm{M}_+(G)$

In this section, we turn our attention to the conjecture $T(G) \leq \mathrm{M}_+(G)$ [2]. We prove that $T(G) \leq \mathrm{M}_+(G)$ for certain line graphs, for $G^{\triangle}$ (defined below) where G is any graph, for graphs whose complements have sufficiently small tree-width (defined below), and for graphs with a sufficiently large edge density.

We first show that for any connected graph G on $n \geq 2$ vertices, $T(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

Lemma 2.1. *Let G be a connected graph on $n \geq 3$ vertices. Then there exists an induced subgraph H of G such that $H = K_{1,p}$ for some $p \geq 1$ and $G - H$ is connected.* (See Figure 1.)

Proof. We prove the lemma by induction. For $n = 3$ the claim holds. Let G be a graph on $n \geq 4$ vertices and suppose the lemma holds for all graphs on $3 \leq k \leq n-1$ vertices. It is known that every connected graph has at most $n-2$ cut-vertices (since a spanning tree of the graph has at least two pendant vertices and the removal of these vertices will not disconnect the graph). Let v be a vertex in $V(G)$ that is not a cut-vertex. By hypothesis, there exists, for some $p \geq 1$, an induced subgraph $H' = K_{1,p}$ in $G - v$ whose deletion does not disconnect $G - v$. First we consider the case with $p = 1$, and then we consider the case with $p \geq 2$.

Case 1. Suppose $p = 1$ (i.e., $H' = K_{1,1}$) and let a and b be the vertices of H'. If v has a neighbor that is not a or b, then $G[V(G) \setminus \{a, b\}]$ is connected, and the claim holds with $H = H'$. Otherwise, v has a neighbor in $\{a, b\}$. Assume first that v is adjacent to exactly one of a and b. Without loss of generality, suppose v is adjacent to a and not adjacent to b. Then $G - H$ is connected for $H = G[\{a, b, v\}] = K_{1,2}$. Now suppose that v is adjacent to both a and b. Since $G-v$ is connected, then either a or b has a neighbor in $G[V(G) \setminus \{v, a, b\}]$. Without loss of generality, let a have a neighbor in $G[V(G) \setminus \{v, a, b\}]$. Then $G - H$ is connected for $H = G[\{v, b\}] = K_{1,1}$.

Case 2. Suppose $p \geq 2$. If v has a neighbor in $G[V(G) \setminus V(H')]$, then set $H = H'$ and the claim holds. Otherwise, v has neighbors only in $V(H')$. Recall that H' is a star. First suppose that v is adjacent to a pendant vertex w of H'. If w is not a cut-vertex of $G - v$, then $G - H$ is connected for $H = G[\{v, w\}] = K_{1,1}$. If w is a cut-vertex of $G - v$, then w has a neighbor outside of $V(H') \cup \{v\}$. Then $H = G[V(H') \setminus \{w\}] = K_{1,q}$ for some $q \geq 1$ and $G - H$ is connected. Next suppose that v is not adjacent to a pendant vertex in H'. Then it must be adjacent to the center vertex. Then $H = G[V(H') \cup \{v\}]$ is a star, and $G - H$ is connected. □

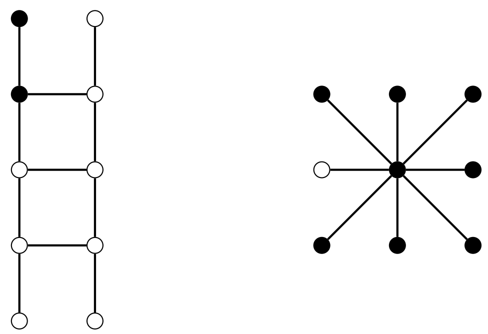

Figure 1: Two examples of Lemma 2.1, where the vertices of the induced subgraphs H are shown in black.

Theorem 2.2. *For any simple connected graph G on $n \geq 2$ vertices, $T(G) \leq \left\lceil \frac{n}{2} \right\rceil$.*

Proof. The theorem holds for $n = 2$. Let G be a connected graph on $n \geq 3$ and assume that the claim holds for all graphs with fewer than n vertices. By Lemma 2.1, there exists an induced tree $H = K_{1,p}$, for some $p \geq 1$, of G such that $G' = G - H$ is connected. By the induction hypothesis, $T(G') \leq \left\lceil \frac{|G'|}{2} \right\rceil \leq \left\lceil \frac{n-2}{2} \right\rceil$. Then $T(G) \leq T(G') + 1 \leq \left\lceil \frac{n}{2} \right\rceil$. □

It follows from [6] that for a triangle-free graph G, $\mathrm{M}_+(G) \leq \frac{n}{2}$. The next corollary is a result of Theorem 2.2 and the fact that $\mathrm{M}_+(G) = \mathrm{T}(G)$ [2] for outerplanar graphs.

Corollary 2.3. *If G is a connected outerplanar graph on n vertices, then* $\mathrm{M}_+(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

Some examples of graphs with $T(G) = \lceil \frac{n}{2} \rceil$ are the *complete graphs* K_n (the graphs on n vertices and all possible edges) and the well-known *friendship graphs* (graphs on $n = 2k+1$ vertices consisting of exactly k triangles all joined at a single vertex). Connected outerplanar graphs having $T(G) = \lceil \frac{n}{2} \rceil$ are characterized in Section 4.

For a graph G, the *line graph* of G, denoted $L(G)$, is the graph whose vertex set is the edge set of G, such that two vertices are adjacent in $L(G)$ if and only if the corresponding edges share an endpoint in G. The positive semidefinite maximum nullity of line graphs is studied in [10]. In particular, it is shown there that if G is a connected graph on n vertices and m edges, then $\mathrm{M}_+(L(G)) \geq m - n + 2$. We use this to get the following theorem.

Theorem 2.4. *If G is a connected graph on n vertices and $m \geq 2n-3$ edges, then $T(L(G)) \leq \mathrm{M}_+(L(G))$.*

Proof. By hypothesis, $\frac{m+3}{2} \geq n$, so $\mathrm{M}_+(L(G)) \geq m - n + 2 \geq \lceil \frac{m}{2} \rceil \geq T(L(G))$. □

The next theorem shows that the conjecture $T(G) \leq \mathrm{M}_+(G)$ holds true for any graph with a sufficiently large edge density.

Theorem 2.5. *Let G be a connected graph on $n \geq 4$ vertices and $m \geq \binom{n}{2} - (\frac{3n}{2} - 4)$ edges. Then $T(G) \leq \mathrm{M}_+(G)$.*

Proof. Let T be a spanning tree of $\overline{G}$. Then G is a subgraph of $\overline{T}$, and $\mathrm{M}_+(\overline{T}) \geq n-3$ (see [11, Theorem 3.16]). Note that $\overline{G}$ can be obtained from T by adding $\binom{n}{2} - m - (n-1)$ edges, so G can be obtained from $\overline{T}$ by deleting $\binom{n}{2} - m - (n-1)$ edges. By Theorem 1.5, edge deletion decreases the positive semidefinite maximum nullity by at most 1, so

$$\mathrm{M}_+(G) \geq \mathrm{M}_+(\overline{T}) - \left(\binom{n}{2} - m - (n-1)\right) \geq (n-3) - \left(\binom{n}{2} - m - (n-1)\right) \geq \frac{n}{2},$$

where the last inequality follows from the fact that $m \geq \binom{n}{2} - (\frac{3n}{2} - 4)$. Since $\mathrm{M}_+(G)$ is an integer, by Theorem 2.2, $\mathrm{M}_+(G) \geq T(G)$. □

Definition 2.6. For a graph $G = (V, E)$, let $G^{\triangle}$ be the graph constructed from G by adding, for each edge $e = \{u, v\} \in E$, add a new vertex w_e such that w_e is adjacent to exactly u and v. The vertices w_e are called the *edge-vertices* of $G^{\triangle}$.

Theorem 2.7. *For a connected graph G on n vertices and m edges, $T(G^{\triangle}) \leq \mathrm{M}_+(G^{\triangle})$.*

Proof. We show that $\mathrm{mr}_+(G^{\triangle}) = \alpha(G^{\triangle})$ and then apply Proposition 1.3. It is always the case that a connected graph H has $\alpha(H) \leq \mathrm{mr}_+(H)$ (see Corollary 2.7 in [4]), so we show that $\mathrm{mr}_+(G^{\triangle}) \leq \alpha(G^{\triangle})$. Let B be the vertex-edge incidence matrix of G, and let $X = \begin{pmatrix} I_m \\ B \end{pmatrix}$, where I_m is the $m \times m$ identity matrix. Then

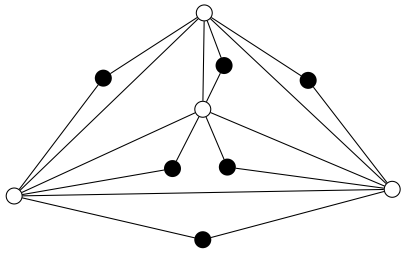

Figure 2: $K_4^{\triangle}$, where the edge-vertices are shown in black.

$XX^T = \begin{pmatrix} I_m & B^T \\ B & BB^T \end{pmatrix} \in \mathcal{S}_+(G^{\triangle})$, where the first m rows and columns are indexed by the edge-vertices and the last n rows and columns are indexed by the vertices in V. Note that the set of edge-vertices of $G^{\triangle}$ is an independent set of size m and that the rank of XX^T is m. So $\mathrm{mr}_+(G^{\triangle}) \leq m \leq \alpha(G)$, and therefore $\mathrm{mr}_+(G^{\triangle}) = \alpha(G^{\triangle})$. By Proposition 1.3, $\mathrm{T}(G^{\triangle}) \leq m + n - \mathrm{mr}_+(G^{\triangle}) = \mathrm{M}_+(G^{\triangle})$. □

The *tree-width* of a graph G, denoted $\mathrm{tw}(G)$, is a widely studied parameter, and there are multiple ways in which it is defined. Here we define the tree-width in terms of chordal completions. A graph is *chordal* if it has no induced cycle on four or more vertices. If G is a subgraph of G' such that $V(G) = V(G')$ and G' is chordal, then G' is called a *chordal completion* of G. The tree-width of G is defined as

$$\mathrm{tw}(G) = \min\{\omega(G') - 1 \mid G' \text{ is a chordal completion of } G\}.$$

Proposition 2.8. *Let G be a graph on n vertices with $\mathrm{tw}(G) \leq \frac{n-4}{2}$. Then $T(\overline{G}) \leq \mathrm{M}_+(\overline{G})$.*

Proof. If $\mathrm{tw}(G) \leq k$, then $\mathrm{mr}_+(\overline{G}) \leq k+2$ [13], i.e., $\mathrm{M}_+(\overline{G}) \geq n-k-2$. For $k = \frac{n-4}{2}$, it follows that $\mathrm{M}_+(\overline{G}) \geq \frac{n}{2}$. It follows that $\mathrm{M}_+(\overline{G}) \geq \left\lceil \frac{n}{2} \right\rceil \geq T(\overline{G})$, where the last inequality is given by Theorem 2.2. □

A *k-tree* is constructed inductively by starting with a complete graph K_{k+1} and at each step adding a new vertex that is adjacent to exactly k vertices in an existing K_k subgraph. It is shown in [9] that $T(G) = \mathrm{M}_+(G)$ for all 2-trees. We now show that $T(G) \leq \mathrm{M}_+(G)$ for 3-trees and 5-trees. To do so, we use the following theorem.

Theorem 2.9. *Let G be a graph with $T(G) \leq 3$. Then $T(G) \leq \mathrm{M}_+(G)$.*

Proof. It is shown in [15] that $\mathrm{M}_+(G) = 1$ if and only if G is a tree. If $T(G) = 1$, then G is a tree and $\mathrm{M}_+(G) = 1$. If $T(G) \geq 2$, then G is not a tree, so $\mathrm{M}_+(G) \geq 2$.

Suppose now that $T(G) = 3$. If $\mathrm{M}_+(G) = 2$, then $\mathrm{Z}_+(G) = 2$ [8] (where $\mathrm{Z}_+(G)$ is the positive semidefinite zero forcing number of G, defined in [1]). It is shown in [8] that $T(G) \leq \mathrm{Z}_+(G)$, and this contradicts $T(G) = 3$. Thus $\mathrm{M}_+(G) \geq 3$. □

Corollary 2.10. *If G is a 3-tree or 5-tree, then $T(G) \leq \mathrm{M}_+(G)$.*

Proof. The tree cover number of a k-tree is $\frac{k+1}{2}$ when k is odd [14], so $T(G) \in \{2, 3\}$. The result then follows from Theorem 2.9. □

3 Tree cover number of graphs with girth at least five

For many graphs, the tree cover number is much lower than the upper bound of $\lceil \frac{n}{2} \rceil$ given in Theorem 2.2. The next theorem improves this bound for graphs with girth at least 5.

Theorem 3.1. *Let G be a connected graph on $n \geq 6$ vertices with girth at least 5. Then $T(G) \leq \frac{n}{3}$.*

Proof. The proof is by induction on n. A connected graph on 6 vertices with girth at least 5 is either a tree, C_6, or C_5 with a pendant vertex adjacent to one of the vertices on the cycle. In each case, the tree cover number is at most 2, so the theorem holds. Let $n \geq 7$. If G has a pendant vertex v, then $T(G) = T(G - v) \leq \frac{n-1}{3}$. Suppose G has no pendant vertices. Let $P = (x, y, z)$ be an induced path in G. We consider the connected components of $G - P$.

Note that G has no pendant vertices and no 3- or 4-cycles, so $G - P$ cannot have an isolated vertex as a connected component. We now show that if $G - P$ has a connected component H with $|H| \in \{3, 4, 5\}$, then the theorem holds.

Suppose $G - P$ has a connected component H of order 3. Then H is a path on three vertices. Note that $G - H$ is a connected graph (since the remaining components of $G - P$ are all connected to P), so if $|G - H| \geq 6$, by applying the induction hypothesis to $G - H$ and covering H with a path, we get that $T(G) \leq 1 + \frac{n-3}{3} = \frac{n}{3}$. Otherwise, since $n \geq 7$ and $G - P$ does not have an isolated vertex as a component, $|G - H| = 5$ and therefore $G - H - P = K_2$. By assumption G has no pendant vertices and no 3- or 4-cycles, so $G - H = C_5$, G is one of the two graphs shown in Figure 3 and the theorem holds.

Suppose that $G - P$ has a connected component H of order 4. Then H is a tree. If $|G - H| \geq 6$, then $T(G) \leq 1 + \frac{n-4}{3} = \frac{n-1}{3}$. If $G - H = P$, then $T(G) = 2$, $n = 7$, and the theorem holds. Otherwise $G - H = C_5$, $T(G) \leq 3$ (since $G - H$ may be covered with 2 trees and H is a tree), $n = 9$, and the theorem holds.

Consider $G - P$ having a connected component H of order 5. Then H is either a tree or $H = C_5$. Assume first that H is a tree. If $|G - H| \geq 6$, then $T(G) \leq 1 + \frac{n-5}{3} = \frac{n-2}{3}$. If $G - H = P$, then $T(G) = 2, n = 8$, and the theorem holds. Otherwise, $G - H = C_5$, $T(G) \leq 3$, $n = 10$, and the theorem holds.

Now suppose $H = C_5 = (u_1, \ldots, u_5)$, and assume without loss of generality that u_1 has a neighbor on $P = (x, y, z)$. If $G - H = P$, then $n = 8$ and for $T_1 = G[\{u_2, u_3, u_4, u_5\}]$ and $T_2 = G[\{x, y, z, u_1\}]$, $\mathcal{T} = \{T_1, T_2\}$ is a tree cover of size 2. Otherwise, for path $P' = (u_2, u_3, u_4, u_5)$, $G - P'$ is a connected graph on at least 6 vertices, so $T(G) \leq 1 + \frac{n-4}{3} = \frac{n-1}{3}$.

We may now assume that each component of $G - P$ is K_2 or has at least 6 vertices. If all components of $G - P$ are of order at least 6, then by applying the induction hypothesis to each of the components, it follows that $T(G) \leq 1 + \frac{n-3}{3} = \frac{n}{3}$. Suppose $G - P$ has exactly one component that is $K_2 = (u, v)$. Since G has no pendant vertices, then each of u and v must be adjacent to a vertex of P. We may also assume without loss of generality that u is adjacent to x and v is adjacent to z since G has no 3- or 4-cycles. Furthermore, $G - P$ must have some other component H with at least 6 vertices since $n \geq 7$. Note that H has a vertex that is adjacent to some $r \in \{x, y, z\}$. By adding r to H, we partition G into a tree (namely, the tree with vertex set $\{x, y, z, u, v\} \setminus \{r\}$) and connected components of order at least 6. Thus, $T(G) \leq 1 + \frac{n-4}{3} = \frac{n-1}{3}$.

Suppose $G - P$ has $s \geq 2$ components that are K_2. We first show that the vertices of $P = (x, y, z)$ and the vertices of all s of the K_2 components can be covered with two trees: recall that G has no pendant vertices, so each vertex of a K_2 must be adjacent to a vertex of P, and since G has no 3- or 4-cycles, for each K_2, one vertex must be adjacent to x and the other must be adjacent to z. Let X be the set of vertices from the K_2 components that are adjacent to x and let Z be the set of of vertices from the K_2 components that are adjacent to z. Then for $T_1 = G[X \cup \{x\}]$ and $T_2 = G[Z \cup \{z, y\}]$, $\mathcal{T} = \{T_1, T_2\}$ is a tree cover of size two that covers the vertices of P and the vertices of all K_2 components of $G - P$. We apply the induction hypothesis to each component of $G - P$ with at least 6 vertices to get that $T(G) \leq 2 + \frac{n-3-2s}{3} \leq \frac{n-1}{3}$. □

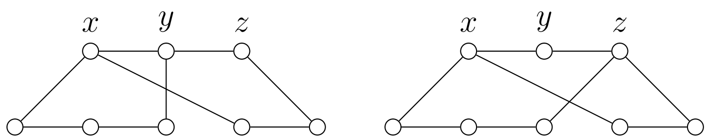

Figure 3: Graphs mentioned in the proof of Theorem 3.1

Corollary 3.2. *If G is a connected outerplanar graph on n vertices with girth at least 5, then* $\mathrm{M}_+(G) \leq \frac{n}{3}$.

Triangle-free graphs are a family of widely studied graphs, so a natural question is whether or not the bound given in Corollary 3.2 holds when the girth is at least 4. The cycle on four vertices demonstrates that the bound no longer holds. However, we offer the following conjecture.

Conjecture 3.3. *For all connected triangle-free graphs,* $T(G) \leq \left\lceil \frac{n}{3} \right\rceil$.

4 Tree cover number for connected outerplanar graphs

We now turn our attention specifically to connected outerplanar graphs on $n \geq 2$ vertices. We have seen that $\mathrm{M}_+(G) = T(G) \leq \left\lceil \frac{n}{2} \right\rceil$ for these graphs, and in this section we characterize graphs that achieve this upper bound.

Let $\mathcal{F}$ denote the family of block-clique graphs such that each clique is K_3 (see Figure 4). Observe that every graph in $\mathcal{F}$ has an odd number of vertices. We show that for n odd, the family of graphs whose tree cover number achieves the upper bound is exactly the family $\mathcal{F}$. The family $\mathcal{F}$ also plays a vital role in characterizing outerplanar graphs on an even number of vertices whose tree cover number achieves the upper bound.

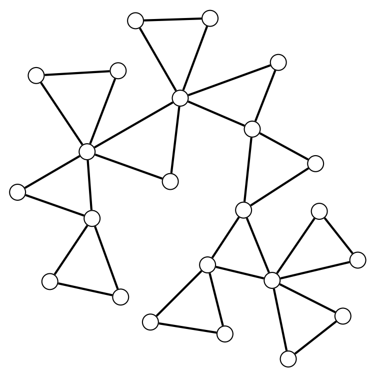

Figure 4: An example of a graph in $\mathcal{F}$.

Theorem 4.1. *Let G be a connected outerplanar graph of odd order $n \geq 3$. Then $T(G) = \left\lceil \frac{n}{2} \right\rceil$ if and only if $G \in \mathcal{F}$.*

In order to prove Theorem 4.1, we need Lemmas 4.2 and 4.3, Corollary 4.4, and Lemma 4.5 below.

Note that the graphs in $\mathcal{F}$ are block-clique graphs in which any two blocks share at most one common vertex. Two blocks are said to be *adjacent* if they have one common vertex. A *pendant block* is a block that is adjacent to exactly one other block.

The following lemma is a special case of [14, Lemma 2].

Lemma 4.2. *Any block-clique graph in $\mathcal{F}$ has at least two pendant blocks.*

Lemma 4.3. *If G is a connected graph with $n \geq 3$ vertices and $T(G) = \left\lceil \frac{n}{2} \right\rceil$, then there exist adjacent vertices $u, v \in V(G)$ such that $G' = G[V(G) \setminus \{u, v\}]$ remains connected. Furthermore, $T(G') = \left\lceil \frac{n-2}{2} \right\rceil$.*

Proof. By Lemma 2.1, we may remove an induced subgraph $H = K_{1,p}$ such that $G - H$ remains connected. First note that if $p \geq 3$, then $T(G) \leq 1 + \left\lceil \frac{n-4}{2} \right\rceil < \left\lceil \frac{n}{2} \right\rceil$, which is a contradiction, so $p \leq 2$. If $p = 1$, then we are done. Suppose $p = 2$ (i.e., H is a path (x, y, z)). If x and z are both pendant vertices in G, then $T(G) = T(G - \{x, z\}) \leq \left\lceil \frac{n-2}{2} \right\rceil$, which is a contradiction to $T(G) = \frac{n}{2}$. So, without loss of generality, x has a neighbor in $G - H$, and the lemma holds with $u = y$ and $v = z$.

It is easy to see that if $T(G') < \left\lceil \frac{n-2}{2} \right\rceil$, then $T(G) < \left\lceil \frac{n}{2} \right\rceil$. So, $T(G') = \left\lceil \frac{n-2}{2} \right\rceil$. $\square$

Corollary 4.4. *If G is a connected graph with $n \geq 3$ vertices and $T(G) = \lceil \frac{n}{2} \rceil$, then there exists a tree cover of G in which each tree is K_2 except for possibly a single K_1.*

Lemma 4.5. *Let G be a connected graph and suppose $u, v \in V(G)$ are adjacent vertices such that $G' = G[V(G) \setminus \{u, v\}]$ is connected. Let $\mathcal{T}'$ be a minimum tree cover of G', and suppose there exists $w \in V(G')$ such that*

(1) *$V(T_w) = \{w, x\}$ for some x (where T_w is the tree in $\mathcal{T}'$ containing w), and*
(2) *there exists $y \in N(w) \cap N(x)$ such that $N(x) \cap V(T_y) = \{y\}$.*

If u is adjacent to w and v is not adjacent to w, then $T(G) \leq T(G')$.

Proof. Note that $\mathcal{T} = (\mathcal{T}' \setminus \{T_w, T_y\}) \cup \{G[\{u, v, w\}], G[V(T_y) \cup \{x\}]\}$ is a tree cover of G of size $T(G')$. □

Proof of Theorem 4.1. Let G be a connected outerplanar graph on $n = 2k + 1$ vertices and first suppose $T(G) = \lceil \frac{n}{2} \rceil$. We prove that $G \in \mathcal{F}$ by induction on k. If $k = 1$, then $G = K_3 \in \mathcal{F}$. Let $k \geq 2$ and suppose that the claim holds for graphs with $2(k-1)+1$ vertices. By Lemma 4.3, we can delete an edge H (including the endpoints) such that $G' = G - H$ is connected and $T(G') = \lceil \frac{n-2}{2} \rceil$. By the induction hypothesis, $G' \in \mathcal{F}$ (see Figure 5).

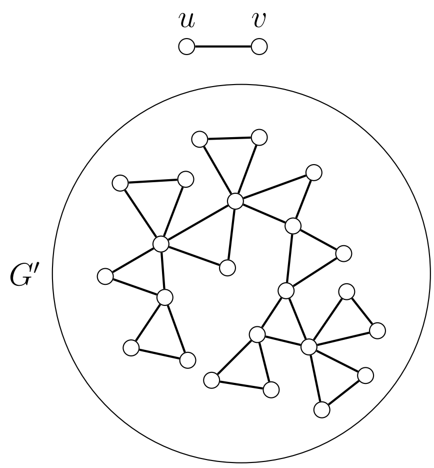

Figure 5: Graph G' mentioned in proof of Theorem 4.1

Furthermore, by Corollary 4.4, G' has a minimum tree cover $\mathcal{T}$ such that one tree is K_1 and the remaining trees are K_2. Let $V(H) = \{u, v\}$. We show that $G \in \mathcal{F}$ by showing (1) u is adjacent to a vertex $w \in V(G')$ if and only if v is adjacent to w and (2) u (and therefore v) is adjacent to exactly one vertex in $V(G')$.

To see (1), suppose u is adjacent to $w \in V(G')$ and v is not adjacent to w. If $V(T_w) = \{w\}$, then $\mathcal{T}' = (\mathcal{T} \setminus \{T_w\}) \cup \{G[\{w, v, u\}]\}$ is a tree cover of G of size $\lceil \frac{n-2}{2} \rceil$, which is a contradiction.

Otherwise, $T_w = P_2 = (w, x)$ for some $x \in V(G')$. Each edge of G' belongs to a triangle, so there exists $y \in V(G')$ such that $y \in N(w) \cap N(x)$. Since any two triangles in G' share at most one vertex, it follows from Lemma 4.5 that $T(G) \leq \lceil \frac{n-2}{2} \rceil$, which is a contradiction. (See Figure 6.) So v must be adjacent to w, and by symmetry, it follows that $N(v) \cap V(G') = N(u) \cap V(G')$.

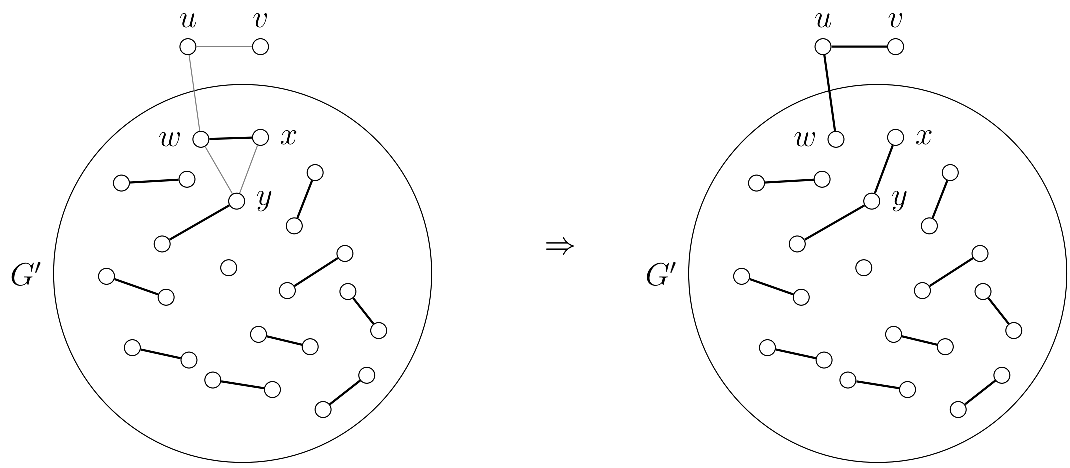

Figure 6: This figure demonstrates how a tree cover of G' is used to form a tree cover of G on $\lceil \frac{n-2}{2} \rceil$ vertices when u is adjacent to w, v is not adjacent to w, and $T_w = P_2 = (w, x)$. The bold edges are used to highlight a minimum tree cover.

To see (2), suppose that v and u are adjacent to $x, y \in V(G')$. By the connectivity of G', there is a path P in G' with endpoints x and y. By Corollary 1.8, this contradicts G being outerplanar.

We show the converse by induction on k. Let $G \in \mathcal{F}$. For $k = 1$, $G = K_3$, so $T(G) = \lceil \frac{n}{2} \rceil$. For $k \geq 2$, by Lemma 4.2, G has a pendant block, so $G = G' \oplus_v K_3$ for some $v \in V$, where G' is a graph on $n - 2$ vertices and $G' \in \mathcal{F}$. It follows from the induction hypothesis and Proposition 1.2 that $T(G) = T(G') + T(K_3) - 1 = \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil$. □

Corollary 4.6. *Let G be a connected outerplanar graph of odd order n. Then $\mathrm{M}_+(G) = \lceil \frac{n}{2} \rceil$ if and only if $G \in \mathcal{F}$.*

A tree cover in which each tree is a path is called a *path cover*, and the minimum cardinality of a path cover, denoted by $P(G)$, is the *path cover number of* G. Since $T(G) = \lceil \frac{n}{2} \rceil$ for any graph $G \in \mathcal{F}$, by Corollary 4.4, we can form a minimum tree cover of G where one tree is K_1 and every other tree is K_2. This shows that $P(G) \leq T(G)$, and it follows, since $P(G) \geq T(G)$ is always true, that $T(G) = P(G)$ for $G \in \mathcal{F}$. The graphs in $\mathcal{F}$ are special cases of the block-cycle graphs studied in [14]. It is shown in [14] that $\mathrm{Z}(G) = P(G)$ for all block-cycle graphs. Thus, we have that

$$\mathrm{Z}(G) = P(G) = T(G) = \mathrm{M}_+(G) \leq \mathrm{Z}_+(G) \leq \mathrm{Z}(G)$$

for any $G \in \mathcal{F}$, so all of the parameters are equal.

We now turn to the characterization for even n. The only connected graph of order $n = 2$ is K_2, which has tree cover number $\frac{n}{2} = 1$.

Theorem 4.7. *For a connected outerplanar graph $G = (V, E)$ of even order $n \geq 4$, $T(G) = \frac{n}{2}$ if and only if one of the following holds.*

(1) *G is obtained from some $G' \in \mathcal{F}$ by adding one pendant vertex.*
(2) *G is obtained from some $G_1, G_2 \in \mathcal{F}$ by connecting them with a bridge.*
(3) *G is constructed from the following iterative process: Start with $G^{[0]} \in \{C_4, K_4 - e\} \cup \{C_r^{\triangle} : r \geq 3\}$. For $i \geq 1$, choose some $v \in V(G^{[i-1]})$ and let $G^{[i]} = G^{[i-1]} \oplus_v K_3$.*

To prove Theorem 4.7, we use Lemmas 4.8 and 4.9.

Lemma 4.8. *For $r \geq 3$, $T(C_r^{\triangle}) = \frac{|C_r^{\triangle}|}{2} = r$.*

Proof. Let G' be the multigraph obtained from C_r by duplicating each edge of C_r once. Then $C_r^{\triangle}$ can be obtained from G' by subdividing each duplicate edge once. Then from Proposition 3.3 of [2], $T(C_r^{\triangle}) = T(G')$, where the tree cover number of a multigraph G is defined to be the minimum number of vertex disjoint simple trees occurring as induced subgraphs of G that cover all of the vertices of G. It follows that $T(C_r^{\triangle}) = T(G') = |G'| = r$. □

Lemma 4.9. *Let $G = (V, E)$ be a connected outerplanar graph of even order n with $T(G) = \frac{n}{2}$ that satisfies the following conditions.*

(a) *G does not have a bridge.*
(b) *The neighbors of each degree-two vertex are adjacent.*

Let $u, v \in V$ be adjacent vertices in G such that $G' = G[V \setminus \{u, v\}]$ remains connected and $T(G') = \frac{n-2}{2}$. If G' does not have a pendant vertex, then one of the following holds.

(1) *G' satisfies (a) and (b).*
(2) *G' satisfies (3) of Theorem 4.7.*
(3) *G satisfies (3) of Theorem 4.7.*

Proof. Assume G' has no pendant vertices. Suppose first that G' does not satisfy (a). Then some edge $e = \{g_1, g_2\}$ is a bridge in G'. Let G_1 and G_2 be the connected components of $G' - e$, where $g_1 \in V(G_1)$ and $g_2 \in V(G_2)$. We show that G satisfies (3). First note that $|G_1|$ and $|G_2|$ must have the same parity since $|G'| = |G_1| + |G_2| = n - 2$ is even. By hypothesis, $T(G') = \frac{n-2}{2}$ and by Lemma 1.4, $T(G') = T(G_1) + T(G_2) - 1$. If $|G_i|$ is even, then $T(G') = T(G_1) + T(G_2) - 1 \leq \frac{|G_1|}{2} + \frac{|G_2|}{2} - 1 = \frac{n-2}{2} - 1$, which is a contradiction. It follows that $|G_i|$ is odd and $T(G_i) = \left\lceil \frac{|G_i|}{2} \right\rceil$ for $i = 1, 2$. Since G is outerplanar, so are G_1 and G_2. Furthermore, since G' has no pendant vertices, $|G_i| \geq 3$ for $i = 1, 2$, and by Theorem 4.1, $G_i \in \mathcal{F}$.

Throughout the remainder of the proof, let $\mathcal{T}'$ be a tree cover of G' of size $\lceil \frac{n}{2} \rceil$ such that each tree has exactly two vertices, as given by Corollary 4.4. Furthermore, note that since $|G_1|$ and $|G_2|$ are both odd, then it must be the case that $\{g_1, g_2\}$ is one of the trees in $\mathcal{T}'$.

Let W be the set of vertices in $V(G') \setminus \{g_1, g_2\}$ that are adjacent to either u or v. We first show that each $w \in W$ is adjacent to both u and v. If not, then without loss of generality, let $w \in W$ be adjacent to u and not v, and suppose that $w \in V(G_1)$. Let $V(T_w) = \{w, x\}$, where T_w is the tree containing w. Note that $x \in V(G_1)$ since $w \neq g_1$. Since $G_1 \in \mathcal{F}$, w and x have a common neighbor y in G_1. Let $V(T_y) = \{y, y'\}$, and note that $y' \notin N(x)$ (if $y' \in V(G_1)$ then this follows from the fact that $G_1 \in \mathcal{F}$, and if $y' \in V(G_2)$, then y must be g_1, $y' = g_2$ and since $e = \{y = g_1, y' = g_2\}$ is a bridge, then $y' = g_2 \notin N(x)$). By Lemma 4.5, $T(G) \leq T(G') = \frac{n-2}{2}$, which contradicts $T(G) = \frac{n}{2}$. Thus, v is adjacent to w, and this shows that u and v have the same set of neighbors in $V(G') \setminus \{g_1, g_2\}$.

It then follows from Corollary 1.8 that $|W| \leq 1$. We show that if $W = \emptyset$, then $G[\{u, v, g_1, g_2\}]$ is $K_4 - e$: Since G is connected, we assume without loss of generality that $\{u, g_1\}$ is an edge in G. Suppose first that g_1 is also adjacent to v. Since G has no bridge, then either $\{g_2, u\}$ or $\{g_2, v\}$ is an edge in G, but not both since G is outerplanar (Corollary 1.8). In either case, $G[\{u, v, g_1, g_2\}]$ is $K_4 - e$. Suppose now that g_1 is not adjacent to v. Since $\{u, v\}$ is not a bridge, v must be adjacent to g_2, and since neighbors of degree-two vertices are adjacent, u and g_2 are adjacent. Thus $G[\{u, v, g_1, g_2\}]$ is $K_4 - e$. Recall that G_1 and G_2 are in $\mathcal{F}$, so it follows that G satisfies (3) with $G^{[0]}$ being K_4 minus an edge.

We may now assume $|W| = 1$. Let $W = \{w\}$, and suppose without loss of generality that $w \in V(G_1)$. Note that either u or v must be adjacent to a vertex in $V(G_2)$ since e is not a bridge in G, and this vertex must be g_2 since $|W| = 1$. Without loss of generality, suppose u is adjacent to g_2, and note that since G is outerplanar, it follows from Corollary 1.8 that v cannot also be adjacent to g_2 . Suppose for the sake of contradiction that u is not adjacent to g_1. Note that since $w \neq g_1$, then $V(T_w) \subseteq V(G_1)$, and since $|W| = 1$, $N(v) \cap V(T_w) = \{w\}$. Then by extending T_w to contain v and extending $T_{g_2} = \{g_1, g_2\}$ to contain u, we form a tree cover of G of size $\frac{n-2}{2}$, which contradicts $T(G) = \frac{n}{2}$. It follows that u must be adjacent to g_1.

Note that since G is outerplanar, by Corollary 1.8, v is neither adjacent to g_1 nor to g_2. Since $G_1, G_2 \in \mathcal{F}$, it follows that G satisfies (3) with $G^{[0]} = C_r^{\triangle}$, where $C_r = (u, w, x_1, \ldots, x_j, g_1)$ and $(w, x_1, \ldots, x_j, g_1)$ is a shortest path between w and g_1 in G_1 (see Figure 7).

Suppose now that G' does not satisfy (b). Then there must exist a vertex $z \in V(G')$ of degree 2 whose neighbors z' and z'' are not adjacent. We show that G' satisfies (3). By Proposition 1.1, contracting the edge $\{z, z'\}$ in G' results in a graph H on $n-3$ vertices with $T(H) = T(G') = \frac{n-2}{2} = \lceil \frac{n-3}{2} \rceil$, so by Theorem 4.1, $H \in \mathcal{F}$. Then there is some triangle (z', z'', y) in H, and it follows that (z', z, z'', y) is a 4-cycle in G', and G' satisfies (3) of Theorem 4.7 with $G^{[0]} = C_4$. □

Proof of Theorem 4.7. Let G be a graph on $n = 2k$ ($k \geq 2$) vertices and first suppose

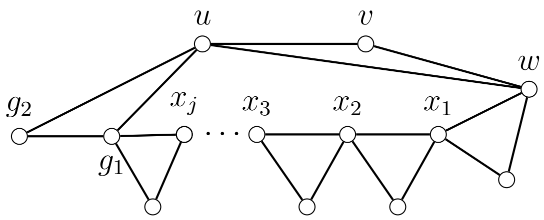

Figure 7: Graph mentioned in the proof of Lemma 4.9

$T(G) = \frac{n}{2}$.

If G has a pendant vertex v, then $T(G) = T(G-v)$, and $G-v$ is in $\mathcal{F}$ by Theorem 4.1. Thus (1) holds. If G has a bridge e and the connected components of $G-e$ are G_1 and G_2, then by Lemma 1.4, $T(G) = T(G_1)+T(G_2)-1$. Note that $|G_1|$ and $|G_2|$ must both be even or both odd since n is even. If both are even then

$$T(G) = T(G_1) + T(G_2) - 1 \leq \frac{|G_1|}{2} + \frac{|G_2|}{2} - 1 = \frac{n}{2} - 1,$$

which contradicts $T(G) = \frac{n}{2}$. Thus, $|G_1|$ and $|G_2|$ are both odd, and

$$\frac{n}{2} = T(G) = T(G_1)+T(G_2)-1 \leq \left\lceil \frac{|G_1|}{2} \right\rceil + \left\lceil \frac{|G_2|}{2} \right\rceil - 1 = \frac{|G_1|+1}{2} + \frac{|G_2|+1}{2} - 1 = \frac{n}{2}.$$

It follows that for $i = 1, 2$, $T(G_i) = \left\lceil \frac{|G_i|}{2} \right\rceil$, and by Theorem 4.1, $G_i \in \mathcal{F}$, so (2) holds. Now suppose G can be obtained from some graph G' by subdividing an edge of G'. By Proposition 1.1, $\frac{n}{2} = T(G) = T(G')$ and by Theorem 4.1, $G' \in \mathcal{F}$. Note that subdividing an edge of a graph in $\mathcal{F}$ results in a graph satisfying (3) with $G^{[0]} = C_4$, so G satisfies (3) of Theorem 4.7.

We may now assume that G has no pendant vertices, G does not have a bridge, and for each $v \in V$ with $\deg(v) = 2$, the neighbors of v are adjacent. For the remainder of the proof, let u and v be the adjacent vertices from Lemma 4.3 such that $G' = G[V(G) \setminus \{u, v\}]$ is connected and $T(G') = \frac{n-2}{2}$. We consider two cases, G' has a pendant vertex and G' does not have a pendant vertex.

Case 1. Suppose G' has a pendant vertex ℓ and let ℓ' be its neighbor. We show G satisfies (3). We first show that u and v have the same set of neighbors in $V(G') \setminus \{\ell, \ell'\}$. Note that $T(G'-\ell) = T(G') = \lceil \frac{n-3}{2} \rceil$ and by Theorem 4.1, $G'-\ell$ is in $\mathcal{F}$. Suppose u has a neighbor w in $V(G') \setminus \{\ell, \ell'\}$ and v is not adjacent to w. Let $\mathcal{T}'$ be a tree cover of G' such that each tree has exactly two vertices, as given by Corollary 4.4, and let $T_w = \{w, x\}$ be the tree containing w. Since $G'-\ell \in \mathcal{F}$, there must be a common neighbor y of w and x such that $V(T_y) = \{y, z\}$ and $z \notin N(x)$. By Lemma 4.5, $T(G) \leq \frac{n-2}{2}$, contradicting $T(G) = \frac{n}{2}$. Therefore u and v have the same set of neighbors in $V(G') \setminus \{\ell, \ell'\}$. Since G is outerplanar, it follows from Corollary 1.8 that this set has cardinality at most 1.

Recall that G has no pendant vertices, so we may assume that u is adjacent to ℓ. Suppose first that v is also adjacent to ℓ. Then either u or v must have a neighbor in

$G' - \ell$ since $\{\ell, \ell'\}$ is not a bridge in G, and since G is outerplanar, u and v cannot both have a neighbor in $G' - \ell$ (since if we were to contract each edge of $G' - \ell$ to obtain a single vertex t, the graph induced on $\{u, v, \ell, t\}$ would form a K_4). Without loss of generality, let u have a neighbor $w \in V(G' - \ell)$. Since u and v have the same neighbors in $V(G') \setminus \{\ell, \ell'\}$, it follows that $w = \ell'$, $G[\{u, v, \ell, \ell'\}]$ is $K_4 - e$, and G satisfies (3) with $G^{[0]} = K_4 - e$.

Now assume v is not adjacent to ℓ. By the assumption that the neighbors of each degree-two vertex are adjacent, it follows that u is adjacent to ℓ' since they are the only neighbors of ℓ in G, and since G has no pendant vertices, v has a neighbor $w \in V(G' - \ell)$. If $w \neq \ell'$, we have already seen that u must also be adjacent to w and that w is the only neighbor of u or v in $V(G) \setminus \{\ell, \ell'\}$. Also note that if $w \neq \ell'$, then by Corollary 1.8, v cannot also be adjacent to ℓ' since G is outerplanar, so $N(u) = \{v, \ell, \ell', w\}$ and $N(v) = \{u, w\}$. To see that G satisfies (3), let $(w, x_1, \ldots, x_j, \ell')$ be a shortest path from w to ℓ' in G' (see Figure 8). Since $G' - \ell \in \mathcal{F}$, it follows that G satisfies (3) with $G^{[0]} = C_r^{\triangle}$ and $C_r = (w, x_1, \ldots, x_j, \ell', u)$.

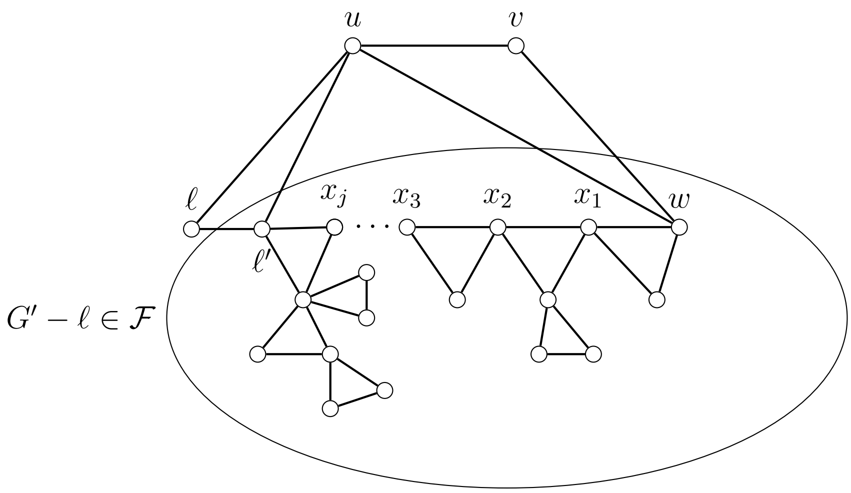

Figure 8: Graph mentioned in proof of Theorem 4.7

Now suppose v is adjacent to ℓ'. Then u and v share the same set of neighbors in $V(G') \setminus \{\ell\}$, and since G is outerplanar, by Corollary 1.8 we must have that $N(u) = \{v, \ell, \ell'\}$ and $N(v) = \{u, \ell'\}$ (see Figure 9). Thus, G satisfies (3) with $G^{[0]} = K_4 - e$.

Case 2. Suppose G' does not have a pendant vertex. Note that this implies that $n \geq 6$. We prove this case by induction on n. Let $n = 6$. Then G' is a graph on four vertices with tree cover number two. Since G' does not have a pendant vertex, then G' is $K_4 - e$ or C_4.

Suppose first that $G' = C_4$. If u has a neighbor $w \in V(C_4)$ and v is not adjacent to w, then for $T_1 = G[\{u, v, w\}]$ and $T_2 = G[V(C_4) \setminus \{w\}]$, $\{T_1, T_2\}$ is a tree cover of G of size 2, contradicting $T(G) = 3$. So u and v have the same set of neighbors in $V(C_4)$, and since G is outerplanar, by Corollary 1.8, u and v have exactly one

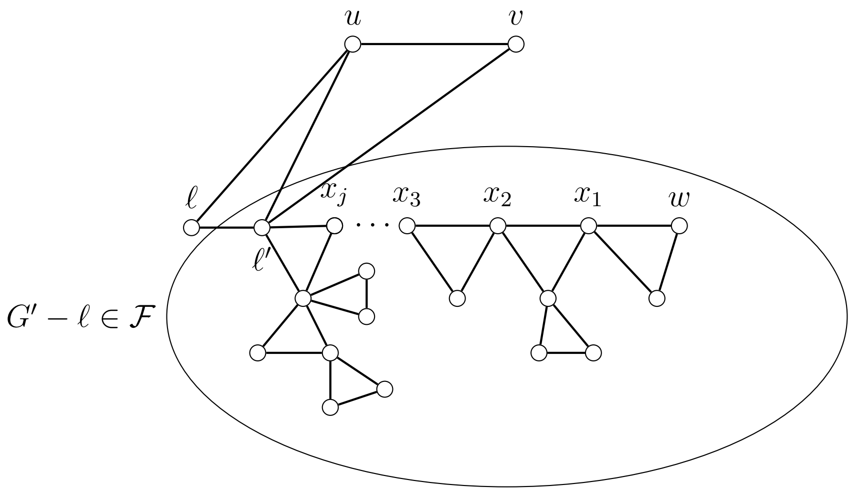

Figure 9: Graph mentioned in proof of Theorem 4.7

neighbor in $V(C_4)$, and (3) holds.

Now assume $G' = K_4 - e$. It is well known that an outerplanar graph on n vertices has at most $2n - 3$ edges (this can be proved by deleting a vertex of degree-two and using induction on n). Thus G has at most nine edges. There are five edges in G' and one edge between u and v, so there are at most three edges between the sets $\{u, v\}$ and $V(G')$, so either u or v has degree two (since G has no pendant vertices). Suppose without loss of generality that $N(u) = \{v, w\}$ for some $w \in V(G')$. By the assumption that the neighbors of each vertex of degree two are adjacent, v and w are adjacent. Note that since G has at most nine edges, v can have at most one additional neighbor. Suppose v has an additional neighbor in $V(G')$. Then G is one of the graphs given in Figure 10 (since the other two possibilities are not outerplanar as one has a K_4 minor and the other has a $K_{2,3}$ minor), and $T(G) = 2$, contradicting $T(G) = \frac{n}{2} = 3$. Thus, v has no additional neighbors, and G satisfies (3) with $G^{[0]} = K_4 - e$.

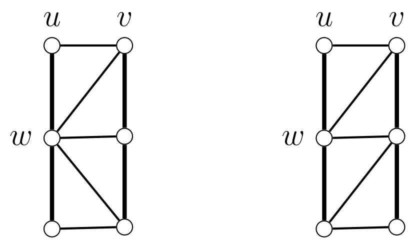

Figure 10: Tree covers of size two given in bold

Now let $n \geq 8$. The graph G' has no pendant vertices, so by Lemma 4.9 we either have that G satisfies (3) (in which case the proof is complete), G' has no bridge and the neighbors of each degree-two vertex in $V(G')$ are adjacent within G', or G'

satisfies (3).

Suppose that G' has no bridge and that the neighbors of each degree-two vertex in $V(G')$ are adjacent. We show that G' satisfies (3). Let G'' be the graph obtained from G' after one more application of Lemma 4.3. If G'' has a pendant vertex, G' satisfies (3) by Case 1. If G'' does not have a pendant vertex, then by the induction hypothesis G' satisfies (3).

We now use the fact that G' satisfies (3) to show that G satisfies (3) by showing that $N(u) = \{v, w\}$ and $N(v) = \{u, w\}$, for some $w \in V(G')$ (i.e., $G = K_3 \oplus_w G'$). Since u is not a pendant vertex, there exists some $w \in V(G')$ that is a neighbor of u. Suppose first that v is not adjacent to w. We show that this contradicts $T(G) = \frac{n}{2}$. Let $\mathcal{T}'$ be a minimum tree cover of G' with each tree having exactly two vertices, as given by Corollary 4.4, and let $V(T_w) = \{w, x\}$. We consider two cases: first, that w and x have a common neighbor y, and second that w and x have no common neighbor. Let $y \in N(w) \cap N(x)$ and let $V(T_y) = \{y, z\}$. If x is not adjacent to z, then by Lemma 4.5, $T(G) \leq \frac{n-2}{2}$ (contradicting that $T(G) = \frac{n}{2}$), so x is adjacent to z and $G[\{w, x, y, z\}]$ is $K_4 - e$. Note that $G[\{u, v, w, x, y, z\}]$ has at most nine edges since it is outerplanar, so there are at most two additional edges between the sets $\{u, v\}$ and $\{x, y, z\}$. Edges $\{u, z\}$ and $\{v, z\}$ each create a K_4 minor and therefore are prohibited. Thus, any additional edge has one endpoint in $\{u, v\}$ and the other in $\{x, y\}$. Edges $\{u, x\}$ and $\{u, y\}$ cannot simultaneously exist, edges $\{v, x\}$ and $\{u, y\}$ cannot simultaneously exist, edges $\{v, x\}$ and $\{v, y\}$ cannot simultaneously exist, and edges $\{v, y\}$ and $\{u, x\}$ cannot simultaneously exist since each of these cases creates a K_4 minor. It can be seen by examination of the remaining six possible graphs (adding edge $\{u, x\}$ only, adding edge $\{u, y\}$ only, adding edge $\{v, x\}$ only, adding edge $\{v, y\}$ only, adding edges $\{u, x\}$ and $\{v, x\}$, and adding edges $\{u, y\}$ and $\{v, y\}$) that $G[\{u, v, w, x, y, z\}]$ can be covered with two trees. These two trees, together with the $\frac{n-2}{2} - 2$ trees given by $\mathcal{T}' \setminus \{T_w, T_y\}$ form a tree cover of G of size $\frac{n-2}{2}$, contradicting $T(G) = \frac{n}{2}$.

Now suppose $N(w) \cap N(x) = \emptyset$. Note that if G' were to satisfy (3) with $G^{[0]} \in \{K_4 - e, C_r^{\triangle}\}$, then every edge of G' would belong to a triangle, so $N(w) \cap N(x) = \emptyset$ implies that G' satisfies (3) with $G^{[0]} = C_4$. Furthermore, every edge of G' that is not an edge of C_4 belongs to a triangle, so $\{w, x\}$ is an edge on C_4. Let (w, x, y, y'') be the 4-cycle, and let $V(T_y) = \{y, z\}$. (It is possible that $z = y''$). By our assumptions on G', y and x have no common neighbors, so $G[\{x, y, z\}]$ is a tree. Since v is not adjacent to w, $G[\{u, v, w\}]$ is a tree. It follows that $G[V(G) \setminus \{u, v, w, x, y, z\}]$ is covered with the $\frac{n-2}{2} - 2$ trees given by $\mathcal{T}' \setminus \{T_w, T_y\}$ and $G[\{u, v, w, x, y, z\}]$ is covered with two trees. This contradicts $T(G) = \frac{n}{2}$.

We have now established that v must be adjacent to w. It follows by symmetry that u and v have the same set of neighbors in G'. By Corollary 1.8, u and v have exactly one common neighbor in G' since G is outerplanar and G' is connected. This shows that G satisfies (3).

We now show the converse. Suppose first that G can be obtained from some $G' \in \mathcal{F}$ by adding one pendant vertex. By Proposition 1.1, $T(G) = T(G')$, and by

Theorem 4.1, $T(G) = T(G') = \frac{n}{2}$.

Suppose G satisfies (2). By Theorem 4.1, $T(G_1) = \left\lceil \frac{|G_1|}{2} \right\rceil$ and $T(G_2) = \left\lceil \frac{|G_2|}{2} \right\rceil$, and by Lemma 1.4, $T(G) = T(G_1) + T(G_2) - 1 = \left\lceil \frac{|G_1|}{2} \right\rceil + \left\lceil \frac{|G_2|}{2} \right\rceil - 1 = \frac{n}{2}$.

Suppose G satisfies (3), with $G = G^{[k]}$ for some $k \geq 0$. If $k = 0$, either $G \in \{C_4, K_4 - e\}$ in which case $T(G) = \frac{n}{2}$ is clear, or $G = C_r^{\triangle}$ for some $r \geq 3$ and $T(G) = \frac{n}{2}$ by Lemma 4.8. Let $G = G^{[k]}$ for some $k \geq 1$. By Proposition 1.2, and by induction, $T(G) = T(G^{[k-1]}) + T(K_3) - 1 = \frac{n}{2}$. □

Corollary 4.10. *Let G be a connected outerplanar graph of even order n. Then $\mathrm{M}_+(G) = \frac{n}{2}$ if and only if one of* (1), (2), (3) *of Theorem 4.7 holds.*

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