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A note on the minimum degree of minimal Ramsey graphs

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Abstract

In this note, we briefly rectify oversights in the works of several authors on $s_r(K_k)$, the Ramsey parameter introduced by Burr, Erdős and Lovász in 1976, which is defined as the smallest minimum degree of a graph G such that any r-colouring of the edges of G contains a monochromatic K_k, whereas no proper subgraph of G has this property. We show that $s_r(K_{k+1}) = O(k^3 r^3 \ln^3 k)$, improving the best known bounds when $k \geqslant 8$ and $k^2 \leqslant r \leqslant O(k^4/\ln^6 k)$.

1 Introduction

A graph G is called r-Ramsey for another graph H, denoted by $G \to (H)_r$, if every r-colouring of the edges of G contains a monochromatic copy of H. Observe that if $G \to (H)_r$, then every graph containing G as a subgraph is also r-Ramsey for H. Some very interesting questions arise when we study graphs G which are minimal with respect to $G \to (H)_r$, that is, $G \to (H)_r$ but there is no proper subgraph G' of G such that $G' \to (H)_r$. We call such graphs *r-Ramsey minimal for H*, and we denote the set of all r-Ramsey minimal graphs for H by $\mathcal{M}_r(H)$. It follows from the

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classical result of Ramsey [12] that $\mathcal{M}_r(H)$ is non-empty for any choices of graph H and positive integer r.

Many questions on $\mathcal{M}_r(H)$ have been explored; for example, the Ramsey number $R_r(H)$ denotes the smallest number of vertices of any graph in $\mathcal{M}_r(H)$ and the size Ramsey number $\hat{R}_r(H)$ denotes the smallest number of edges. We refer the reader to [2, 4, 10, 13] for various results on Ramsey minimal problems. In this paper, we will be interested in the *smallest minimum degree of an r-Ramsey minimal graph*, defined by

$$s_r(H) \coloneqq \min_{G \in \mathcal{M}_r(H)} \delta(G),$$

for a finite graph H and positive integer r, where $\delta(G)$ denotes the minimum degree of G. Trivially, we have $s_r(H) \leqslant R_r(H) - 1$, since the complete graph on $R_r(H)$ vertices is r-Ramsey for H and is $(R_r(H) - 1)$-regular (taking minimal Ramsey subgraphs of this graph cannot increase the minimum degree). This parameter was introduced by Burr, Erdős and Lovász [3] in 1976. They were able to show the rather surprising exact result, $s_2(K_{k+1}) = k^2$, where K_{k+1} is the complete graph on $k+1$ vertices, which is far away from the trivial exponential bound of $s_2(K_{k+1}) \leqslant R_2(k+1) - 1$.

While no precise values are known for $s_r(K_{k+1})$ for $r > 2$, Fox, Grinshpun, Liebenau, Person, and Szabó [6] showed that $s_r(K_{k+1})$ is quadratic in r, up to a polylogarithmic factor, when the size of the clique is fixed. Formally, they showed that for all $k \geqslant 2$ there exist constants $c_k, C_k > 0$ such that for all $r \geqslant 3$, we have

$$c_k r^2 \frac{\ln r}{\ln \ln r} \leqslant s_r(K_{k+1}) \leqslant C_k r^2 (\ln r)^{8k^2}. \tag{1.1}$$

When $k = 2$, Guo and Warnke [7] settled the exact polylogarithmic factor, following earlier work in [6]. The constant in the upper bound of (1.1) is rather large ($C_k \sim k^2 2^{8k^2}$), and in particular not polynomial in k. To remedy this, Fox, Grinshpun, Liebenau, Person, and Szabó [6] also proved an upper bound which is polynomial in both k and r and is applicable for small values of r and k.

Theorem 1.1 (Fox, Grinshpun, Liebenau, Person, Szabó)**.** *For all $k \geqslant 2$, $r \geqslant 3$, $s_r(K_{k+1}) \leqslant 8k^6 r^3$.*

In the other regime, when the number of colours is fixed, Hàn, Rödl, and Szabó [8] showed that $s_r(K_{k+1})$ is quadratic in the clique size k, up to a polylogarithmic factor. They showed that there exists a constant k_0 such that for every $k > k_0$ and $r < k^2$, we have $s_r(K_{k+1}) \leqslant 80^3 (r \ln r)^3 (k \ln k)^2$. Combined with (1.1), this result implies the existence of a large absolute constant C and a polynomial upper bound for $s_r(K_{k+1})$.

Theorem 1.2 (Hàn, Rödl, Szabó)**.** *There exists an absolute constant C such that for every $k \geqslant 2$ and $r < k^2$,*

$$s_r(K_{k+1}) \leqslant C (r \ln r)^3 (k \ln k)^2.$$

Finally, using a group theoretic model of generalised quadrangles introduced by Kantor in 1980 [9], Bamberg and the authors [1] proved another polynomial bound, reducing the dependency in r, and improving on Theorem 1.1 for any k, r and on Theorem 1.2 when $r > k^6$.

Theorem 1.3 (Bamberg, Bishnoi, Lesgourgues)**.** *There exists an absolute constant C such that for all $k \geqslant 2$, $r \geqslant 3$, $s_r(K_{k+1}) \leqslant Ck^5r^{5/2}$.*

These theorems all use the equivalence between $s_r(K_k)$ and another extremal function, called the *r-colour k-clique packing number* [6]. Theorems 1.1 and 1.3 further use some 'triangle-free' point-line geometries, for which, under certain conditions on their parameters, any packing of these geometries implies an upper bound on the r-colour k-clique packing number. This argumentation, initially developed by Dudek and Rödl [5] and then by Fox et al. in [6], has been slightly optimized by Bamberg et al. in [1, Lemma 3.1], allowing for the use of more general geometries and optimising the choice of some parameters. We then rectify the oversight of [1, 6], showing that, by using the optimized argumentation in [1] and the finite geometric construction of Fox et al. in [6], we immediately obtain the following upper bound, improving on the best known bounds for $k \geqslant 8$ and r in the range $k^2 \leqslant r \leqslant O(k^4/\ln^6 k)$.

Theorem 1.4. *For all $k \geqslant 2$, $r \geqslant 3$, $s_r(K_{k+1}) \leqslant (8kr \ln k)^3$.*

Table 1 contains a summary of the bounds presented above, explaining which theorem gives the best known upper bound for $s_r(K_{k+1})$, depending on the range of r as a function of k.

Range for r	$r < k^2$	$k^2 \leqslant r \leqslant O(k^4/\ln^6 k)$	$r = \Omega(k^4/\ln^6 k)$
Upper bound	$C(r \ln r)^3(k \ln k)^2$	$(8kr \ln k)^3$	$Ck^5r^{5/2}$
Source	Theorem 1.2 [8]	Theorem 1.4	Theorem 1.3 [1]

Table 1: Upper bounds for $s_r(K_{k+1})$.

2 Packing partial linear spaces

A partial linear space is an incidence structure of points $\mathcal{P}$ and lines $\mathcal{L}$, with an incidence relation such that there is at most one line through every pair of distinct points. If every line is incident with exactly $s+1$ points and every point is incident with exactly $t+1$ lines, then the partial linear space has order (s,t). If there are no three distinct lines pairwise meeting each other in three distinct points, then the partial linear space is *triangle-free*. *Generalised quadrangles* are standard examples of triangle-free partial linear spaces, with the additional property that for every non-incident point-line pair x, ℓ there exists a unique point x' incident to ℓ such that x and x' are collinear (see the book by Payne and Thas [11] for a standard reference on finite generalised quadrangles).

The next lemma can be found in [1, Lemma 3.1]. Its proof follows a methodology initially developed by Dudek and Rödl [5], using the r-colour k-clique packing number developed in [6].

Lemma 2.1 (Bamberg, Bishnoi, Lesgourgues)**.** *Let r, k, s, t be positive integers. Say there exists a family $(\mathcal{I}_i)_{i=1}^r$ of triangle-free partial linear spaces of order (s, t), on the same point set $\mathcal{P}$ and with pairwise disjoint line sets $\mathcal{L}_1, \ldots, \mathcal{L}_r$, such that the point-line geometry $(\mathcal{P}, \bigcup_{i=1}^r \mathcal{L}_i)$ is also a partial linear space. If $s \geqslant 3rk \ln k$ and $t \geqslant 3k(1 + \ln r)$, then $s_r(K_{k+1}) \leqslant |\mathcal{P}|$.*

While Theorem 1.1 from [6] was the motivation behind the general Lemma 2.1 and its use in conjunction with the new group-based construction in [1], the authors did not check then the impact of their improved Lemma 2.1 directly on the motivating construction of [6]. This note aims at rectifying this oversight. The following lemma is a reformulation in the language of (triangle-free) partial linear space of the construction that can be found in [6, Proof of Lemma 4.4]. Theorem 1.4 is then a direct consequence of Lemmas 2.1 and 2.2.

Lemma 2.2. *Let q be any prime power. There exists a family $(\mathcal{I}_i)_{i=1}^{q-1}$ of triangle-free partial linear spaces of order $(q-1, q-2)$, on the same point set $\mathcal{P}$ of size q^3 and with pairwise disjoint line-sets $\mathcal{L}_1, \ldots, \mathcal{L}_{q-1}$, such that the point-line geometry $\big(\mathcal{P}, \bigcup_{i=1}^{q-1} \mathcal{L}_i\big)$ is also a partial linear space.*

We include a short proof of Theorem 1.4 for completeness. We note again that this is a replica of the proof of Fox et al. [6], with an optimal choice of q allowed by the work of Bamberg et al [1] presented in Lemma 2.1. The proof of [6] (using their own construction) works word by word if the prime q is chosen to be at least $Ckr \ln k$, for some constant C, instead of $k^2 r$ as done [6].

Proof of Theorem 1.4. Let $k \geqslant 2$, $r \geqslant 3$, and let q be the smallest prime such that $q \geqslant 4kr \ln k$. By Bertrand's postulate, $q \leqslant 8kr \ln k$. By Lemma 2.2, there exists a family of $r < q$ triangle-free partial linear spaces of order $(q-1, q-2)$, on the same point set $\mathcal{P}$ and pairwise disjoint line-sets $\mathcal{L}_1, \ldots, \mathcal{L}_r$, such that the point-line geometry $(\mathcal{P}, \bigcup_{i=1}^r \mathcal{L}_i)$ is also a partial linear space. Note that with $k \geqslant 2$ and $r \geqslant 3$, we have $q - 1 \geqslant 3rk \ln k$ and $q - 2 \geqslant 3k(1 + \ln r)$. By Lemma 2.1, $s_r(K_{k+1}) \leqslant |\mathcal{P}|$, and then $|\mathcal{P}| = q^3$ yields the desired bound. □

A careful review of the arguments in [1, Lemma 3.1 and 5.2] would allow a small optimisation on the multiplicative constant of this corollary. However, in light of the conjectured quadratic upper bound [1, Conjecture 5.2], we did not push this further.

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