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Comparing degeneracy and odd cycle bounds for chromatic number

Allan Bickle

Department of Mathematics
Purdue University
610 Purdue Mall, West Lafayette
IN 47907, U.S.A.
aebickle@purdue.edu

Abstract

A graph G is k-degenerate if the vertices of G can be successively deleted, so that when each vertex v is deleted, it has degree at most k in the remaining graph. The degeneracy $D(G)$ is the smallest k such that G is k-degenerate. The chromatic number $\chi(G)$ is bounded by both $1+D(G)$ and $1+l(G)$, where $l(G)$ is the length of its longest odd cycle of G. We show that for 2-connected graphs, $D(G) \leq l(G)$, and $K_{l(G)+1}$ is the only extremal graph.

1 Introduction

There are many upper bounds for the chromatic number $\chi(G)$ of a graph. Let G be a graph with maximum degree $\Delta(G)$, independence number $\alpha(G)$, longest path $lp(G)$, degree sequence $d_1 \geq \cdots \geq d_n$, and largest eigenvalue $\lambda_1(G)$. The bounds $\chi(G) \leq 1+\Delta(G)$, $\chi(G) \leq n+1-\alpha(G)$, $\chi(G) \leq 1+lp(G)$, $\chi(G) \leq 1+\max_i \min\{d_i, i-1\}$, and $\chi(G) \leq 1+\lambda_1(G)$ are well-established [1, 4]. However, all of these bounds are inferior to a bound based on the degeneracy of a graph.

Definition 1.1. A graph G is *k-degenerate* if the vertices of G can be successively deleted, so that when each vertex v is deleted, it has degree at most k in the remaining graph. The *degeneracy* $D(G)$ is the smallest k such that G is k-degenerate.

See [3] for a survey of degeneracy and related topics. The *degeneracy bound* $\chi(G) \leq 1+D(G)$ follows immediately from the definition. Further, it is not hard to show that $D(G) \leq \min\{\Delta(G), n-\alpha(G), lp(G), \max_i \min\{d_i, i-1\}, \lambda_1(G)\}$ (see [1, 2]). Thus the degeneracy bound is superior to the other bounds for chromatic number.

Let $l(G)$ be the *odd circumference* of a non-bipartite graph G, the length of its longest odd cycle. Erdös and Hajnal [5] showed that for a non-bipartite graph,

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$\chi(G) \leq 1+l(G)$. This bound can be superior to the degeneracy bound. For example, $K_{r,r} \cup K_3$ has degeneracy r when $r \geq 2$ and odd circumference 3. But this example requires two blocks, one bipartite block to make the degeneracy large, and another containing a relatively short odd cycle. We will show that in the nontrivial case of 2-connected graphs, the odd circumference is at least the degeneracy.

Definitions of terms and notation not defined here appear in [3]. In particular, if vertices u and v are adjacent, we write $u \leftrightarrow v$, and if they are nonadjacent, we write $u \not\leftrightarrow v$.

2 Main Results

We begin with two lemmas on the existence of odd cycles.

Lemma 2.1. *If G is a 2-connected graph containing an odd cycle, then any edge $e = uv$ of G is on a odd cycle.*

Proof. Let C be an odd cycle of G and e be not on C. By the vertex form of Menger's Theorem, there are independent paths from u and v to some vertices x and y on C. There are two $x-y$ paths in C whose lengths have opposite parity. Combining one of them with the $x-y$ path containing e produces an odd cycle in G containing e. □

Lemma 2.2. [6] *If a non-bipartite 2-connected graph G contains a cycle C of length $2r$, $r \geq 2$, then it contains an odd cycle of length at least $r+1$.*

Proof. Let C be a cycle of length $2r$ containing edge e. Now e is on an odd cycle C', which must leave and return to C at some vertices x and y for which the $x-y$ distance on C' has opposite parity from the $x-y$ distance on C. Then combining the $x-y$ path on C' and the longer $x-y$ path on C forms an odd cycle with length at least $r+1$. □

Theorem 2.3. *If G is a 2-connected graph containing an odd cycle, then $l(G) \geq D(G)$.*

Proof. Let $d = D(G)$, and H be the maximal subgraph of G with minimum degree $\delta(H) = d$. Consider a path $P = v_0 v_1 v_2 \ldots$ with maximum length in H, one of whose ends is $v = v_0$. Now v has at least d neighbors on P (else it could be extended).

Case 1. If v has an even neighbor v_{2i} with $2i \geq d-1$, we have an odd cycle $v_0 v_1 \ldots v_{2i} v_0$ with length at least d.

Case 2. If v has no even neighbor, then v has an odd neighbor v_{2i-1}, $i \geq d$. Thus G contains a cycle C of length at least $2d$. By Lemma 2.2, there is an odd cycle with length at least $d+1$.

Case 3. The previous cases show that v has both an even neighbor and an odd neighbor. Suppose v has $k \geq 1$ odd neighbors, and at least $d-k \geq 1$ even neighbors up to v_{d-2}. Let v_{2j-1}, $j \geq k$, be the largest (odd) neighbor. Then an even neighbor v_{2i}, $2i \leq 2j-1-(d-2)$, would create an odd cycle $v_0 v_{2i} \ldots v_{2j-1} v_0$ with length at least d. Otherwise all the even neighbors are between $2j - d + 2$ and $d - 2$, which leaves at most $\frac{(d-2)-(2j-d+2)}{2} + 1 = d - j - 1 \leq d - k - 1$ of them, a contradiction. □

This implies that the degeneracy bound is superior to the odd cycle bound for 2-connected graphs.

Corollary 2.4. *Let G be a 2-connected graph containing an odd cycle with odd circumference $l(G)$. Then $\chi(G) \leq 1 + D(G) \leq 1 + l(G)$.*

Kenkre and Vishwanathan [6] characterized the extremal graphs for the bound $\chi(G) \leq 1 + l(G)$ as those containing $K_{l(G)+1}$. Their proof depends on several other theorems on cycles in graphs. We can provide a proof that depends only on the previous theorem. This is also a stronger result, since it characterizes graphs with $1 + D(G) = 1 + l(G)$, rather than $\chi(G) = 1 + l(G)$.

Theorem 2.5. *Let G be a 2-connected graph with odd circumference $l = l(G)$. Then $D(G) = l(G)$ if and only if $G = K_{l+1}$.*

Proof. ($\Leftarrow$) Certainly $D(K_{l+1}) = l(K_{l+1}) = l$.

($\Rightarrow$) Consider the proof of Theorem 2.3. In Case 3, there must be an odd cycle with length at least $l + 2$. In Case 2, there must be an odd cycle of length at least $l + 1$. Thus the longest odd cycle has length l only when Case 1 holds. Then $d(v) = D(G) = d$, and v has consecutive neighbors v_i and v_{i+1} on P. Then $P_1 = v_1 v_2 \ldots v_i v_0 v_{i+1} \ldots$ has the same length as P. Then v_1 has a neighbor v_j on P, $j \geq d$, and $v_1 v_2 \ldots v_j v_1$ and $v_1 \ldots v_i v_0 v_{i+1} \ldots v_j v_1$ are cycles of length at least d and $d + 1$. Then $d(v_1) = d$, d is odd, and $v_1 \leftrightarrow v_k$, $2 \leq k \leq d$.

Now the path $P_i = v_i v_{i-1} \ldots v_1 v_0 v_{i+1} \ldots$, $i < d$, has the same length as P. The same argument as for v shows that $v_i \leftrightarrow v_j$, $0 \leq j \leq d$, $i \neq j$. Thus $v_0, \ldots, v_d$ induce K_{d+1}. If G contains any other vertex x, then it is on a path between some v_i and v_j, and this can be used to construct a longer odd cycle. Thus $G = K_{l+1}$. □

Kenkre and Vishwanathan [6] proved that if G does not contain $K_{l(G)}$, then $\chi(G) \leq l(G) - 1$. To characterize the graphs with $D(G) = l(G) - 1$, we first consider $l(G) = 3$.

Theorem 2.6. *Let G be a 2-connected graph with $l(G) = 3$. Then $D(G) = 2$ if and only if $G = K_2 + \overline{K}_r$, where $r \geq 1$.*

Proof. ($\Leftarrow$) If $G = K_2 + \overline{K}_r$, certainly $D(G) = 2$ and $l(G) = 3$.

($\Rightarrow$) Assume the hypothesis and consider a path $P = v_0 v_1 v_2 \ldots$ with maximum length. Now v_0 is not adjacent to an even-numbered vertex other than v_2. We consider two cases.

Suppose $v_0 \leftrightarrow v_2$ and v_3 exists. Now v_2 is not a cutvertex. Then one of v_0 or v_1, say v_1, is adjacent to another vertex, which must be v_3, or else there is a longer odd cycle. Then there cannot be a v_4, since some later vertex on P must be adjacent to one of v_0, v_1, or v_2, which would create a longer odd cycle. There could be another vertex u adjacent to v_1 or v_2 (say v_2). Now u cannot be adjacent to v_0 or v_3, but it can be adjacent to v_1. As before, u cannot have any other neighbors not on P. Similarly, we find that any other vertex must neighbor v_1 and v_2, so $G = K_2 + \overline{K}_r$, $r \geq 1$.

Now suppose $v_0 \nleftrightarrow v_2$. By Lemma 2.1, v_0v_1 is on a triangle with v_0v_{2i+1} for some i. But then v_1v_{2i+1} is also on the triangle, so $v_{2i+1} = v_3$, or else there is a longer odd cycle. Now we have a copy of $K_2 + \overline{K}_2$ in G, so we can relabel the vertices and repeat the previous case. □

When $l(G) \geq 5$, we determine graphs with $D(G) = l(G) - 1$ given the added restriction that $\delta(G) = l(G) - 1$.

Theorem 2.7. *Let G be a 2-connected graph with $l = l(G) \geq 5$ and $\delta(G) = l - 1$. Then $D(G) = l - 1$ if and only if $G = K_{l+1} - rK_2$, $1 \leq r \leq \frac{l+1}{2}$, or $G = K_l$.*

Proof. ($\Leftarrow$) These graphs have $\delta(G) = D(G) = l(G) - 1$.

($\Rightarrow$) Let G be 2-connected graph with $l \geq 5$ and $D(G) = \delta(G) = l - 1$. Consider a path $P = v_0v_1v_2\ldots$ with maximum length. We say a *long odd cycle* is one with length more than l. Now v_0 is not adjacent to an even-numbered vertex beyond v_l, since this would create a long odd cycle. Also, v_0 is not adjacent to an odd-numbered vertex beyond v_{2l-3}, since by Lemma 2.2 an even cycle of length at least $2l$ would force a long odd cycle.

Case 1. Suppose v_0 has adjacent neighbors v_i and v_{i+1} on P, $i \leq l - 1$. Then $v_1v_2\ldots v_iv_0v_{i+1}\ldots$ has the same length as P. Now v_1 has a neighbor v_j on P, $j \geq l - 1$. Thus v_1 has no neighbor beyond v_l, or else one of $v_1v_2\ldots v_jv_1$ and $v_1\ldots v_iv_0v_{i+1}\ldots v_jv_1$ is a long odd cycle.

If v_0 has a neighbor beyond v_{l+2}, v_1 being adjacent to v_3 or v_5 would create a long odd cycle. Then v_1 has at most $l - 2$ neighbors, so v_0 has no neighbor beyond v_{l+2}.

Suppose further that $v_0 \leftrightarrow v_{l+2}$. If $v_1 \leftrightarrow v_3$, $v_0v_1v_3\ldots v_{l+2}v_0$ is a long odd cycle, so v_1 is adjacent to $v_4, \ldots, v_l$. If $v_0 \leftrightarrow v_2$, $v_0v_2\ldots v_{l+2}v_0$ is a long odd cycle, and if $v_0 \leftrightarrow v_3$, $v_0v_3v_4v_1v_5\ldots v_{l+2}v_0$ is a long odd cycle. Thus v_0 is adjacent to $v_4, \ldots, v_l$. Now v_2 has no neighbor beyond v_{l+1}, and $v_2 \nleftrightarrow v_4$, so v_2 is adjacent to $v_5, \ldots, v_{l+1}$. Then $v_2v_3v_4v_0v_1v_5\ldots v_{l+1}v_2$ is a long odd cycle, a contradiction.

Case 2. If v_0 has no adjacent neighbors, its neighbors are the odd vertices $v_1, v_3, \ldots, v_{2l-3}$. Now $v_{2l-4}v_{2l-5}\ldots v_0v_{2l-3}\ldots$ is a path with the same length as P. If v_{2l-4} has any neighbor beyond v_{2l-3}, we find a cycle with length more than $2l - 2$, which by Lemma 2.2 forces a long odd cycle. If v_{2l-4} has adjacent neighbors on P, we can relabel P and repeat the argument from Case 1. Thus v_{2l-4} is adjacent to odd vertices $v_1, v_3, \ldots, v_{2l-3}$. Similarly, we find paths of maximum length that start at $v_2, v_4, \ldots, v_{2l-6}$. Adding edges while avoiding creating a long odd cycle eventually induces $K_{l-1,l-1}$ on $\{v_0, \ldots, v_{2l-3}\}$, but this contains no odd cycle. Adding any path to create an odd cycle will create a long odd cycle, a contradiction.

Thus v_0 has adjacent neighbors and no neighbor beyond v_l, so v_0 is nonadjacent to at most one vertex in $\{v_1, \ldots, v_l\}$. We want to show that each vertex before v_l on P is the start of a path with the same length as P. If $v_0 \leftrightarrow v_{i+1}$, then $v_i \ldots v_0 v_{i+1} \ldots$ is such a path. If $v_0 \not\leftrightarrow v_2$, $v_1 v_2 v_3 v_0 v_4 \ldots$ is a maximum length path.

If $v_0 \not\leftrightarrow v_j$, $j < l$, there is a maximum length path $v_j v_{j-1} \ldots v_0 v_{j+1} \ldots$ starting at v_j. By the previous paragraph, there is also a maximum length path starting at v_{j-1}. If $v_0 \not\leftrightarrow v_l$ and $v_i \leftrightarrow v_l$, $i \leq l-2$, then $v_l \ldots v_{i+1} v_0 \ldots v_i v_l \ldots$ is a maximum length path.

Since each vertex in $\{v_0, \ldots, v_{l-1}\}$ is the start of a path of maximum length, there are no vertices outside $\{v_0, \ldots, v_l\}$. Thus $G \subseteq K_{l+1}$. To have $D(G) = l - 1$, a matching with at least one edge must be deleted.

If $v_i \not\leftrightarrow v_l$, $0 \leq i \leq l-2$, then v_{l-1} is a cut-vertex. Thus v_l does not exist, so $G \subseteq K_l$. To have $D(G) = l - 1$, $G = K_l$. □

Any graph with $\chi(G) = l(G)$ contains a critical subgraph with $\delta(G) \geq l(G) - 1$. Of the graphs in Theorem 2.7, the only l-critical subgraph is K_l. Thus Theorem 2.7 implies the result in [6] that if G does not contain $K_{l(G)}$, then $\chi(G) \leq l(G) - 1$.

Without the restriction that $\delta(G) = l - 1$, there are other graphs with $D(G) = l - 1$. One example is the graphs formed from K_l by adding $r \geq 0$ vertices, each adjacent to the same two vertices in K_l.

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