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$(27, 6, 5)$ designs with a nice automorphism group

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Abstract

According to the Handbook of Combinatorial Designs, the only known example of a $(27, 6, 5)$ design was given by Hanani. Even though a census of the designs with these parameters appears to be unfeasible, in this paper we show how some algebraic methods and the aid of a computer allow us to say that their number is at least 459. We show, in particular, that two of them are doubly point-transitive with their full automorphism groups being $A\Gamma L(1, 27)$ and $AGL(1, 27)$. These two special designs are both flag-transitive and additive.

1 Introduction

A 2-(v, k, λ)-*design*, or simply a (v, k, λ)-*design*, is a pair $\mathcal{D} = (V, \mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a collection of k-subsets of V called *blocks* having the property that each pair of distinct points of V is contained in precisely λ blocks. It is well known that every point appears in exactly $r := \lambda\frac{v-1}{k-1}$ blocks and that the total number of blocks is $b := \lambda\frac{v(v-1)}{k(k-1)}$. This gives the so-called admissibility conditions

$$\lambda(v-1) \equiv 0 \mod k-1 \qquad \text{and} \qquad \lambda v(v-1) \equiv 0 \mod k(k-1). \tag{1}$$

An automorphism of a design $\mathcal{D} = (V, \mathcal{B})$ is a permutation on V leaving $\mathcal{B}$ invariant. The following definition is quite important in this paper.

Definition 1. A (v, k, λ)-design is 1-*rotational* if it admits an automorphism consisting of a fixed point and a cycle of length $v - 1$.

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Any subgroup of the group $Aut(\mathcal{D})$ of all the automorphisms of a design $\mathcal{D}$ is said to be *an automorphism group* of $\mathcal{D}$. One refers to $Aut(\mathcal{D})$ as *the full automorphism group* of $\mathcal{D}$. Two designs with the same parameters are *isomorphic* if there exists a bijection (*isomorphism*) between their point sets mapping blocks into blocks.

For general background on designs we refer to the excellent textbooks [3, 22].

Saying that the number of (v, k, λ) designs is n one tacitly assumes "up to isomorphism". To establish the precise number n is unfeasible apart from very specific cases. In general, the number of (v, k, λ) designs whose automorphism group is trivial (*asymmetric* designs) is hugely greater than the number of the others. For instance, there are precisely 11,084,874,829 $(19, 3, 1)$ designs but the number of those with a non-trivial automorphism group is only 164,758 [16]. In spite of this fact the designs with a "nice" automorphism group are more easily catchable since they are enlightened by their algebraic properties. For an evidence of the above, we observe that for a given admissible triple (v, k, λ) with $r = \frac{\lambda(v-1)}{k-1} \leq 41$, the tables of the Handbook [17] give, in most cases, the best known lower bound for the number of designs with these parameters, not the precise number. Also, this lower bound often refers to designs none of which is asymmetric.

We have been surprised to see that the lower bound for the number of $(27, 6, 5)$ designs given in these tables is only 1. The related reference is an example given by Hanani [15] and we have checked that its full automorphism group has order 78. In this paper we significantly improve this bound by finding 458 new examples. As a matter of fact one of these 458 was already known and constructed by Abel, but it was not checked that it is not isomorphic to that of Hanani. The paper will be organized as follows.

In the next section we determine two $(27, 6, 5)$ designs whose related groups of automorphisms, $A\Gamma L(1, 27)$ and $AGL(1, 27)$, are very rich. We will show that both these designs are *flag-transitive* and *additive*. The one admitting $A\Gamma L(1, 27)$ is worthy of special attention considering that very little is known about (v, k, λ) designs with $\gcd(r, \lambda) = 1$ and a flag-transitive automorphism group $G \leq A\Gamma L(1, q)$ for some q.

In Section 3 we give the background concerning the 1-*rotational difference families* which are fundamental in the construction of 1-rotational designs (see Definition 1).

In Section 4 we revisit the $(27, 6, 5)$ design by Hanani showing that it is 1-rotational and we compare it with the other 1-rotational $(27, 6, 5)$ design constructed by Abel.

In Section 5 we determine the structure of the 1-rotational difference families giving rise to $(27, 6, 5)$ designs. Considering that they depend on too many parameters, we limit our search to those having a *multiplier* of order 3 — namely to those giving rise to a 1-rotational $(27, 6, 5)$ design with an automorphism group of order 78 as the design by Hanani. We found that their number is 2760. But the number of pairwise non-isomorphic designs generated by them is 230.

In Section 6, by suitably "distorting" the 1-rotational difference families obtained before, we find 228 1-rotational difference families without multipliers giving rise to as many $(27, 6, 5)$ designs with an automorphism group of order 26.

In the final section we give a table summarizing the results obtained in this paper.

These results were first presented by the second author at the Combinatorics conference in Carovigno, Italy, in June 2024.

2 Doubly point-transitive $(27, 6, 5)$ designs

We assume familiarity with the basic notions about *group action*. As it is standard, if G is a group of permutations on a set V and B is a subset of V, we denote by G_B and B^G, the G-stabilizer and the G-orbit of B, respectively.

A design is said to be *doubly point-transitive* when its full automorphism group acts 2-transitively on the points.

The following result is very well-known (see, e.g., Theorem 3.4.3 in [4]).

Theorem 2.1. *Let G be a group of permutations acting t-transitively on a v-set V with $t \geq 2$, and let B be a k-subset of V with $1 < k < v - 1$. Then the pair $\mathcal{D} = (V, B^G)$ is a t-(v, k, λ) design for a suitable λ.*

It is evident that the design $\mathcal{D}$ of the above theorem admits G as an automorphism group acting t-transitively on the points. We apply this theorem with $t = 2$ to find doubly point-transitive $(27, 6, 5)$ designs. First, we recall that an *affine transformation* of the field $\mathbb{F}_q$ is any map of the form

$$\alpha_{m,t} : x \in \mathbb{F}_q \longrightarrow mx + t \in \mathbb{F}_q$$

where m and t are elements of $\mathbb{F}_q$ with $m \neq 0$. We also recall that the affine transformations of $\mathbb{F}_q$ form a group under composition which acts (sharply) doubly transitively on $\mathbb{F}_q$. This group, denoted by $AGL(1, q)$, is called the *one dimensional general affine group over* $\mathbb{F}_q$.

Theorem 2.2. *There exist at least two doubly transitive* $(27, 6, 5)$ *designs.*

Proof. Let $\mathbb{F}_{27}$ be the finite field of order 27 and let $\mathbb{F}_{27}^*$ be its multiplicative group. Take any element $x \in \mathbb{F}_{27}$ not belonging to the subfield of order 3 and consider the 6-subset of $\mathbb{F}_{27}$

$$B_x = \{0, 1, 2, x, x + 1, x + 2\}.$$

In the following G will denote the group $AGL(1, 27)$ of the affine transformations of $\mathbb{F}_{27}$. Given that this group is 2-transitive on $\mathbb{F}_{27}$, by Theorem 2.1 we can claim that $\mathcal{D}_x := (\mathbb{F}_{27}, B_x^G)$ is a $(27, 6, \lambda)$ design for a suitable λ. Let us prove that $\lambda = 5$.

The number b of blocks of a $(27, 6, \lambda)$ design is $\lambda\frac{27\times 26}{30}$. On the other hand, the number of blocks of our design $\mathcal{D}_x$ is the size of B_x^G, that is $\frac{|G|}{|G_{B_x}|}$. Note that G has size 27×26 and check that

$$\{\alpha_{1,0},\ \alpha_{1,1},\ \alpha_{1,2},\ \alpha_{2,x},\ \alpha_{2,x+1},\ \alpha_{2,x+2}\} \subset G_{B_x}$$

so that G_{B_x} has order at least 6. Thus we have:

$$\lambda\frac{27\times 26}{30}=\frac{27\times 26}{|G_{B_x}|}\leq\frac{27\times 26}{6}$$

which gives $\lambda\leq 5$. On the other hand, λ must be a multiple of 5 by the first condition in (1). It necessarily follows that $\lambda=5$.

We conclude that $\mathcal{D}_x$ is a doubly point-transitive $(27,6,5)$ design for any $x\in\mathbb{F}_{27}\setminus\{0,1,2\}$.

Note that $\mathcal{D}_x$ is 1-rotational. Indeed, denoting by ω a primitive element of $\mathbb{F}_{27}$, it is clear that the affine transformation $\alpha_{\omega,0}$ fixes zero and cyclically permutes all the other elements of the field. In the following, as primitive element ω we agree to take a root of the primitive polynomial z^3-z+1.

We observe that $B_x=B_{x+1}=B_{x+2}$ for every $x\in\mathbb{F}_{27}\setminus\{0,1,2\}$. We also observe that the 3-subsets of $\mathbb{F}_{27}\setminus\{0,1,2\}$ of the form $\{x,x+1,x+2\}$ are precisely

$$X_i=\{x_i,x_i+1,x_i+2\},\qquad 1\leq i\leq 8$$

with

$$(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8)=(\omega,\omega^2,\omega^4,\omega^5,\omega^6,\omega^8,\omega^{14},\omega^{19}).$$

One can easily check that we have:

$$X_1=\{\omega,\omega^9,\omega^3\};\qquad X_2=\{\omega^2,\omega^{21},\omega^{12}\};$$
$$X_3=\{\omega^4,\omega^{18},\omega^7\};\qquad X_4=\{\omega^5,\omega^{17},\omega^{20}\};$$
$$X_5=\{\omega^6,\omega^{11},\omega^{10}\};\qquad X_6=\{\omega^8,\omega^{15},\omega^{25}\};$$
$$X_7=\{\omega^{14},\omega^{16},\omega^{22}\};\qquad X_8=\{\omega^{19},\omega^{23},\omega^{24}\}.$$

Thus, from the previous observations, our construction leads to at most eight designs that are $\mathcal{D}_{x_1}$, ..., $\mathcal{D}_{x_8}$. Now check that we have:

$$B_{x_1}+\omega^{14}=B_{x_7};\qquad B_{x_2}+\omega^8=B_{x_6};$$
$$B_{x_3}+\omega^5=B_{x_4};\qquad B_{x_5}+\omega^{19}=B_{x_8}.$$

It follows that $B_{x_i}^G=B_{x_j}^G$ for each pair $(i,j)\in\{(1,7),(2,6),(3,4),(5,8)\}$ so that $\mathcal{D}_{x_i}=\mathcal{D}_{x_j}$. Thus our construction leads to at most four distinct designs that are $\mathcal{D}_{x_1}$, $\mathcal{D}_{x_2}$, $\mathcal{D}_{x_3}$ and $\mathcal{D}_{x_5}$.

Now let $\phi: x\in\mathbb{F}_{27}\longrightarrow x^3\in\mathbb{F}_{27}$ be the *Frobenius automorphism* of $\mathbb{F}_{27}$ and check that it cyclically permutes B_{x_2}, B_{x_5} and B_{x_3}. It follows that ϕ is an isomorphism between $\mathcal{D}_{x_i}$ and $\mathcal{D}_{x_j}$ for each pair $(i,j)\in\{(2,5),(5,3),(3,2)\}$. Indeed, if (i,j) is any of these pairs and (m,t) any pair of $\mathbb{F}_{27}^*\times\mathbb{F}_{27}$, keeping in mind that $\mathbb{F}_{27}$ has characteristic 3 we can write:

$$\phi(\alpha_{m,t}(B_{x_i}))=\phi(mB_{x_i}+t)=(mB_{x_i}+t)^3=m^3B_{x_j}+t^3=\alpha_{m^3,t^3}(B_{x_j}).\qquad(2)$$

Thus ϕ maps every block of $B_{x_i}^G$ into a block of $B_{x_j}^G$, i.e., ϕ turns the block set of $\mathcal{D}_{x_i}$ into the block set of $\mathcal{D}_{x_j}$. Hence $\mathcal{D}_{x_2}$, $\mathcal{D}_{x_3}$ and $\mathcal{D}_{x_5}$ are pairwise isomorphic.

Finally note that ϕ fixes B_{x_1} so that, reasoning similarly as in (2), we can see that ϕ is an automorphism of $\mathcal{D}_{x_1}$. It follows that the group generated by G and ϕ, that is the semidirect product $G \rtimes \langle\phi\rangle$, is an automorphism group of $\mathcal{D}_{x_1}$ of order $27 \times 26 \times 3 = 2106$. This group, denoted by $A\Gamma L(1, 27)$, is called *the one-dimensional general affine semilinear group over* $\mathbb{F}_{27}$.

Using GAP [14] we have checked that the full automorphism group of $\mathcal{D}_{x_1}$ is actually $G \rtimes \langle\phi\rangle$ whereas the full automorphism group of $\mathcal{D}_{x_2}$ is "only" G. We conclude that our construction leads to exactly two non-isomorphic doubly transitive $(27, 6, 5)$ designs. The first one is $\mathcal{D}_\omega = (\mathbb{F}_{27}, B_\omega^G)$ where

$$B_\omega = \{0, 1, 2, \omega, \omega + 1, \omega + 2\} = \{0, 1, 2, \omega, \omega^3, \omega^9\} \tag{3}$$

and its full automorphism group of order 2106 is $A\Gamma L(1, 27)$. The second one is $\mathcal{D}_{\omega^2} = (\mathbb{F}_{27}, B_{\omega^2}^G)$ where

$$B_{\omega^2} = \{0, 1, 2, \omega^2, \omega^2 + 1, \omega^2 + 2\} = \{0, 1, 2, \omega^2, \omega^{12}, \omega^{21}\} \tag{4}$$

and its full automorphism group of order 702 is $AGL(1, 27)$. □

We note that both the designs $\mathcal{D}_\omega$ and $\mathcal{D}_{\omega^2}$ are point-primitive and block-imprimitive. The primitivity on the points obviously derives from the fact that they are doubly point-transitive. The imprimitivity on the blocks is a consequence of the well-known fact that a transitive permutation group is primitive if and only if a point stabilizer is a maximal subgroup (see, e.g., [12, Corollary 1.5A]).

2.1 A connection with flag-transitive designs

A *flag* of a design $(V, \mathcal{B})$ is any pair $(x, B) \in V \times \mathcal{B}$ with $x \in B$. A design is *flag-transitive* if it admits an automorphism group acting transitively on its flags. There is very extensive literature on this topic (see, e.g., [2, 19, 5]).

Proposition 2.3. *The designs $\mathcal{D}_\omega$ and $\mathcal{D}_{\omega^2}$ constructed above are flag-transitive. In particular, they are sharply flag-transitive under the action of $AGL(1, 27)$.*

Proof. Keeping the same notation used in Theorem 2.2, we note that G_{B_x} is isomorphic to the dihedral group of order 6 and acts sharply transitively on the points of B_x for any $x \in \mathbb{F}_{27} \setminus \{0, 1, 2\}$. Fix $x \in \{\omega, \omega^2\}$ and take any two flags f_1, f_2 of the design $\mathcal{D}_x$. For $i = 1, 2$, we have $f_i = (y_i^{g_i}, B_x^{g_i})$ for a suitable $y_i \in B_x$ and a suitable $g_i \in G$ by definition of $\mathcal{D}_x$. Given that the points y_1, y_2 are in B_x, from the previous observation about G_{B_x} there is exactly one element $h \in G_{B_x}$ such that $y_1^h = y_2$. Then we see that $g_1^{-1} h g_2$ brings the flag f_1 into the flag f_2 so that $\mathcal{D}_x$ is flag-transitive.

Clearly, a (v, k, λ) design has precisely $\frac{\lambda v(v-1)}{k-1}$ flags. Thus, in particular, the number of flags of a $(27, 6, 5)$ design is $27 \cdot 26$ which is also the order of the group $G = AGL(1, 27)$. It follows that G is sharply flag-transitive on both $\mathcal{D}_\omega$ and $\mathcal{D}_{\omega^2}$. □

The (v, k, λ) designs with $\gcd(r, \lambda) = 1$ having a flag-transitive automorphism group G have been completely classified except when G is a subgroup of a semilinear affine group. Some examples lying in this exceptional situation are provided in Section 4 of [2]. We note that our design $\mathcal{D}_\omega$ is one more example in view of Proposition 2.3.

2.2 A connection with additive designs

A design is said to be *additive* if, up to isomorphism, its point set V is a subset of an abelian group G and its blocks are all zero-sum in G. In particular, it is *strictly additive* if V is the whole group G. This is a very interesting topic introduced in [10], further developed in [11, 13, 21], and generalized in [6]. It seems that an additive (v, k, λ) design is harder to construct the smaller the λ. For instance, the only known classes of additive $(v, k, 1)$ designs are the point-line designs associated to an affine or projective geometry [9]. Infinitely many other constructions can be found in [8] but they are unwieldy since v is huge in comparison with k. A sporadic example of an additive $(124, 4, 1)$ design was constructed in [6]. An example of a strictly additive $(81, 6, 2)$ design was constructed in [20].

Now we prove that the two $(27, 6, 5)$ designs $\mathcal{D}_\omega$ and $\mathcal{D}_{\omega^2}$ constructed above are strictly additive.

Proposition 2.4. *The doubly transitive designs $\mathcal{D}_\omega$ and $\mathcal{D}_{\omega^2}$ are strictly additive.*

Proof. For $i = 1, 2$, the point set of $\mathcal{D}_{\omega^i}$ is the additive group of $\mathbb{F}_{27}$ and a block of $\mathcal{D}_{\omega^i}$ is of the form $mB_{\omega^i} + t$ for a suitable pair $(m, t) \in \mathbb{F}_{27}^* \times \mathbb{F}_{27}$. Recalling that $B_{\omega_i} = \{0, 1, 2, \omega^i, \omega^i + 1, \omega^i + 2\}$, we can write

$$mB_{\omega^i} + t = \{t,\ m + t,\ 2m + t,\ m\omega^i + t,\ m\omega^i + m + t,\ m\omega^i + 2m + t\}.$$

Thus we see that the elements of $mB_{\omega^i} + t$ sum up to $3(2m + m\omega^i + 2t)$ which is clearly equal to zero since we are in characteristic 3. The assertion follows. □

The results of this section can be summarized as follows.

Theorem 2.5. *Let $G = AGL(1, 27)$ and let B_ω and B_{ω^2} be the subsets of $\mathbb{F}_{27}$ defined in (3) and (4), respectively.*

The pair $\mathcal{D}_\omega = (\mathbb{F}_{27}, {B_\omega}^G)$ is a $(27, 6, 5)$ design with full automorphism group $A\Gamma L(1, 27)$ which is doubly point-transitive, flag-transitive, and strictly additive.

The pair $\mathcal{D}_{\omega^2} = (\mathbb{F}_{27}, B_{\omega^2}^G)$ is a $(27, 6, 5)$ design with full automorphism group G which is doubly point-transitive, flag-transitive, and strictly additive.

3 1-rotational difference families

The fundamental tool for getting 1-rotational designs are the so-called 1-*rotational difference families* (see §16.6 in [1]). We recall here all the basic definitions for the comprehension of this matter.

In the following, given any positive integer v, we consider the action of the group $G = (\mathbb{Z}_{v-1}, +)$ on the set $\mathbb{Z}_{v-1} \cup \{\infty\}$ (with ∞ being a symbol not in $\mathbb{Z}_{v-1}$) defined by $g(x) = x + g$ for every pair $(g, x) \in \mathbb{Z}_{v-1} \times \mathbb{Z}_{v-1}$ and by $g(\infty) = \infty$ for every $g \in \mathbb{Z}_{v-1}$. Again, given a k-subset B of $\mathbb{Z}_{v-1} \cup \{\infty\}$, we denote by G_B and B^G the G-stabilizer and the G-orbit of B, respectively.

The *list of differences* of B is the multiset ΔB defined as follows:

$$\Delta B = \{b - b' \mid b, b' \in B; b \neq b'\} \qquad \text{if } \infty \notin B,$$

or

$$\Delta B = \{b - b' \mid b, b' \in B \setminus \{\infty\}; b \neq b'\} \cup \underbrace{\{\infty, \infty, \ldots, \infty\}}_{|B|-1 \text{ times}} \qquad \text{if } \infty \in B.$$

It is possible to see that the list ΔB defined above is the multiset union of $|G_B|$ copies of a multiset ∂B which is called *the list of partial differences of* B.

Remark 2. It is very easy to see that the following facts hold.

(i) $\partial B = \Delta B \iff G_B = \{0\}$;

(ii) $|G_B|$ is a divisor of $|B \setminus \{\infty\}|$;

(iii) $|\partial B| = \begin{cases} \frac{k(k-1)}{|G_B|} & \text{if } \infty \notin B \\ \frac{(k-1)^2}{|G_B|} & \text{if } \infty \in B \end{cases}$

Definition 3. A 1-*rotational* (v, k, λ) *difference family* is a collection $\mathcal{F} = \{B_1, \ldots, B_n\}$ of k-subsets (*base blocks*) of $\mathbb{Z}_{v-1} \cup \{\infty\}$ such that every non-zero element of $\mathbb{Z}_{v-1} \cup \{\infty\}$ occurs λ times in $\partial B_1 \cup \ldots \cup \partial B_n$. A base block B_i is *full* or *short* according to whether B_i^G has full order $v - 1$ or not, respectively.

Equivalently, a base block is full if and only if its G-stabilizer is trivial. The importance of 1-rotational difference families is clarified by the following theorem.

Theorem 3.1. *Let $\mathcal{F} = \{B_1, \ldots, B_n\}$ be a 1-rotational (v, k, λ) difference family and let $\mathcal{B} = B_1^G \cup \ldots \cup B_n^G$. Then $\mathcal{D} = (\mathbb{Z}_{v-1} \cup \{\infty\}, \mathcal{B})$ is a 1-rotational (v, k, λ) design.*

An automorphism of the design $\mathcal{D}$ in Theorem 3.1 with a fixed point and a cycle of length $v - 1$ is the permutation π on $\mathbb{Z}_{v-1} \cup \{\infty\}$ defined by $\pi(\infty) = \infty$ and $\pi(x) = x + 1$ for every $x \in \mathbb{Z}_{v-1}$. The group $\langle \pi \rangle$ of permutations generated by π is clearly a group of automorphisms of $\mathcal{D}$ isomorphic to $\mathbb{Z}_{v-1}$ acting sharply transitively on all but one point.

The converse of Theorem 3.1 holds, namely every 1-rotational (v, k, λ) design $\mathcal{D} = (V, \mathcal{B})$ is generated, up to isomorphism, by a suitable 1-rotational (v, k, λ) difference family $\mathcal{F}$ which can be constructed in five steps as follows.

1ST STEP. Take an automorphism α of $\mathcal{D}$ fixing one point z and cyclically permuting the others (it exists by definition of a 1-rotational design).

2ND STEP. Choose a point $y \in V \setminus \{z\}$.

3RD STEP. Take the function $f : V \longrightarrow \mathbb{Z}_{v-1} \cup \{\infty\}$ mapping z to ∞, and mapping $\alpha^i(y)$ to i for $0 \leq i \leq v-2$.

4TH STEP. Take a complete system S of representatives for the $\langle\alpha\rangle$-orbits on $\mathcal{B}$.

5TH STEP. $\mathcal{F} = \{f(B) \mid B \in S\}$ is a 1-rotational (v, k, λ) difference family giving rise to a design isomorphic to $\mathcal{D}$.

Example 4. Let us examine which is the 1-rotational difference family $\mathcal{F}_\omega$ giving rise to the doubly transitive design $\mathcal{D}_\omega$ constructed in Section 2.

1ST STEP. Take the permutation α on $\mathbb{F}_{27}$ defined by $\alpha(x) = \omega x$ for every $x \in \mathbb{F}_{27}$. This is an automorphism fixing the point $z = 0$ and cyclically permuting all the others.

2ND STEP. Take $y = 1$ as auxiliary point in $\mathbb{F}_{27} \setminus \{0\}$.

3RD STEP. We have $\alpha^i(y) = \alpha^i(1) = \omega^i$. Hence we have to consider the function $f : \mathbb{F}_{27} \longrightarrow \mathbb{Z}_{26} \cup \{\infty\}$ defined by $f(0) = \infty$ and $f(\omega^i) = i$ for $0 \leq i \leq 25$.

4TH STEP. We have checked that

$$S = \{B_\omega,\ B_\omega + \omega,\ B_\omega + \omega^2,\ B_\omega + \omega^4,\ B_\omega + \omega^5\}$$

is a complete system of representatives for the $\langle\alpha\rangle$-orbits on $\mathcal{B}$ and we have:

$$B_\omega = \{0, \omega^0, \omega^{13}, \omega, \omega^3, \omega^9\};$$

$$B_\omega + \omega = \{\omega, \omega^9, \omega^3, \omega^{14}, \omega^{16}, \omega^{22}\}; \quad B_\omega + \omega^2 = \{\omega^2, \omega^{21}, \omega^{12}, \omega^{10}, \omega^6, \omega^{11}\};$$

$$B_\omega + \omega^4 = \{\omega^4, \omega^{18}, \omega^7, \omega^2, \omega^{21}, \omega^{12}\}; \quad B_\omega + \omega^5 = \{\omega^5, \omega^{17}, \omega^{20}, \omega^{19}, \omega^{23}, \omega^{24}\}.$$

5TH STEP. Taking the images of the members of S under f we obtain the 1-rotational $(27, 6, 5)$ difference family $\mathcal{F}_\omega$ whose base blocks are:

$$B_1 = \{\infty, 0, 13, 1, 3, 9\};$$
$$B_2 = \{1, 9, 3, 14, 16, 22\};$$
$$B_3 = \{2, 21, 12, 10, 6, 11\};$$
$$B_4 = \{4, 18, 7, 2, 21, 12\};$$
$$B_5 = \{5, 17, 20, 19, 23, 24\}.$$

Example 5. By proceeding as in Example 4 one finds that the base blocks of a 1-rotational $(27, 6, 5)$ difference family $\mathcal{F}_{\omega^2}$ giving rise to $\mathcal{D}_{\omega^2}$ are the following:

$$B_1 = \{\infty, 0, 13, 2, 21, 12\};$$
$$B_2 = \{2, 21, 12, 15, 25, 8\};$$
$$B_3 = \{1, 9, 3, 10, 6, 11\};$$
$$B_4 = \{4, 18, 7, 23, 24, 19\};$$
$$B_5 = \{5, 17, 20, 3, 1, 9\}.$$

It is quite obvious that if $\mathcal{F} = \{B_1, \dots, B_n\}$ is a 1-rotational (v, k, λ) difference family, then $\mathcal{F}' = \{uB_1 + t_1, \dots, uB_n + t_n\}$ is a 1-rotational (v, k, λ) difference family as well, for any unit u of $\mathbb{Z}_{v-1}$ and for any n-tuple $(t_1, \dots, t_n)$ of elements of $\mathbb{Z}_{v-1}$ (it is understood that $u\infty = \infty$). Two difference families $\mathcal{F}$ and $\mathcal{F}'$ as above are said to be *equivalent*. It is also evident that equivalent difference families generate isomorphic designs.

A *multiplier* of a 1-rotational (v, k, λ) difference family $\mathcal{F} = \{B_1, \dots, B_n\}$ is a unit u of $\mathbb{Z}_{v-1}$ such that uB_i is a translate of $B_{\pi(i)}$ for a suitable permutation π of $\{1, \dots, n\}$.

We note that 3 is a multiplier of the difference family $\mathcal{F}_\omega$ constructed in Example 4. This is a consequence of the fact that the Frobenius automorphism of $\mathbb{F}_{27}$ is an automorphism of $\mathcal{D}_\omega$. More concretely, we can see that multiplying by 3 the base blocks $B_1, \dots, B_5$ of Example 4 we get the following: $3B_1 = B_1$; $3B_2 = B_2$; $3B_3 = B_5 + 13$; $3B_4 = B_3$; $3B_5 = B_4 + 13$.

On the contrary, one can check that the difference family $\mathcal{F}_{\omega^2}$ constructed in Example 5 does not have any non-trivial multiplier.

All the multipliers of a 1-rotational (v, k, λ) difference family $\mathcal{F}$ form a subgroup of the group $U(\mathbb{Z}_{v-1})$ of all the units of $\mathbb{Z}_{v-1}$. We also note that if u is a multiplier of $\mathcal{F}$, then the permutation μ_u on $\mathbb{Z}_{v-1} \cup \{\infty\}$ defined by $\mu_u(\infty) = \infty$ and $\mu_u(x) = ux$ for every $x \in \mathbb{Z}_{v-1}$ is an automorphism of the design $\mathcal{D}$ generated by $\mathcal{F}$. It follows that if M is the group of the multipliers of $\mathcal{F}$, then the semidirect product $\mathbb{Z}_{v-1} \rtimes M$ is an automorphism group of $\mathcal{D}$.

4 Previously known $(27, 6, 5)$ designs

Up to our knowledge, there were only two known $(27, 6, 5)$ designs in the literature.

The first example was given by Hanani [15, Table 5.18]; it arises from the 1-rotational $(27, 6, 5)$ difference family $\mathcal{H} = \{H_1, \dots, H_5\}$ whose base blocks are the following:

$$\begin{aligned}
H_1 &= \{\infty, 0, 13, 1, 3, 9\};\\
H_2 &= \{14, 16, 22, 1, 3, 9\};\\
H_3 &= \{2, 6, 13, 1, 3, 9\};\\
H_4 &= \{6, 18, 13, 1, 3, 9\};\\
H_5 &= \{18, 2, 13, 1, 3, 9\}.
\end{aligned}$$

We note that the second block is short; indeed it is readily seen that $G_{H_2} = \{0, 13\}$. All the other blocks are full.

Remark 6. We warn the reader that Hanani presented the above example using a different notation. More precisely, the group $\mathbb{Z}_{26}$ was given as direct product $\mathbb{Z}_2 \times \mathbb{Z}_{13}$. The elements 0 and 1 of $\mathbb{Z}_2$ were represented with $\emptyset$ and 0, respectively. The zero of $\mathbb{Z}_{13}$ was also represented with the symbol $\emptyset$ whereas any other element x of $\mathbb{Z}_{13}$ was represented with its "logarithm" in base 2 (if $x = 2^y$ mod 13, then x was represented by y).

Another $(27, 6, 5)$ design has been constructed by R.J.R. Abel [1, Example 16.86] and it is generated by the 1-rotational $(27, 6, 5)$ difference family $\mathcal{A} = \{A_1, \ldots, A_5\}$ whose base blocks are the following:

$$\begin{aligned} A_1 &= \{\infty, 0, 7, 11, 13, 21\}; \\ A_2 &= \{2, 5, 6, 15, 18, 19\}; \\ A_3 &= \{0, 1, 2, 8, 22, 23\}; \\ A_4 &= \{0, 3, 6, 24, 14, 17\}; \\ A_5 &= \{0, 9, 18, 20, 16, 25\}. \end{aligned}$$

As in Hanani's example, only the second block is short.

Both the 1-rotational designs by Abel and Hanani have an automorphism group isomorphic to $\mathbb{Z}_{26} \rtimes \mathbb{Z}_3$. Indeed their related 1-rotational difference families admit 3 as a multiplier of order 3; μ_3 fixes H_i and A_i for $i = 1, 2$ whereas it cyclically permutes the triples (H_3, H_4, H_5) and (A_3, A_4, A_5).

On the other hand, using GAP [14] we have checked the block intersection numbers of these two designs to show that they are not isomorphic.

	Abel[1]	Hanani[15]
$\lvert B \cap B' \rvert = 0$	1040	702
$\lvert B \cap B' \rvert = 1$	3198	3900
$\lvert B \cap B' \rvert = 2$	2067	1911
$\lvert B \cap B' \rvert = 3$	481	117
$\lvert B \cap B' \rvert = 4$	0	78
$\lvert B \cap B' \rvert = 5$	0	78

Using `nauty`, [18], we have also checked that the full automorphism group of both the designs by Abel and Hanani has order 78 so that none of them is isomorphic to one of the two doubly point-transitive designs $\mathcal{D}_\omega$, $\mathcal{D}_{\omega^2}$ considered in Section 2.

5 The structure of a 1-rotational $(27, 6, 5)$ difference family

Let us determine the size n and the structure of a 1-rotational $(27, 6, 5)$ difference family $\mathcal{F} = \{B_1, \ldots, B_n\}$.

First, it is obvious that each B_i must be a 6-subset of $G \cup \{\infty\}$ where $G = \mathbb{Z}_{26}$.

It is also obvious that at least one base block contains ∞. So, without loss of generality, we can assume that $\infty \in B_1$. Note that $|G_{B_1}|$ must divide both 26 (by the theorem of Lagrange) and $|B_1 \setminus \{\infty\}| = 5$ by Remark 2(ii). It follows that $|G_{B_1}| = 1$ so that ∂B_1 contains ∞ exactly $\frac{|B_1 \setminus \{\infty\}|}{|G_{B_1}|} = 5$ times. Therefore there is no other base block of $\mathcal{F}$ containing ∞, as the list of partial differences of B_1 already covers all the required number of occurrences of ∞ in $\partial B_1 \cup \cdots \cup \partial B_n$, that is 5.

Given that the set of non-zero elements of $G \cup \{\infty\}$ has size 26, by Definition 3 we have

$$|\partial B_1| + |\partial B_2| + \cdots + |\partial B_n| = 5 \times 26 = 130$$

and then, considering that $|\partial B_1| = 25$ by Remark 2(iii), we can write

$$|\partial B_2| + \cdots + |\partial B_n| = 130 - 25 = 105. \tag{5}$$

It follows that there is at least one B_i with $2 \leq i \leq n$ that is short. Indeed, in the opposite case, we would have $|\partial B_i| = |\Delta B_i| = 6 \times 5 = 30$ for $2 \leq i \leq n$ and then, by (5), $30(n-1) = 105$ which is absurd.

Let us assume, without loss of generality, that B_2 is short. Thus $|G_{B_2}|$ is a divisor of 26 greater than 1. On the other hand, by Remark 2(ii), $|G_{B_2}|$ is also a divisor of $|B_2| = 6$. It necessarily follows that G_{B_2} has order 2, hence $G_{B_2} = \{0, 13\}$ so that we have

$$B_2 = \{a, b, c, a+13, b+13, c+13\}$$

for suitable elements $a, b, c \in G$. Now note that 13 appears six times in the list ΔB_2 of differences of B_2. Indeed we have:

$$\begin{aligned} 13 &= (a+13) - a = (b+13) - b = (c+13) - c \\ &= a - (a+13) = b - (b+13) = c - (c+13). \end{aligned}$$

Thus 13 appears $\frac{6}{|G_{B_2}|} = 3$ times in ∂B_2. Then none of the remaining $n-2$ blocks $B_3, \ldots, B_n$ can be short. Indeed, if for instance B_3 was short, repeating the same reasoning done for B_2 we would conclude that 13 appears 3 times in ∂B_3, hence 6 times in $\partial B_2 \cup \partial B_3$. This contradicts the fact that the number of occurrences of every non-zero element of G in $\partial B_1 \cup \cdots \cup \partial B_n$ is 5. Thus for $3 \leq i \leq n$ the block B_i is full and then $|\partial B_i| = |\Delta B_i| = 6 \times 5 = 30$. Taking into account (5) we also deduce that $\mathcal{F}$ has size $n = 5$. To summarize, up to equivalence, any 1-rotational $(27, 6, 5)$ difference family is of the form $\{B_1, B_2, B_3, B_4, B_5\}$ with

$$\begin{aligned} B_1 &= \{\infty, 0, x_1, x_2, x_3, x_4\}, \\ B_2 &= \{0, x_5, x_6, 13, x_5 + 13, x_6 + 13\}, \\ B_3 &= \{0, x_7, x_8, x_9, x_{10}, x_{11}\}, \\ B_4 &= \{0, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}\}, \\ B_5 &= \{0, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}\}. \end{aligned} \tag{6}$$

So it depends on a massive twenty-one parameters. Although these parameters are not completely independent [1], it is understandable that an exhaustive computer search for all the inequivalent 1-rotational $(27, 6, 5)$ difference families would require a long time.

We are going to see that if we add the requirement that 3 is a multiplier (as it happens for the difference families by Hanani and Abel), the number of parameters essentially falls to six.

[1]For instance, it is possible to see that, up to equivalence, we can take $0 < x_1 < x_2 < x_3 < x_4 < 26$, $0 < x_5 < x_6 < 13$, $x_7 = 1 \leq x_{12} \leq x_{17} \leq 21$, and $x_i < x_{i+1} < x_{i+2} < x_{i+3} < x_{i+4} < 26$ for $i = 7, 12, 17$.

Proposition 5.1. *Up to equivalence, a* 1*-rotational* $(27, 6, 5)$ *difference family admitting* 3 *as a multiplier is necessarily of the form* $\{X, Y, Z, 3Z, 9Z\}$ *where:*

- $X = \{\infty, 0, 13, x, 3x, 9x\}$ *with* $x \in \{\pm 1, \pm 2, \pm 4, \pm 5\}$*;*
- $Y = \{1, 3, 9, 14, 16, 22\}$*;*
- Z *is a suitable* 6*-subset of* G*.*

Proof. Let $\mathcal{F}$ be a 1-rotational $(27, 6, 5)$ difference family as in (6) and assume that 3 is a multiplier of $\mathcal{F}$. Thus we have $3B_i = B_{\pi(i)} + t_i$ for a suitable permutation π of $\{1, \ldots, 5\}$ and a suitable 5-tuple $(t_1, \ldots, t_5) \in G^5$.

We have $\infty \in B_1$ so that $\infty \in 3B_1 = B_{\pi(1)} + t_1$ which obviously implies that $\infty \in B_{\pi(1)}$. This gives $\pi(1) = 1$ since B_i does not contain ∞ for each $i \neq 1$. Thus we have $3B_1 = B_1 + t_1$ for a suitable t_1. Up to translations, one can see that this is possible only if we have $B_1 = X$ with X as in the statement.

Given that $3 \cdot 13 = 13$ and that $B_2 + 13 = B_2$, we can write

$$3B_2 + 13 = 3(B_2 + 13) = 3B_2$$

which means that $13 \in G_{3B_2}$. Thus, given that $3B_2 = B_{\pi(2)} + t_2$, we have $G_{B_{\pi(2)}} = G_{B_{\pi(2)}+t_2} = \{0, 13\}$. This gives $\pi(2) = 2$ since G_{B_i} is trivial for each $i \neq 2$. We conclude that B_2 is a 6-subset of G such that

- $G_{B_2} = \{0, 13\}$;
- $3B_2$ is a translate of B_2 itself.

Up to translations, one can see that a 6-subset with these properties necessarily is of the form mY with $m \in \{1, 5, 7, 17\}$ and Y as in the statement.

Assume that we have $\pi(i) = i$ for some $i \in \{3, 4, 5\}$. In this case B_i would be a 6-subset of G such that

- $G_{B_i} = \{0\}$;
- $3B_i$ is a translate of B_i itself.

Up to translations one can see that there are twelve 6-subsets of G with these properties. They are those of the form $\{t, 3t, 9t, \tau, 3\tau, 9\tau\}$ or of the form $\{t, 3t, 9t, 2\tau, 6\tau, 18\tau\}$ where, in both cases, $\{t, \tau\}$ is a 2-subset of $\{\pm 1, \pm 5\}$. In the following, $\mathcal{T}$ will denote the set of these twelve subsets of G.

Now assume that π switches two elements $j, k \in \{3, 4, 5\}$ so that $3B_j$ is a translate of B_k and $3B_k$ is a translate of B_j. Thus, up to equivalence, we may assume that we have $B_j = 3B_k$ and $B_k = 3B_j$. These equalities give $B_j = 9B_j$ and $B_k = 9B_k$. Multiplying by 3 we get $3B_j = B_j$ and $3B_k = B_k$ so that both B_j and B_k belong to $\mathcal{T}$.

From the above two paragraphs, we deduce that if π fixes an element of $\{3, 4, 5\}$, then, up to translations, the three blocks B_3, B_4, B_5 belong to $\mathcal{T}$, hence $\mathcal{F} =$

$\{X, mY, T, T', T''\}$ with X as in the statement, $m \in \{1, 5, 7, 17\}$, and $T, T', T'' \in \mathcal{T}$. On the other hand, this is contradicted by the fact that we have checked by elementary calculation that every $\mathcal{F}$ of this form is not a 1-rotational difference family.

We conclude that, up to a reordering of the indices, π cyclically permutes the three blocks B_3, B_4, B_5. Then, up to translations, we have $\mathcal{F} = \{X, mY, Z, 3Z, 9Z\}$ for a suitable 6-subset Z of $\mathbb{Z}_{26}$. Let μ be the inverse of m modulo 26 and set $\mathcal{F}' = \mu\mathcal{F}$. Of course, by definition, $\mathcal{F}$ and $\mathcal{F}'$ are equivalent 1-rotational $(27, 6, 5)$ difference families. We also note that $\mathcal{F}' = \{X', Y', Z', 3Z', 9Z'\}$ with $X' = \mu X$, $Y' = \mu m Y = Y$, and $Z' = \mu Z$. It is easy to check that $X' = \{\infty, 0, 13, x', 3x', 9x'\}$ for a suitable $x' \in \{\pm 1, \pm 2, \pm 4, \pm 5\}$ so that $\mathcal{F}'$ has the form claimed in the statement and the assertion follows. □

Note that Hanani's difference family $\mathcal{H}$ has precisely the form described by Proposition 5.1.

This is not true for Abel's difference family $\mathcal{A}$ only because its second block $A_2 = \{2, 5, 6, 15, 18, 19\}$ is different from $Y = \{1, 3, 9, 14, 16, 22\}$. Note that we have $A_2 = mY$ with $m = 5$. Thus $Y = \mu A_2$ where $\mu = 21$ is the inverse of m modulo 26. It follows that $\mathcal{A}$ is equivalent to $\mathcal{A}' = \{X, Y, Z, 3Z, 9Z\}$ with $X = \mu A_1 = \{0, 13, 17, 23, 25\}$ and $Z = \mu A_3 = \{0, 21, 16, 12, 20, 15\}$. We finally note that $\mathcal{A}'$ has the form described by Proposition 5.1 since $X = \{0, 13, x, 3x, 9x\}$ with $x = -1$.

By using GAP [14] we counted precisely 2760 difference families as prescribed by Proposition 5.1 up to translations of the block Z. After that, we constructed the designs generated by them and, using `nauty` [18], we have established that their number up to isomorphism is 230, and they include the design D_w constructed in Section 2.

6 More $(27, 6, 5)$ designs by flipping the signs

Even though to determine all the 1-rotational $(27, 6, 5)$ difference families without multipliers appears to be hard, a small fraction of them can be determined easily by using *similar difference families* as in [7].

It is quite clear that $\partial B = \partial(-B)$ for any subset B of $\mathbb{Z}_{v-1} \cup \{\infty\}$. It immediately follows that by flipping the sign of some (possibly none) base blocks of a given 1-rotational difference family $\mathcal{F}$ we get another difference family $\mathcal{F}'$ with the same parameters. We say that $\mathcal{F}$ and $\mathcal{F}'$ are *similar*. The related designs $\mathcal{D}(\mathcal{F})$ and $\mathcal{D}(\mathcal{F}')$ are also said to be *similar*. It is possible to have similar designs which are not isomorphic. Of course, there are precisely 2^n difference families which are similar to $\mathcal{F}$, where n is the size of $\mathcal{F}$. Note, however, that if $\mathcal{F}'$ is obtained from $\mathcal{F}$ by flipping the sign of all the base blocks, then the permutation of $\mathbb{Z}_{v-1} \cup \{\infty\}$ mapping any x into $-x$ is an isomorphism between $\mathcal{D}(\mathcal{F})$ and $\mathcal{D}(\mathcal{F}')$. Thus the number of non-isomorphic designs arising from the difference families similar to $\mathcal{F}$ is at most 2^{n-1}.

Let F be the set of the 230 1-rotational $(27, 6, 5)$ difference families constructed in the above section. For each $\mathcal{F} \in \mathsf{F}$, we have constructed the 32 designs arising from all difference families similar to $\mathcal{F}$ and then, using `nauty` [18] again, we have got 228 pairwise non-isomorphic designs none of which is isomorphic to the starting 230.

7 Conclusion

We have found 459 $(27, 6, 5)$ designs $\mathcal{D}$ with a "nice" automorphism group. Our results about them are summarized in the following table.

<table>
<tr><th>$Aut(\mathcal{D})$</th><th>$|Aut(\mathcal{D})|$</th><th>$\sharp\mathcal{D}$</th><th>properties</th></tr>
<tr><td>$A\Gamma L(27, 1)$</td><td>2106</td><td>1</td><td rowspan="2">doubly point-transitive; flag-transitive; 1-rotational; additive.</td></tr>
<tr><td>$AGL(27, 1)$</td><td>702</td><td>1</td></tr>
<tr><td>$\mathbb{Z}_{26} \rtimes \mathbb{Z}_3$</td><td>78</td><td>229</td><td>1-rotational with a multiplier of order 3.</td></tr>
<tr><td>$\mathbb{Z}_{26}$</td><td>26</td><td>> 228</td><td>1-rotational without multipliers.</td></tr>
</table>

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