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Non-conflicting nowhere-zero $Z_2 \times Z_2$-flows in cubic graphs

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Abstract

Let $Z_2 \times Z_2 = \{0, \alpha, \beta, \alpha + \beta\}$. Suppose G is a bridgeless cubic graph, F a perfect matching of G and $\overline{F}$ the complementary 2-factor of F. A nowhere-zero $Z_2 \times Z_2$-flow θ of $G/\overline{F}$ is called non-conflicting with respect to $\overline{F}$ if there is no edge $e = uv$ of $\overline{F}$ such that u is incident to an edge with θ-value α and v is incident to an edge with θ-value β. In this paper, we demonstrate the usefulness of non-conflicting flows by showing that if a cubic graph G admits such a flow with respect to some 2-factor $\overline{F}$ then G admits a normal 6-edge-coloring. We use this observation in order to show that claw-free bridgeless cubic graphs as well as bridgeless cubic graphs possessing a 2-factor having at most two cycles admit a normal 6-edge-coloring. We demonstrate the usefulness of non-conflicting flows further by relating them to a recent conjecture of Thomassen about edge-disjoint perfect matchings in highly connected regular graphs. Finally, we construct infinitely many 2-edge-connected cubic graphs such that $G/\overline{F}$ does not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to any perfect matching F.

1 Introduction

Graphs considered in this paper are finite and undirected. They do not contain loops, though they may contain parallel edges. We also consider pseudo-graphs, which may contain both loops and parallel edges, and simple graphs, which contain neither loops nor parallel edges. As usual, a loop contributes to the degree of a vertex by two. For a graph G, $V = V(G)$ and $E = E(G)$ will denote the sets of vertices and edges of G, respectively. A matching in a graph G is a subset F of edges such that no two edges of F share a vertex. A matching F is perfect if every vertex of the graph is incident to an edge from F. A graph G is k-regular if every vertex of G is of degree k. A graph is cubic if it is 3-regular. For $k \geq 1$, a k-factor of a graph G is a spanning

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k-regular subgraph of G. Note that if K is a 1-factor of G, then $E(K)$ is a perfect matching in G. If K is a 2-factor of G, then K is comprised of vertex disjoint circuits such that every vertex of G lies on one of these circuits.

If G is a cubic graph then F is a perfect matching in G if and only if $G - F$ is a 2-factor in G. This 2-factor will be called a complementary 2-factor of F in G. For a perfect matching F of a cubic graph G, its complementary 2-factor will be denoted by $\overline{F}$.

If G is a graph and $X \subseteq V(G)$ then $\partial_G(X)$ denotes the set of edges of G that connect a vertex of X to one from $V \backslash X$. In particular, for a graph G and a vertex v the set of edges of G that are incident to v in G is $\partial_G(v) = \partial_G(\{v\})$. In the paper, we will use contractions in order to obtain small graphs from the one under consideration. If C is a circuit in a graph G then $G/V(C)$ denotes the graph whose vertices are identical to those of G except that $V(C)$ is replaced with a new vertex v_C, and the set of edges is the same except that v_C is incident to all edges of $\partial_G(V(C))$ in $G/V(C)$. Note that if C has a chord then in $G/V(C)$ we do not get a loop corresponding to this edge. Moreover, if G is a bridgeless cubic graph and T a triangle in G then G/T denotes the bridgeless cubic graph in which T is replaced with a new vertex v_T which is incident to the three edges of $\partial_G(V(T))$.

In the paper, we will deal with contractions in cubic graphs. If G is a cubic graph and $\overline{F}$ is a 2-factor in G then $G/\overline{F}$ denotes the graph whose vertices are the circuits of $\overline{F}$ and edge-set is F. Note that if a circuit C contains k chords then we get k loops around the vertex C in $G/\overline{F}$. Moreover, if C_1 and C_2 are two circuits of $\overline{F}$ that are joined with t edges of F then we get t parallel edges joining C_1 and C_2 in $G/\overline{F}$.

If G is a graph then its girth is the length of the shortest cycle in G. For $n \geq 1$ let K_n denote the unique graph on n vertices where every pair of vertices is an edge in it. Such a graph is called complete and usually is denoted by K_n (Figure 5). A graph G is bipartite if $V(G)$ can be partitioned into two sets V_1 and V_2, such that every edge of G joins a vertex from V_1 to V_2. A bipartite graph is called complete if every vertex of V_1 is joined to every vertex of V_2. When G is a complete bipartite graph with $|V_1| = m$ and $|V_2| = n$ then it will be denoted by $K_{m,n}$.

For $k \geq 1$ a graph G is called cyclically k-edge-connected if we have to delete at least k edges of G so that the resulting graph contains at least two components containing a cycle. A simple path of a graph G is called hamiltonian if all vertices of G lie on it. Similarly, a simple cycle of a graph G is called hamiltonian if all vertices of G lie on it. A graph G is called hamiltonian if it contains a hamiltonian cycle.

A k-edge-coloring of a graph G is an assignment of colors $\{1, \ldots, k\}$ to edges of G such that adjacent edges receive different colors [51]. The smallest k for which G admits a k-edge-coloring is called the chromatic index and is denoted by $\chi'(G)$. Vizing's classical theorem in the area states that if G is a simple graph then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ [53]. Here $\Delta(G)$ denotes the maximum degree of a vertex in G. For cubic graphs this becomes $3 \leq \chi'(G) \leq 4$. Holyer's theorem states that the problem of testing a given cubic graph for $\chi'(G) = 3$ is an NP-complete problem [11].

Let G and H be two cubic graphs. If there is a mapping $\phi : E(G) \to E(H)$ such that for each $v \in V(G)$ there is $w \in V(H)$ such that $\phi(\partial_G(v)) = \partial_H(w)$ then ϕ is called an H-coloring of G. If G admits an H-coloring then we will write $H \prec G$. It can be easily seen that if $H \prec G$ and $K \prec H$, then $K \prec G$. In other words, $\prec$ is a transitive relation defined on the set of cubic graphs. If G is the complete bipartite graph $K_{3,3}$ and H is the complete graph K_4 then Figure 1 shows an example of an H-coloring of G. Here $V(H) = \{1, 2, 3, 4\}$ and $E(H) = \{a_1, a_2, a_3, a_4, a_5, a_6\}$. Figure 1 shows the colors of edges of G with the edges of H and the labels of vertices of G are the vertices of K_4 that this vertex is mapped by the H-coloring of G.

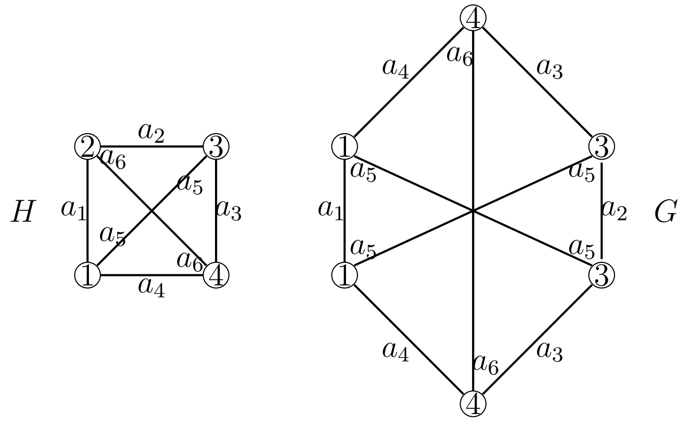

Figure 1: An example of an H-coloring of G.

Let P_{10} be the well-known Petersen graph (Figure 2). The main topic of this paper is the Petersen Coloring Conjecture of Jaeger. It is a striking conjecture in graph theory that asserts that the edge-set of every bridgeless cubic graph G can be colored by $E(P_{10})$ in such a way that adjacent edges of G receive, as colors, adjacent edges of P_{10}.

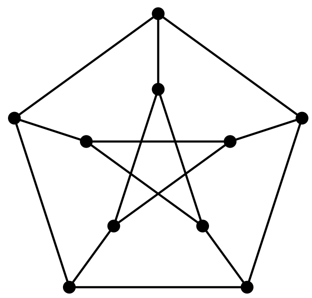

Figure 2: The graph P_{10}.

Conjecture 1.1. *(Jaeger, 1988 [16]) For any bridgeless cubic graph G, one has $P_{10} \prec G$.*

The conjecture is well-known and it is largely considered hard to prove since it implies some other classical conjectures in the field such as the Berge-Fulkerson Conjecture (Conjecture 1.2 below), the Cycle Double Cover Conjecture, the $(5,2)$-cycle-cover conjecture (Conjecture 1.3 below) and the Shortest Cycle Cover Conjecture (see [6, 15, 55]).

Conjecture 1.2. *(Berge-Fulkerson, 1972 [6, 47]) Any bridgeless cubic graph G contains six (not necessarily distinct) perfect matchings $F_1, \dots, F_6$ such that any edge of G belongs to exactly two of them.*

Conjecture 1.3. *($(5,2)$-cycle-cover conjecture, [3, 40]) Any bridgeless graph G (not necessarily cubic) contains five even subgraphs such that any edge of G belongs to exactly two of them.*

It can be shown that the Petersen graph is the only 2-edge-connected cubic graph that can color all bridgeless cubic graphs [37]. In [31] Mazzuoccolo proves that Conjecture 1.2 is equivalent to proving that the edge-set of all bridgeless cubic graphs can be covered with five perfect matchings. Moreover, see the recent paper [21] which shows that every bridgeless cubic graph G has a pair of perfect matchings F_1 and F_2 such that $G - (F_1 \cup F_2)$ is bipartite. This statement is a corollary of Conjecture 1.2 which was conjectured by Mazzuoccolo in [33]. More conjectures similar to Conjecture 1.1 can be found in [9, 37]. In [26, 36] some new results about H-colorings are presented when the graphs under consideration are regular and not necessarily cubic.

Jaeger in [15] introduced an equivalent formulation of Conjecture 1.1. Let c be an edge-coloring of G. For a vertex v of G let $S_c(v)$ be the set of colors that edges incident to v receive. If uv is an edge of a cubic graph G then $3 \leq |S_c(u) \cup S_c(v)| \leq 5$.

Definition 1.4. *Suppose ab is an edge of a cubic graph G and c is an edge-coloring of G. Then:*

- *ab is called poor with respect to c if $|S_c(a) \cup S_c(b)| = 3$,*
- *ab is called abnormal with respect to c if $|S_c(a) \cup S_c(b)| = 4$,*
- *ab is called rich with respect to c if $|S_c(a) \cup S_c(b)| = 5$.*

Edge-colorings having only poor edges are trivially 3-edge-colorings of G. Also edge-colorings having only rich edges have been considered in recent years, and they are called strong edge-colorings. In [15] Jaeger focused on the case when all edges must be either poor or rich.

Definition 1.5. *An edge-coloring c of a cubic graph is normal if G does not contain abnormal edges with respect to c. In other words, any edge is rich or poor with respect to c.*

It is straightforward that an edge-coloring which assigns a different color to every edge of a simple cubic graph is normal since all edges are rich. Hence, we can define the normal chromatic index of a simple cubic graph G denoted by $\chi'_N(G)$ as the smallest k for which G admits a normal k-edge-coloring.

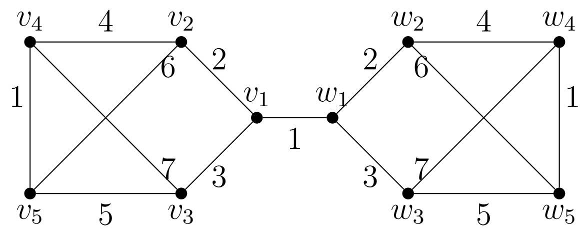

Figure 3: A cubic graph that requires 7 colors in a normal edge-coloring. The bridge is poor. All other edges are rich. It can be shown that $\chi'_N(G) = 7$.

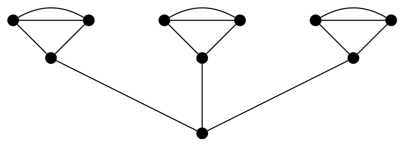

Figure 4: A cubic graph that does not admit a normal k-edge-coloring for any $k \geq 1$.

Figure 3 provides an example of a normal 7-edge-coloring of a cubic graph. All edges of this graph are rich in this coloring except the unique bridge which is poor. In [35] it is shown that $\chi'_N(G) = 7$ if G is the graph from Figure 3. Moreover, [35] argues that any cubic graph having a subgraph that is isomorphic to the complete graph K_4 with one edge subdivided has $\chi'_N(G) = 7$.

Not all cubic graphs admit a normal k-edge-coloring for some $k \geq 1$. Consider a cubic multi-graph containing a triangle in which one edge is of multiplicity two (see Figure 4). It can be easily seen that such a cubic graph cannot have a normal k-edge-coloring because an edge of multiplicity two is going to be abnormal in any edge-coloring.

Using the notion of normal edge-colorings in [15] Jaeger has shown that:

Proposition 1.6. *(Jaeger, [15]) If G is a cubic graph then $P_{10} \prec G$ if and only if G admits a normal 5-edge-coloring.*

This implies that Conjecture 1.1 can be stated as follows:

Conjecture 1.7. *For any bridgeless cubic graph G, $\chi'_N(G) \leq 5$.*

In these terms, the Petersen Coloring Conjecture is equivalent to saying that every bridgeless cubic graph has normal chromatic index at most 5. Observe that Conjecture 1.7 is trivial for 3-edge-colorable cubic graphs. This is true because in any 3-edge-coloring c of a cubic graph G any edge e is poor, and hence c is a normal edge-coloring of G. Thus non-3-edge-colorable cubic graphs are the main obstacle for Conjecture 1.7. Structural properties of non-3-edge-colorable bridgeless cubic graphs sometimes called snarks are investigated in [48, 50]. Conjecture 1.7 is verified for some non-3-edge-colorable bridgeless cubic graphs in [5, 8, 45, 46]. Finally, in [42] the percentage of edges of a bridgeless cubic graph which can be made poor or rich in a 5-edge-coloring is investigated. See [30] for a recent paper about the percentage of normal edges (that is, not abnormal edges) in 5-edge-colored cubic graphs. Other recent results in this direction are obtained in [34] and [39].

If we consider the larger class of simple cubic graphs without any assumption on connectivity, some interesting questions naturally arise. Indeed, examples of simple cubic graphs with $\chi'_N(G) > 5$ can be constructed in this class (Figure 3). Hence it is natural to ask for a possible upper bound for this parameter.

Let us remark that any strong edge-coloring is a normal edge-coloring. Andersen has shown in [1] that any simple cubic graph admits a strong edge-coloring with ten colors. Hence ten is also an upper-bound for the normal chromatic index. The result was improved following the approach of Andersen. In [2] it is shown that any simple cubic graph admits a normal edge-coloring with nine colors. In [35] it is proved that if G is any simple cubic graph then $\chi'_N(G) \leq 7$. This result is complemented with an infinite family of simple cubic graphs in which $\chi'_N(G) = 7$ [35]. Thus, the upper bound seven is (asymptotically) best-possible.

Once the upper bound seven for all simple cubic graphs is established, one may wonder about improving it for some interesting graph classes. Of course, the first class that comes to one's mind is the class of bridgeless cubic graphs. Conjecture 1.7 predicts an upper bound five which is difficult to prove. Thus the following intermediate conjecture could be an excellent step in this direction.

Conjecture 1.8. *(R. Šámal, [43]) For any bridgeless cubic graph G, $\chi'_N(G) \leq 6$.*

In this paper, we focus on Conjecture 1.8. For a given perfect matching F of a bridgeless cubic graph G we introduce the notion of a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow of $G/\overline{F}$ (Definition 2.4). Recall that if A is an abelian group and H is a graph then a nowhere-zero A-flow of H is a pair (D, f) where D is an orientation of H and f is a mapping $f : E(H) \to A - \{0\}$ such that for any vertex $v \in V(H)$

$$f(\partial^+_H(v)) = f(\partial^-_H(v)).$$

Here 0 stands for the zero of A, $\partial^+_H(v)$ is the set of arcs (with respect to D) that leave the vertex v and $\partial^-_H(v)$ is the set of arcs (with respect to D) that enter the vertex v.

We demonstrate the usefulness of non-conflicting nowhere-zero $Z_2 \times Z_2$-flows by showing that if a bridgeless cubic graph G has a perfect matching F such that $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow then G admits a normal 6-edge-coloring (Lemma 2.6). Moreover, we relate non-conflicting nowhere-zero $Z_2 \times Z_2$-flows in cubic graphs to a recent conjecture of Thomassen about the existence of a pair of edge-disjoint perfect matchings in highly edge-connected regular graphs [52] (see Lemma 2.9). Then we obtain the main results of the paper. The first one states that claw-free bridgeless cubic graphs G have a perfect matching F with respect to which $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow (see Theorem 2.17). Moreover, one can find such a perfect matching in bridgeless cubic graphs which have a 2-factor that contains at most two cycles if G is not the Petersen graph (see Theorem 2.19). In the end of the paper we construct infinitely many 2-edge-connected cubic graphs G that do not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ for any perfect matching F of G (see Proposition 2.21 and Theorem 2.22). We conclude the paper in Section 3 where we summarize the paper and present some questions that could be a direction of future research. Non-defined terms and concepts can be found in [54].

2 Main results

In this section we obtain the main results of the paper. We will need some definitions. Let T be a triangle in a cubic graph G such that each edge of T is of multiplicity one. If e is an edge of T, then let f be the edge of G that is incident to a vertex of T and is not adjacent to e. The edges e and f will be called opposite. We start with the following proposition:

Proposition 2.1. *The minimum counterexample to Conjecture 1.8 is a* 3*-edge-connected cubic graph of girth at least four.*

Proof. Let G be a counterexample to the statement minimizing $|V(G)|$. Clearly, G is connected. Let us show that it has no 2-edge-cuts. Assume that $C = \{e_1, e_2\}$ is a 2-edge-cut. Let G_1 and G_2 be the two smaller bridgeless cubic graphs arising from the two components of $G - C$ by adding one edge connecting the two degree-two vertices in the same component. We let h_1 and h_2 be the two added edges of these two graphs, respectively. Since the graphs G_1 and G_2 are smaller we have that they admit normal 6-edge-colorings f_1 and f_2. By renaming the colors in G_2 we can always assume that the colors of h_1 and h_2 are the same and moreover, the colors appearing in the ends of e_1 are also the same. Now, if we color e_1 and e_2 with the color of h_1 then we will have that e_1 is always poor, moreover if at least one of h_1 and h_2 is normal then e_2 will also be normal. This means that the resulting 6-edge-coloring of G will be normal, too. Thus, G must be 3-connected.

Since we do not have 2-edge-cuts in G we cannot have 2-cycles in G. Let us show that G cannot contain a triangle. On the opposite assumption, assume that G contains a triangle T. Consider the bridgeless cubic graph G/T. It is smaller than G. Hence it has a normal 6-edge-coloring. We can extend it to a normal 6-edge-coloring of G as follows: for any edge e of T color e with the color of f, where f is the opposite edge of e. This leads to a normal 6-edge-coloring of G in which the edges of T are poor. Thus, G has girth at least four. □

Remark 2.2. *We presented the proof of Proposition 2.1 for the sake of completeness. It is implicit in [34].*

Remark 2.3. *In [13], Jaeger himself proved that the smallest counterexample to Conjectures 1.1 and 1.7 is a cyclically* 4*-edge-connected cubic graph. Unfortunately, we are not able to get rid of non-trivial* 3*-edge-cuts in the smallest counterexample* *for Conjecture 1.8. So if one is able to prove Conjecture 1.8 for cyclically* 4*-edge-connected cubic graphs it is not obvious how to derive the proof of the full conjecture.*

The situation is different for Conjecture 1.2. By answering a question raised back in [47] Máčajová and Mazzuoccolo proved in [27] that it suffices to prove Conjecture 1.2 for cyclically 5-edge-connected cubic graphs.

Conjecture 1.8 predicts an upper bound six for χ'_N in the class of all bridgeless cubic graphs. We would like to mention that seven is relatively easy to prove via flows [2, 35]. The classical 8-flow theorem [55] implies that every bridgeless cubic graph G admits a nowhere-zero $Z_2 \times Z_2 \times Z_2$-flow. It can be easily verified that any such flow yields a normal 7-edge-coloring of G (see [2, 35] for details). In [35], the authors managed to show that all simple, cubic graphs (not necessarily bridgeless) admit a normal 7-edge-coloring. The proof given in [35] heavily uses flows. So an interesting question is whether flows can be helpful in proving Conjecture 1.8. If they are then one may wonder what kind of flow results we need in order to prove Conjecture 1.8. The following approach is due to Mazzuoccolo:

Definition 2.4. *Let G be a bridgeless cubic graph, F a perfect matching of G and $\overline{F}$ the complementary* 2*-factor of F in G. A nowhere-zero $Z_2 \times Z_2$-flow θ of $G/\overline{F}$ is called non-conflicting with respect to F (or with respect to $\overline{F}$) if $\overline{F}$ contains no edge $e = uv$ such that u is incident to an edge with θ-value α and v is incident to an edge with θ-value β.*

In this paper, an edge $e = uv \in E(\overline{F})$ such that u is incident to an edge with θ-value α and v is incident to an edge with θ-value β will be called a conflicting edge or just a conflict.

Remark 2.5. *If $\overline{F}$ contains a triangle then $G/\overline{F}$ does not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$.*

Proof. Suppose $\overline{F}$ contains a triangle T. Since G is bridgeless, the vertex of $G/\overline{F}$ corresponding to T has degree three. Thus, any nowhere-zero $Z_2 \times Z_2$-flow θ of $G/\overline{F}$ has exactly one edge of values α, β and $\alpha + \beta$. Hence T contains a conflict. □

Lemma 2.6 proved below demonstrates the usefulness of non-conflicting flows for Conjecture 1.8.

Lemma 2.6. *Let G be a bridgeless cubic graph and let F be a perfect matching in it. If $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow θ with respect to $\overline{F}$ then $\chi'_N(G) \leq 6$.*

Proof. We follow the approach of the proof of Lemma 5.2 in [12]. Suppose θ is a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow of $G/\overline{F}$ where $Z_2 \times Z_2 = \{0, \alpha, \beta, \alpha + \beta\}$ and $\alpha = (1, 0)$, $\beta = (0, 1)$. Consider a nowhere-zero $Z_2 \times Z_2 \times Z_2$-flow μ of G obtained from θ as follows: for any edge $h \in F$ we define the triple $\mu(h)$ as follows: $\mu(h) = (0, \theta(h))$. Then let C be any cycle of $\overline{F}$. Let x_0 be any element of $Z_2 \times Z_2 \times Z_2$ whose first coordinate is 1 (for example, $x_0 = (1, 0, 1)$). Assign x_0 to an edge of C. Then observe that the rest of the values of edges of C are defined uniquely in μ. Moreover, the first coordinate of the values of μ on C is 1. Hence for any edges $h_1 \in F$ and $h_2 \in \overline{F}$ we have $\mu(h_1) \neq \mu(h_2)$. Also observe that for different cycles of $\overline{F}$ we can choose x_0 differently. It can be easily checked that μ is a nowhere-zero $Z_2 \times Z_2 \times Z_2$-flow of G. Hence, it yields a normal 7-edge-coloring of G as we mentioned before. Now let us consider an edge-coloring of G obtained from μ by changing the values of all edges e with $\mu(e) = (0, \beta)$ to $\mu(e) = (0, \alpha)$. The resulting coloring is not a flow however it is a normal 6-edge-coloring since θ was non-conflicting by assumption. □

As an approach towards Conjecture 1.8, in [34] the following conjecture is presented:

Conjecture 2.7. *([34]) Let G be a 3-edge-connected cubic graph different from the Petersen graph. Then G admits a nowhere-zero $Z_2 \times Z_2 \times Z_2$-flow f such that there are two elements $x, y \in Z_2 \times Z_2 \times Z_2$ with*

(1) $f^{-1}(\{x, y\})$ is a matching in G,

(2) there is no edge $e = uv$ of G such that u is incident to an edge e_u and v is incident to an edge e_v with $f(e_u) = x$ and $f(e_v) = y$.

The paper [34] shows that Conjecture 2.7 implies Conjecture 1.8.

Remark 2.8. *The nowhere-zero $Z_2 \times Z_2 \times Z_2$-flow μ that we constructed in the proof of Lemma 2.6 satisfies the assumption of Conjecture 2.7.*

Lemma 2.6 demonstrates the usefulness of non-conflicting nowhere-zero $Z_2 \times Z_2$-flows for obtaining normal 6-edge-colorings of cubic graphs. Rather surprisingly as our next statement demonstrates they are useful for a recent and influential conjecture of Thomassen [52] that deals with the problem of existence of pairs of edge-disjoint perfect matchings in highly connected regular graphs. In [52], Thomassen conjectured that there is an integer r_0 such that any r-regular r-edge-connected graph on an even number of vertices with $r \geq r_0$ contains a pair of edge-disjoint perfect matchings. Snarks (graphs like P_{10}) demonstrate that $r_0 \geq 4$. Now, using completely different approaches Rizzi in [41] and Mazzuoccolo in [32] constructed a 4-regular 4-edge-connected graph on an even number of vertices in which every pair of perfect matchings have a common edge. The two constructions lead to the same 4-regular graph up to graph isomorphisms. This implies that $r_0 \geq 5$. Now, we prove a statement that shows the usefulness of non-conflicting nowhere-zero $Z_2 \times Z_2$-flows for this conjecture when $r = 5$.

Lemma 2.9. *Let H be any 5-regular 5-edge-connected graph. Consider a bridgeless cubic graph G obtained from H by replacing every vertex of H with a cycle of length five. Let $\overline{F}$ be the 2-factor of G comprised of these 5-cycles corresponding to vertices of H. If $H = G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ then H contains a pair of edge-disjoint perfect matchings.*

Proof. Suppose θ is a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow of $G/\overline{F} = H$ with respect to $\overline{F}$. Since every vertex of H has odd degree (in fact, it is 5-regular) we have that around every vertex of H there should be an odd number of edges with θ-value α, β and $\alpha+\beta$. In particular, there should be edges $e \in F$ and $f \in F$ such that $\theta(e) = \alpha$ and $\theta(f) = \beta$. If around some vertex of H there are at least three edges with θ-value α or at least three edges with θ-value β then since our cycles of $\overline{F}$ are of length five it can be easily checked that there is a conflict. Thus, around every vertex of H there is exactly one edge of θ-value α and exactly one edge of θ-value β. Now, these edges induce a pair of edge-disjoint perfect matchings in H. $\square$

It is an open problem whether there is a 5-edge-connected 5-regular graph that is not 5-edge-colorable [25]. Next, we discuss non-conflicting nowhere-zero $Z_2 \times Z_2$-flows in bipartite graphs and 3-edge-colorable cubic graphs. We start with:

Observation 2.10. *Let G be a bridgeless cubic graph and let F be a perfect matching of G. If all cycles of $\overline{F}$ are even then $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$.*

Proof. Let $x \in \{\alpha, \beta, \alpha+\beta\}$. For any edge $e \in E(G/\overline{F})$ set $\theta(e) = x$. Since all cycles of $\overline{F}$ are even and $x + x = 0$ we have that θ is a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$. $\square$

Observation 2.11. *Let G be an arbitrary bipartite cubic graph and let F be a perfect matching of G. Then $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$.*

Proof. Since G is bipartite, all cycles in $\overline{F}$ are even. Hence the statement follows from Observation 2.10. $\square$

Observation 2.12. *Every 3-edge-colorable cubic graph G has a perfect matching F such that $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$.*

Proof. If G is 3-edge-colorable then there is a perfect matching F such that $\overline{F}$ has only even cycles. Hence the statement follows from Observation 2.10. $\square$

Recall that a graph G is claw-free if it does not contain four vertices such that the subgraph of G induced on these vertices is isomorphic to $K_{1,3}$. Next, we are going to show that claw-free bridgeless cubic graphs have a perfect matching F such that $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$. Since in [34] it is shown that all such graphs admit a normal 6-edge-coloring our statement is going to strengthen the corresponding result from [34]. Proving upper bounds for χ'_N in this class is important as if one shows that all claw-free simple bridgeless cubic graphs admit a normal 5-edge-coloring then Conjecture 1.7 follows (see [34] for all details on this).

We will need some results on claw-free simple cubic graphs. In [4] arbitrary claw-free graphs are characterized. In [38] Oum has characterized simple claw-free bridgeless cubic graphs. In order to formulate Oum's result we need some definitions. In a claw-free simple cubic graph G any vertex belongs to one, two, or three triangles. If a vertex v belongs to three triangles of G then the component of G containing v is isomorphic to K_4 (Figure 5). An induced subgraph of G that is isomorphic to $K_4 - e$ is called a diamond [38]. It can be easily checked that in a claw-free cubic graph no two diamonds intersect.

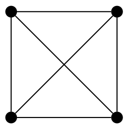

Figure 5: The graph K_4.

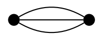

Figure 6: The graph K_2^3.

A string of diamonds of G is a maximal sequence $F_1, \ldots, F_k$ of diamonds in which F_i has a vertex adjacent to a vertex of F_{i+1}, $1 \le i \le k-1$. A string of diamonds has exactly two vertices of degree two which are called the head and the tail of the string. Replacing an edge $e = uv$ with a string of diamonds with the head x and the tail y is to remove e and add edges (u, x) and (v, y).

If G is a connected claw-free simple cubic graph such that each vertex lies in a diamond then G is called a ring of diamonds. It can be easily checked that each vertex of a ring of diamonds lies in exactly one diamond. As in [38] we require that a ring of diamonds contains at least two diamonds.

Proposition 2.13. *(Oum, [38]) G is a connected claw-free simple bridgeless cubic graph if and only if*

(1) G is isomorphic to K_4, or

(2) G is a ring of diamonds, or

(3) there is a connected bridgeless cubic graph H such that G can be obtained from H by replacing some edges of H with strings of diamonds and by replacing any vertex of H with a triangle.

We would like to present the following simple extension of Proposition 2.13 when G may have a parallel edge or just a cycle of length two.

Proposition 2.14. *([10]) G is a connected claw-free bridgeless cubic graph if and only if*

(1) G is isomorphic to K_4 (Figure 5) or to K_2^3 (Figure 6), or

(2) G is a ring of diamonds or 2-cycles, or

(3) there is a connected bridgeless cubic graph H such that G can be obtained from H by replacing some edges of H with strings of diamonds or 2-cycles (Figure 7) and by replacing any vertex of H with a triangle.

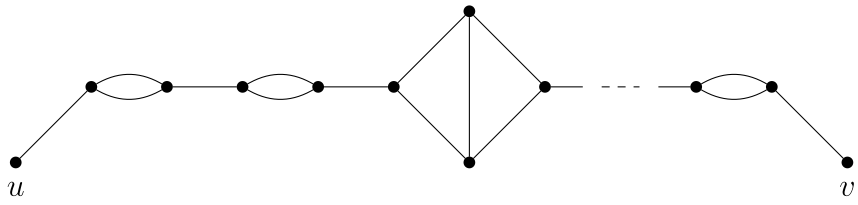

Figure 7: Replacing an edge uv with a string of diamonds or 2-cycles.

It is known that:

Theorem 2.15. *([24, 44]) Every edge of a bridgeless cubic graph G belongs to some perfect matching of G.*

Using the theory of fractional perfect matchings and Edmonds' matching polytope theorem it is not hard to prove:

Theorem 2.16. *([18, 19, 55]) For every edge e of a bridgeless cubic graph G there is a perfect matching F of G such that $e \in F$ and F intersects every 3-edge-cut of G in a single edge.*

We are ready to prove our main result for claw-free bridgeless cubic graphs.

Theorem 2.17. *Let e be an edge of a claw-free bridgeless cubic graph G. Then G has a perfect matching F with $e \in F$ and $G/\overline{F}$ admits a non-conflicting $Z_2 \times Z_2$-flow with respect to $\overline{F}$.*

Proof. Take a perfect matching F of G such that $e \in F$ and F intersects all 3-edge-cuts of G in a single edge (see Theorem 2.16). F intersects all triangles of G in a single edge. Moreover, $G/\overline{F}$ is bridgeless and contains no 3-edge-cuts. Hence by Proposition 10 of [14] $G/\overline{F}$ admits a nowhere-zero $Z_2 \times Z_2$-flow. Consider all nowhere-zero $Z_2 \times Z_2$-flows of $G/\overline{F}$ and among them choose the one with smallest number of conflicts. Let us show that this number is zero.

Suppose there is a conflict with edges $f \in F$ and $f' \in F$. Then at least one of f and f' belongs to a new triangle of G that is this triangle was replacing a vertex of H. Thus, at least one of f and f' is a chord in the corresponding 2-factor $\overline{F}$. Hence it is a loop in $G/\overline{F}$. Thus, by changing the flow value there to $\alpha+\beta$ we will still obtain a nowhere-zero $Z_2 \times Z_2$-flow with less conflicts. This contradicts our choice. Thus, F contains the edge e and $G/\overline{F}$ admits a nowhere-zero $Z_2 \times Z_2$-flow with respect to F as needed. □

Combined with Lemma 2.6 we get the following result from [34]:

Corollary 2.18. *All claw-free bridgeless cubic graphs admit a normal 6-edge-coloring.*

In [34] it is shown that every cubic permutation graph admits a normal 6-edge-coloring. Our next result strengthens this statement by showing that all bridgeless cubic graphs containing a 2-factor having at most two cycles admit such an edge-coloring.

Theorem 2.19. *Let G be a bridgeless cubic graph containing a 2-factor $\overline{F}$ having at most two cycles. Then $G/\overline{F_0}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to some 2-factor $\overline{F_0}$ unless G is the Petersen graph.*

Proof. Our proof is by induction on the number of vertices of G. Clearly, our statement is true when $|V| = 2$. By induction assume that the statement is true for all graphs with less than $|V|$ vertices, and let us consider a bridgeless cubic graph G containing a 2-factor $\overline{F}$ having at most two cycles. If the number of cycles in $\overline{F}$ is one or two and both of the cycles are even then our statement follows from Observation 2.10. Thus, we can focus on the case when $\overline{F}$ has two odd cycles C_1 and C_2. In great contrast with cubic permutation graphs our cycles C_1 and C_2 may have chords. Let n be the number of edges of F joining a vertex of C_1 to a vertex of C_2. Moreover, let $u_1, \ldots, u_n$ be these n vertices of C_1 that are joined to n vertices $v_1, \ldots, v_n$ from C_2, respectively. We assume $u_j v_j \in F$ for $j = 1, 2, \ldots, n$. Since C_1 and C_2 are odd cycles we have that n is odd.

Assume that $u_1, \ldots, u_n$ appear on C_1 in this order. Let n_1 be the number of pairs $u_j u_{j+1}$, such that $u_j u_{j+1} \in E(G)$ $(1 \leq j \leq n)$. If $j = n$ we set $u_{j+1} = u_1$. Similarly, assume that the vertices $v_1, \ldots, v_n$ appear on C_2 in this order. Let n_2 be the number of pairs $v_j v_{j+1}$, such that $v_j v_{j+1} \in E(G)$ $(1 \leq j \leq n)$. If $j = n$ we set $v_{j+1} = v_1$.

We have $n_1, n_2 \leq n$. Let us consider two cases.

Case 1: $n \geq 5$. Hence

$$\binom{n}{2} - n_1 \geq \binom{n}{2} - n \geq n \geq n_2, \tag{1}$$

as $n \geq 5$. Suppose that $\binom{n}{2} - n_1 > n_2$. Then, there is a pair u_i, u_j with $u_i u_j \notin E(G)$ such that $v_i v_j \notin E(G)$. Define a function θ on F as follows: $\theta(u_i v_i) = \alpha$, $\theta(u_j v_j) = \beta$, and on the remaining edges of F we set the value of θ as $\alpha + \beta$. Since C_1 and C_2 are odd cycles there are exactly one edge of θ-value α and β (by definition of θ $u_i v_i$ and $u_j v_j$ are these two edges) we have that θ is a nowhere-zero $Z_2 \times Z_2$-flow in $G/\overline{F}$. Since there is no edge of $\overline{F}$ connecting an endpoint of $u_i v_i$ and $u_j v_j$ we have θ is a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow in $G/\overline{F}$ with respect to the 2-factor $\overline{F}$.

Thus, it remains to consider the case $\binom{n}{2} - n_1 = n_2$. From the chain of inequalities (1) we have that

$$\binom{n}{2} - n_1 = \binom{n}{2} - n = n = n_2.$$

This implies that $n = 5$ and C_1, C_2 have no chords. Thus, either G is 3-edge-colorable hence we have the statement via Observation 2.12 or G is the Petersen graph.

Case 2: $n = 3$. If $\binom{n}{2} - n_1 > n_2$ the same reasoning from Case 1 works. Thus, we can assume $\binom{n}{2} - n_1 \leq n_2$ or

$$3 = \binom{3}{2} = \binom{n}{2} \leq n_1 + n_2.$$

Since $n_1, n_2 \leq n = 3$ this means that either $n_1 \geq 1$ and $n_2 \geq 1$ or $(n_1, n_2) \in \{(3,0),(0,3)\}$.

Case 2a: $n_1 \geq 1$ and $n_2 \geq 1$. Consider the graphs $G/V(C_1)$ and $G/V(C_2)$. Since $n_1 \geq 1$ the new vertex of $G/V(C_2)$ is adjacent to two consecutive vertices of C_1. Similarly, since $n_2 \geq 1$ the new vertex of $G/V(C_1)$ is adjacent to two consecutive vertices of C_2. These observations imply that both of $G/V(C_1)$ and $G/V(C_2)$ are hamiltonian as it is not hard to extend C_1 to a hamiltonian cycle of $G/V(C_2)$ and extend C_2 to a hamiltonian cycle of $G/V(C_1)$. Thus, both of these cubic graphs are 3-edge-colorable. Thus, G is 3-edge-colorable. Hence, our statement for this case follows from Observation 2.12.

Case 2b: $(n_1, n_2) \in \{(3,0),(0,3)\}$. Since the cases are symmetric it suffices to consider the case $n_1 = 3$ and $n_2 = 0$. Since $n_1 = 3$, it means one of the cycles of the 2-factor is a triangle. Let $C_1 = T = u_1u_2u_3$ be this triangle. For $i = 1, 2, 3$ let v_i be the unique neighbor of u_i on C_2. Consider a cubic graph H obtained from G by removing all vertices of T and v_2, and adding the edges $g = v_1v_3$ and f connecting the two neighbors of v_2 on C_2. The cubic graph H is hamiltonian as $D = C_2 - v_2 + f$ is a hamiltonian cycle in it. Consider the perfect matching F given by every other edge of D (which has even length) such that f is not in F. It does not contain the edge g, too. Observe that $H - F$ is 2-edge-colorable hence all circuits of $H - F$ are even. Consider a 2-factor $\overline{F'}$ of G obtained from $\overline{F}$ by replacing the edge f with the two edges of C_2 incident to v_2 and replacing g with four edges v_1u_1, u_1u_2, u_2u_3 and u_3v_3. Let C_f and C_g be the circuits of $\overline{F'}$ such that C_f contains the edges incident to v_2 and C_g be the circuit containing the edge v_1u_1. If $C_f = C_g$ or these circuits are different but both of them are of even length then by defining $\theta(e) = \alpha + \beta$ for every edge $e \in F'$ we will obtain a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow of $G/\overline{F'}$ with respect to the 2-factor $\overline{F'}$. Thus it remains to consider the case when C_f and C_g are different and both of them are of odd length. C_f and C_g are the only odd cycles of $\overline{F'}$. Hence if we set $\theta(e) = \alpha + \beta$ for every edge $e \in F'$ we will have that the definition of nowhere-zero $Z_2 \times Z_2$-flow of $G/\overline{F'}$ is violated only at the two vertices of $G/\overline{F'}$ corresponding to C_f and C_g.

Suppose that C_f has a vertex w not adjacent to v_2 such that the edge e_w of F' incident to w connects w to a vertex outside $V(C_f)$ (Figure 8). Since G is bridgeless we have that $G/\overline{F'}$ is bridgeless, too. Hence $G/\overline{F'}$ contains a simple cycle C' containing u_2v_2 and e_w (Figure 8). We modify the above defined function θ as follows: it is equal to $\theta(e) = \alpha + \beta$ for every edge $e \in F'$ except on edges of C' where we have $\theta(u_2v_2) = \alpha$ and $\theta(e) = \beta$ for all edges of $e \in C' - \{u_2v_2\}$ (Figure 8). θ is a nowhere-zero $Z_2 \times Z_2$-flow of $G/\overline{F'}$. Let us show that it is non-conflicting. We have only one edge with θ-value α which is u_2v_2. The edge u_1u_3 is a chord of $\overline{F'}$ hence we can take its flow value $\alpha + \beta$. This does not violate the definition of the flow. On the other hand, the two edges of F' that are incident to a vertex adjacent to v_2 do not lie on C' the flow values of these edges will be $\alpha + \beta$, too. Hence we will not have a conflict.

Thus, we are left with the case when $V(C_f)$ is joined to $V \backslash V(C_f)$ with exactly

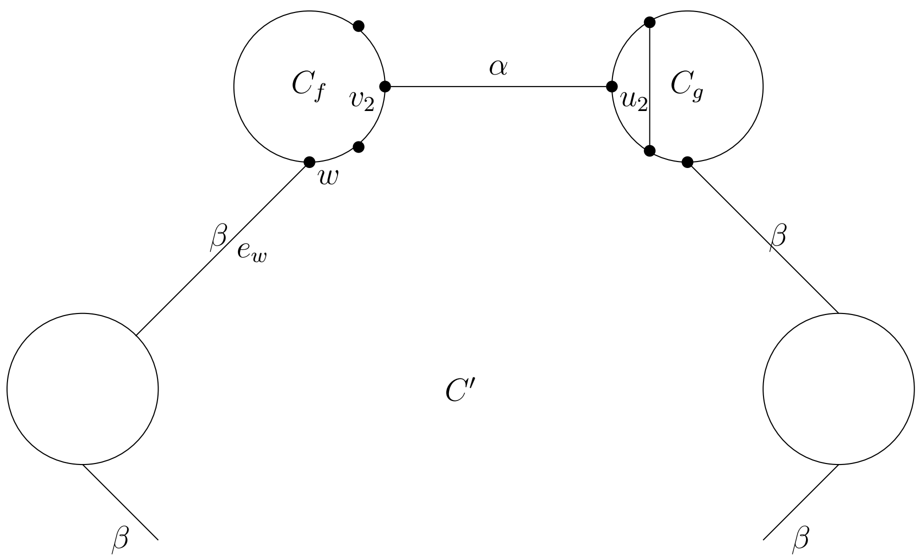

Figure 8: The simple cycle C' of $G/\overline{F'}$ containing the edges u_2v_2 and e_w.

three edges. One of them is $u_2v_2 \in F'$ and the others are $g \in F'$ and $h \in F'$. Moreover, g and h are incident to vertices that are adjacent to v_2 on C_2. Let $K = \{u_2v_2, g, h\}$ be the non-trivial 3-edge-cut joining $V(C_f)$ to $V \backslash V(C_f)$. Consider the graphs

$$H_1 = G/V(C_f) \text{ and } H_2 = G/(V \backslash V(C_f)).$$

Since the new vertex of H_2 corresponding to $V \backslash V(C_f)$ is joined to two neighboring vertices on C_f we have that H_2 has a hamiltonian cycle hence it is 3-edge-colorable. On the other hand, H_1 is a bridgeless cubic graph containing a 2-factor with at most two cycles (C_1 and the restriction of C_2 to H_1). Since $|V(H_1)| < |V|$, H_1 contains a triangle hence it is different from P_{10} by induction we have that H_1 has a perfect matching F_0 such that $H_1/\overline{F_0}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow. By Remark 2.5, F_0 intersects $\{u_1v_1, u_2v_2, u_3v_3\}$ in a single edge. Moreover, trivially F_0 intersects K in a single edge. Let this edge of K be t. Suppose that the value of non-conflicting nowhere-zero $Z_2 \times Z_2$-flow of $H_1/\overline{F_0}$ on t is x. Since H_2 is 3-edge-colorable we have that H_2 has a perfect matching J such that $t \in J$ and $H_2 - J$ is comprised of even cycles. Extend the perfect matching F_0 to a perfect matching J' of our original graph G by adding the edges of J to it and taking the flow value on them as x. We will get a perfect matching J' of G with respect to which $G/\overline{J}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow. $\square$

Combined with Lemma 2.6 we get the following result from [34]:

Corollary 2.20. *All cubic permutation graphs admit a normal* 6*-edge-coloring.*

In the previous statements we explicitly required that the cubic graph under consideration differs from P_{10}. The reader probably guessed that the main reason why we did this is that it does not admit a non-conflicting flow with respect to any 2-factor $\overline{F}$. Since P_{10} is triangle-free the presence of a triangle in a 2-factor (see Remark 2.5) is not the only obstruction for the existence of a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow.

Proposition 2.21. *The Petersen graph P_{10} does not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to any 2-factor $\overline{F}$.*

Proof. The complementary 1-factor F is a 5-edge-cut that separates the two 5-cycles of $\overline{F}$. Since it is an odd edge-cut in any nowhere-zero $Z_2 \times Z_2$-flow of $G/\overline{F}$ an odd number of α, β and $\alpha+\beta$ edges must appear on it. Assume that on one of them the value α appears. Then this edge is joined to the other four edges of F with an edge from $\overline{F}$. Thus, we cannot put the value β on either of them. □

The Petersen graph P_{10} is exceptional in many cases. In other words, there are statements where the only counterexample to them is P_{10} (Theorem 2.19, or the main result of [23] are typical cases of this phenomenon). So one may wonder whether P_{10} is the only 2-edge-connected cubic graph which does not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to any 2-factor $\overline{F}$. Unfortunately, this statement fails as our next result shows.

Theorem 2.22. *There exist infinitely many 2-edge-connected cubic graphs G that do not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ for any perfect matching F of G.*

Proof. For $\ell \geq 1$ take 3ℓ vertex-disjoint copies of $P_{10} - e$. Let $H_1, \ldots, H_{3\ell}$ be these graphs. Join $H_1, \ldots, H_{3\ell}$ cyclically by paths of length two. Now introduce ℓ new vertices $u_1, \ldots, u_\ell$ and join each of them to exactly three central vertices of these paths of length two so that the resulting graph G is cubic (see Figure 9). G is 2-edge-connected.

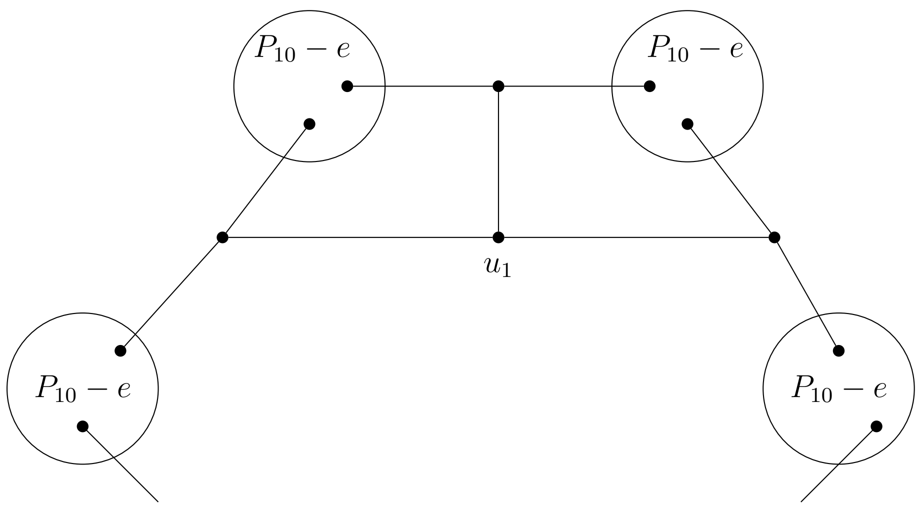

Figure 9: Examples of arbitrary large 2-edge-connected cubic graphs.

Let us show that G does not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ for any perfect matching F of G. Let F be a perfect matching of G and let θ be a nowhere-zero $Z_2 \times Z_2$-flow of $G/\overline{F}$. Consider the vertex u_1. Then, there is an edge u_1w_1 of G such that $u_1w_1 \notin F$. Here w_1 is a central vertex of a path of length two. Thus, there is j, $1 \leq j \leq 3\ell$ such that the two edges of G that connect H_j to $V(G)\backslash V(H_j)$ belong to F. Hence, they have the same flow value with respect to θ. Now, with the same approach as we did in Proposition 2.21 one can show that H_j contains a conflict with respect to the flow θ. □

3 Conclusion and future work

From our perspective there are some questions that deserve further consideration. In Proposition 2.1 we showed that the smallest counterexample to Conjecture 1.8 is 3-edge-connected. Hence they do not contain 2-edge-cuts. This prompted Mazzuoccolo to ask whether P_{10} is the only 3-edge-connected cubic graph which does not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ for any perfect matching F of G. Since P_{10} admits a normal 5-edge-coloring it is not a counterexample to Conjecture 1.8. Thus, Mazzuoccolo's statement combined with Proposition 2.1 and Lemma 2.6 implies Conjecture 1.8.

The author suspects that this statement is not true. Thus, Mazzuoccolo asked whether P_{10} is the only cyclically 4-edge-connected cubic graph G which does not have a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ for any perfect matching F of G. If this statement is true then cyclically 4-edge-connected cubic graphs will admit a normal 6-edge-coloring. Unfortunately, as we stated in Remark 2.3 this does not directly imply Conjecture 1.8. Nevertheless, the author thinks that this is a question deserving further consideration. The author suspects that the answer to this question should be negative, too. As a result he would like to offer:

Conjecture 3.1. *There exist infinitely many cyclically 6-edge-connected cubic graphs G which do not have a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ for any perfect matching F of G.*

In Conjecture 3.1 for cyclic edge-connectivity we write six and not a larger number because of the Jaeger-Swart conjecture [17] which predicts that all cyclically 7-edge-connected cubic graphs are 3-edge-colorable. Thomassen has a related conjecture that predicts the Hamiltonicity of all cyclically 8-edge-connected cubic graphs. The latter conjecture is open even when cyclic edge-connectivity is a constant greater than eight.

Another interesting problem by Mazzuoccolo is the following one:

Problem 3.2. *For every integer $g \geq 3$ construct infinitely many 3-edge-connected cubic graphs G of girth at least g that do not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ for any perfect matching F of G.*

This problem is reminiscent of the girth conjecture that was predicting that all snarks should have bounded girth. Recall that the girth conjecture has been refuted by Kochol in [22] where he constructed snarks of arbitrary large girth. Kochol's approach has been simplified in papers [20] and [28].

As we mentioned in Observation 2.11 every bipartite cubic graph admits a non-conflicting flow with respect to every 2-factor $\overline{F}$. One can state a conjecture that predicts the converse: if a cubic graph G admits a non-conflicting flow with respect to every 2-factor $\overline{F}$ then G is bipartite. Unfortunately, this statement is not true. Consider a cubic graph such that its every 2-factor is a hamiltonian cycle (such as K_4, Figure 5). Then it is an obvious counterexample to our conjecture by Observation 2.10. One can construct more connected examples as follows: consider a graph G which is a ring of diamonds or 2-cycles (see Theorem 2.14). The components of every 2-factor $\overline{F}$ of such a graph are even cycles. Hence, they admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$ for any perfect matching F of G by Observation 2.10. These graphs are not bipartite as diamonds contain triangles. The examples described above are 3-edge-colorable. If one takes the graph K_2^3 (Figure 6) and replaces both of its vertices with a $P_{10} - v$ then the resulting graph is not 3-edge-colorable. However $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to every 2-factor $\overline{F}$. Thus, as an interesting problem we would like to offer:

Problem 3.3. *Characterize 2-edge-connected cubic graphs in which $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to every 2-factor $\overline{F}$.*

As we mentioned in Remark 2.5 if $\overline{F}$ contains a triangle then $G/\overline{F}$ does not admit a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to $\overline{F}$. Thus, graphs satisfying the conditions of Problem 3.3 should not contain a triangle in any of its 2-factor. As rings of diamonds demonstrate this does not necessarily mean that G should not contain a triangle.

Our last conjecture states:

Conjecture 3.4. *Let G be a bridgeless cubic graph containing a hamiltonian path. Then $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to some 2-factor $\overline{F}$ unless G is the Petersen graph.*

Conjecture 3.4 implies that if a bridgeless cubic graph G has a vertex z such that $G - z$ has a hamiltonian cycle then $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to some 2-factor $\overline{F}$ unless G is the Petersen graph. The latter implies that if G is a cubic graph such that for every vertex z the graph $G - z$ contains a hamiltonian cycle then $G/\overline{F}$ admits a non-conflicting nowhere-zero $Z_2 \times Z_2$-flow with respect to some 2-factor $\overline{F}$ unless G is the Petersen graph. Non-3-edge-colorable bridgeless cubic graphs in which for every vertex z the graph $G - z$ contains a hamiltonian cycle are called hypo-hamiltonian snarks and examples of such graphs can be found in [29] and Theorem 2.5 from [49]. The orders for which hypo-hamiltonian snarks of that order exist are characterized in [7].

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References

[1] L. D. Andersen, The strong chromatic index of a cubic graph is at most 10, *Discrete Math.* 108 (1992), 231–252.

[2] H. Bílková, Petersenovské obarvení a jeho varianty, Bachelor thesis, Charles University in Prague, Prague, 2012, (in Czech).

[3] A. U. Celmins, On cubic graphs that do not have an edge-3-colouring, Ph.D. Thesis, Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Canada, 1984.

[4] M. Chudnovsky and P. Seymour, The structure of claw-free graphs, In: *Surveys in Combinatorics 2005*, London Math. Soc. Lec. Note Ser. 327, 153–171, Cambridge Univ. Press, Cambridge, 2005.

[5] L. Ferrarini, G. Mazzuoccolo and V. V. Mkrtchyan, Normal 5-edge-colorings of a family of Loupekhine snarks, *AKCE Int. J. Graphs Comb.* 17(3) (2020), 720–724.

[6] D. R. Fulkerson, Blocking and anti-blocking pairs of polyhedra, *Math. Program.* 1 (1971), 168–194.

[7] J. Goedgebeur and C. T. Zamfirescu, On hypohamiltonian snarks and a theorem of Fiorini, *Ars Math. Contemp.* 14(2) (2018), 227–249.

[8] J. Hägglund and E. Steffen, Petersen-colorings and some families of snarks, *Ars Math. Contemp.* 7 (2014), 161–173.

[9] A. Hakobyan and V. Mkrtchyan, S_{12} and P_{12}-colorings of cubic graphs, *Ars Math. Contemp.* 17(2) (2019), 431–445.

[10] A. Hakobyan and V. Mkrtchyan, On Sylvester Colorings of Cubic Graphs, *Australas. J. Combin.* 72(3) (2018), 472–491.

[11] I. Holyer, The NP-Completeness of Edge-coloring, *SIAM J. Comput.* 10(4) (1981), 718–720.

[12] F. Holyord and M. Škoviera, Colouring of cubic graphs by Steiner triple systems, *J. Combin. Theory Ser. B* 91 (2004), 57–66.

[13] F. Jaeger, On nowhere-zero flows in multigraphs, in "Proc. Fifth British Combin. Conf. Aberdeen, 1975," *Congress. Numer.* XV (1976), 373–378.

[14] F. Jaeger, Flows and generalized coloring theorems in graphs, *J. Combin. Theory Ser. B* 26 (1979), 205–216.

[15] F. Jaeger, On five-edge-colorings of cubic graphs and nowhere-zero flow problems, *Ars Combin.* 20B (1985), 229–244.

[16] F. Jaeger, Nowhere-zero flow problems, in: *Selected topics in graph theory* 3, Academic Press, San Diego, CA, 1988, 71–95.

[17] F. Jaeger and T. Swart, Problem session, *Ann. Discrete Math.* 9 (1980), 305.

[18] T. Kaiser, D. Král and S. Norine, Unions of perfect matchings in cubic graphs, *Electron. Notes Discret. Math.* 22, (2005), 341–345.

[19] T. Kaiser and R. Škrekovski, Cycles Intersecting Edge-Cuts of Prescribed Sizes, *SIAM J. Discrete Math.* 22(3) (2008), 861–874.

[20] J. Karabáš, E. Máčajová, R. Nedela and M. Škoviera, Girth, oddness, and colouring defect of snarks, *Discrete Math.* 345(11) (2022), 113040.

[21] F. Kardoš, E. Máčajová and J.-P. Zerafa, Disjoint odd circuits in a bridgeless cubic graph can be quelled by a single perfect matching, *J. Combin. Theory Ser. B* 160 (2023), 1–14.

[22] M. Kochol, Snarks without small cycles, *J. Combin. Theory Ser. B* 67 (1996), 34–47.

[23] L. Lovász, Matching structure and the matching lattice, *J. Combin. Theory Ser. B* 43(2) (1987), 187–222.

[24] L. Lovász and M. D. Plummer, Matching Theory, *Annals Discrete Math.* 29 North Holland, 1986.

[25] Y. Ma, D. Mattiolo, E. Steffen and I. H. Wolf, Pairwise Disjoint Perfect Matchings in r-Edge-Connected r-Regular Graphs, *SIAM J. Discrete Math.* 37(3) (2023), 1548–1565.

[26] Y. Ma, D. Mattiolo, E. Steffen and I. H. Wolf, Sets of r-graphs that color all r-graphs, arXiv:2305.08619, 2023.

[27] E. Máčajová and G. Mazzuoccolo, Reduction of the Berge-Fulkerson conjecture to cyclically 5-edge-connected snarks, *Proc. Amer. Math. Soc.* 148(11) (2020), 4643–4652.

[28] E. Máčajová and M. Škoviera, Superposition of snarks revisited, *European J. Combin.* 91 (2021), Article 103220.

[29] E. Máčajová and M. Škoviera, Constructing hypohamiltonian snarks with cyclic connectivity 5 and 6, *Electron. J. Combin.* 14 (2007), #R18.

[30] D. Mattiolo, G. Mazzuoccolo and V. V. Mkrtchyan, On sublinear approximations for the Petersen coloring conjecture, *Bull. Inst. Combin. Appl.* 92 (2021), 78–90.

[31] G. Mazzuoccolo, The equivalence of two conjectures of Berge and Fulkerson, *J. Graph Theory* 68(2) (2011), 125–128.

[32] G. Mazzuoccolo, An Upper Bound for the Excessive Index of an r-Graph, *J. Graph Theory* 73(4) (2013), 377–385.

[33] G. Mazzuoccolo, New conjectures on perfect matchings in cubic graphs, *Electron. Notes Discret. Math.* 40, 235–238.

[34] G. Mazzuoccolo and V. V. Mkrtchyan, Normal 6-edge-colorings of some bridgeless cubic graphs, *Discrete Appl. Math.* 277 (2020), 252–262.

[35] G. Mazzuoccolo and V. V. Mkrtchyan, Normal edge-colorings of cubic graphs, *J. Graph Theory* 94(1) (2020), 75–91.

[36] G. Mazzuoccolo, G. Tabarelli and J. P. Zerafa, On the existence of graphs which can colour every regular graph, *Discrete Appl. Math.* 337 (2023), 246–256.

[37] V. Mkrtchyan, A remark on the Petersen coloring conjecture of Jaeger, *Australas. J. Combin.* 56 (2013), 145–151.

[38] S.-il Oum, Perfect matchings in claw-free cubic graphs, *Electron. J. Combin.* 18 2011, #P62.

[39] F. Pirot, J. S. Sereni and R. Škrekovski, Variations on the Petersen Colouring Conjecture, *Electron. J. Combin.* 27(1) (2020), #P1.81.

[40] M. Preissmann, Sur les colorations des arêtes des graphes cubiques, Thèse de troisième cycle, Grenoble, 1981.

[41] R. Rizzi, Indecomposable r-graphs and some other counterexamples, *J. Graph Theory* 32(1) (1999), 1–15.

[42] R. Šámal, New approach to Petersen coloring, *Electron. Notes Discret. Math.* 38 (2011), 755–760.

[43] R. Šámal, Personal communication to G. Mazzuoccolo and V. V. Mkrtchyan, 2016.

[44] T. Schönberger, Ein Beweis des Petersenschen Graphensatzes, *Acta Sci. Math. (Szeged)* 7 (1934), 51–57.

[45] J. Sedlar and R. Škrekovski, Normal 5-edge-coloring of some more snarks superpositioned by the Petersen graph, (2023), arXiv:2312.08739.

[46] J. Sedlar and R. Škrekovski, Normal 5-edge-coloring of some snarks superpositioned by Flower snarks, *European J. Combin.* 122 (2024), 104038.

[47] P. D. Seymour, On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte, *Proc. Lond. Math. Soc.* 38(3) (1979), 423–460.

[48] E. Steffen, Classifications and characterizations of snarks, *Discrete Math.* 188 (1998), 183–203.

[49] E. Steffen, On bicritical snarks, *Math. Slovaca* 51 (2001), 141–150.

[50] E. Steffen, Measurements of edge-uncolorability, *Discrete Math.* 280 (2004), 191–214.

[51] M. Stiebitz, D. Scheide, B. Toft and L. M. Favrholdt, Graph Edge-Coloring, John Wiley and Sons, 2012.

[52] C. Thomassen, Factorizing regular graphs, *J. Combin. Theory Ser. B* 141 (2020), 343–351.

[53] V. Vizing, On an estimate of the chromatic class of a p-graph, *Diskret Analiz* 3 (1964), 25–30.

[54] D. West, Introduction to Graph Theory, Prentice-Hall, Englewood Cliffs, (1996).

[55] C.-Q. Zhang, Integer flows and cycle covers of graphs, Marcel Dekker, Inc., New York Basel Hong Kong, 1997.

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