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Spectral properties of generalized Paley graphs*

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Abstract

We study the spectrum of generalized Paley graphs $\Gamma(k,q) = Cay(\mathbb{F}_q, R_k)$, undirected or not, with $R_k = \{x^k : x \in \mathbb{F}_q^*\}$ where $q = p^m$ with p prime and $k \mid q-1$. We first show that the eigenvalues of $\Gamma(k,q)$ are given by the Gaussian periods $\eta_i^{(k,q)}$ with $0 \le i \le k-1$. Then, we explicitly compute the spectrum of $\Gamma(k,q)$ with $1 \le k \le 4$ and of $\Gamma(5,q)$ for $p \equiv 1 \pmod 5$ and $5 \mid m$. Also, we characterize those GP-graphs having integral spectrum, showing that $\Gamma(k,q)$ is integral if and only if p divides $(q-1)/(p-1)$. Next, we focus on the family of semiprimitive GP-graphs. We show that they are integral strongly regular graphs (of pseudo-Latin square type). Finally, we characterize all integral Ramanujan graphs $\Gamma(k,q)$ with $1 \le k \le 4$ or where (k,q) is a semiprimitive pair.

1 Introduction

In this paper we study the spectrum of generalized Paley graphs (GP-graphs for short), and some properties that can be deduced from the spectrum. The work has three parts. We first study the spectrum of GP-graphs $\Gamma(k,q)$ and put the spectrum in terms of cyclotomic Gaussian periods. This allows us to give $\mathrm{Spec}(\Gamma(k,q))$ explicitly for $1 \le k \le 4$ and to characterize those GP-graphs having integral spectrum (first main result). In the second part, we focus on the family of semiprimitive GP-graphs. These graphs are, in particular, strongly regular graphs with integral spectrum. We study the spectrum, parameters and invariants of these graphs as strongly regular graphs. Finally, in the third part, we study GP-graphs which are Ramanujan. We classify all integral Ramanujan graphs of the form $\Gamma(k,q)$ with $1 \le k \le 4$ and all semiprimitive GP-graphs which are Ramanujan (second main result).

Some results on the spectrum of arbitrary GP-graphs and on the structure of semiprimitive GP-graphs are known, and can be found scattered in the literature. For

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completeness, we have decided to include our proofs (with some additional references) to give a unified treatment and notations to these topics. However, the explicit computation of the spectra for $\Gamma(3,q)$ and $\Gamma(4,q)$, the characterization of GP-graphs with integral spectrum and the classification of Ramanujan semiprimitive GP-graphs are completely new.

Generalized Paley graphs. If G is an abelian group and S is a subset of G not containing 0, the associated Cayley graph $\Gamma = X(G,S)$ is the directed graph (digraph) with vertex set G and where two vertices u, v form a directed edge from u to v in Γ if and only if $v - u \in S$. Since $0 \notin S$ then Γ has no loops. Analogously, the Cayley sum graph $X^+(G,S)$ has the same vertex set G but now $v, w \in G$ are connected in Γ by an arrow from v to w if and only if $v + w \in S$. We will use the notation $X^*(G,S)$ when we want to consider both $X(G,S)$ and $X^+(G,S)$ indistinctly. Notice that if S is symmetric, that is $-S = S$, then $X^*(G,S)$ is an $|S|$-regular simple (undirected without multiple edges) graph. Actually, given any two vertices u, v there are two directed edges, $\vec{uv}$ and $\vec{vu}$. As usual, we consider these two directed edges as a non-directed single one denoted uv. However, the graph $X^+(G,S)$ may contain loops. In this case, there is a loop on vertex x provided that $x + x \in S$.

The *generalized Paley graph* and *generalized Paley sum graph* are the Cayley graphs respectively given by

$$\Gamma(k,q) = X(\mathbb{F}_q, R_k) \quad \text{and} \quad \Gamma^+(k,q) = X^+(\mathbb{F}_q, R_k)$$

with connection set

$$R_k = \{x^k : x \in \mathbb{F}_q^*\}.$$

That is, $\Gamma(k,q)$ is the graph with vertex set $\mathbb{F}_q$ and two vertices $u, v \in \mathbb{F}_q$ are neighbours (directed edge) if and only if $v - u = x^k$ for some $x \in \mathbb{F}_q^*$. We will refer to them simply as *GP-graphs* and *GP$^+$-graphs* respectively (or *GP*-graphs* for both indistinctly).

Notice that if ω is a primitive element of $\mathbb{F}_q$, then $R_k = \langle \omega^k \rangle = \langle \omega^{(k,q-1)} \rangle$. This implies that $\Gamma(k,q) = \Gamma(k',q)$, where

$$k' = \gcd(k, q-1),$$

and that $\Gamma(k,q)$ is a $\frac{q-1}{k'}$-regular graph. Thus, one usually assumes that

$$k \mid q-1$$

(hence $k' = k$), for if not we have that $\Gamma(k,q) = \Gamma(1,q) = K_q$. Summing up, we have

$$\Gamma(k,q) = \begin{cases} K_q & \text{if } k' = 1, \\ \Gamma(k',q) & \text{if } k' > 1. \end{cases}$$

Notice that for q even, we have that $\Gamma^+(k,q) = \Gamma(k,q)$. On the other hand, when q is odd, one can show that $\Gamma^+(k,q)$ has loops, since in this case there are exactly $|R_k|$

elements $x \in \mathbb{F}_q$ such that $x + x = 2x \in R_k$ (multiplication by 2 is a bijection in $\mathbb{F}_q$ for q odd).

The graph $\Gamma(k,q)$ is undirected if and only if q is even or else q is odd and $k \mid \frac{q-1}{2}$. The graph is connected if and only if $\frac{q-1}{k}$ is a primitive divisor of $q-1$ (i.e. $\frac{1}{k}(p^m-1)$ does not divide $p^a - 1$ for any $a < m$, where $q = p^m$). This was proved in [14] for the undirected case, but it also holds in the directed case since $\Gamma(k,q)$ is strongly connected if and only if the Waring number $g(k,q)$ exists, and this happens if and only if $\frac{q-1}{k}$ is a primitive divisor of $q-1$. We recall that a strongly connected digraph is a directed graph in which there is a directed path in each direction between any pair of vertices of the graph.

For some values of k and q, the GP-graphs $\Gamma(k,q)$ are known graphs. For instance, for $k = 1, 2$ we get the complete graph $\Gamma(1,q) = K_q$, the classic (undirected) Paley graph $\Gamma(2,q) = P(q)$ for $q \equiv 1 \pmod 4$, and the directed Paley graph $\vec{P}(q)$ for $q \equiv 3 \pmod 4$. The graphs $\Gamma(3,q)$ and $\Gamma(4,q)$ are of interest too (see [24], where infinite pairs of equienergetic non-isospectral regular graphs $\Gamma(k,q), \bar{\Gamma}(k,q)$, $k = 3, 4$, are obtained). One can see that for p prime, we have that $\Gamma(\frac{p-1}{2}, p) = C_p$ and $\Gamma(p-1,p) = \vec{C}_p$, where C_p and $\vec{C}_p$ are the undirected and directed p-cycles, respectively. Generalized Paley graphs with $k = q^\ell + 1$ are studied in [20]. The connected GP-graphs of the form $\Gamma(\frac{p^{bm}-1}{b(p^m-1)}, p^{bm})$ are the Hamming graphs $H(b,p^m)$ (see [14]).

Spectrum. The spectrum of a graph Γ, denoted $\mathrm{Spec}(\Gamma)$, is the spectrum of its adjacency matrix A (i.e. the set of eigenvalues of A counted with multiplicities). If Γ has different eigenvalues $\lambda_0, \ldots, \lambda_t$ with multiplicities $m_0, \ldots, m_t$, we write as usual

$$\mathrm{Spec}(\Gamma) = \{[\lambda_0]^{m_0}, \ldots, [\lambda_t]^{m_t}\}.$$

It is well-known that an n-regular graph Γ has n as one of its eigenvalues, with multiplicity equal to the number of connected components of Γ. That is, Γ is connected if and only if n has multiplicity 1. The same happens for n-regular digraphs, i.e. those directed graphs such that any vertex has the same in-degree and out-degree equal to n. In this case, Γ is strongly connected if and only if n has multiplicity 1.

The spectrum of few families of GP-graphs are known. The graphs $\Gamma(1,q)$ and $\Gamma(2,q)$ with $q \equiv 1 \pmod 4$ are classic being the complete graphs K_q and the classic Paley graphs $P(q)$, and hence with known spectra. The spectrum of $\Gamma(k,q)$, for $k = 3, 4$, was computed in [24] in the special case $k \mid \frac{q-1}{p-1}$, i.e. in the case with integral spectrum (see Section 4), where $q = p^m$ for some m. Also, in [20] we computed the spectrum of a subfamily of semiprimitive GP-graphs, those of the form $\Gamma(q^\ell + 1, q^m)$ with $\frac{m}{(m,\ell)}$ even.

If Γ is an n-regular graph, then n is the greatest eigenvalue of Γ. A connected n-regular undirected graph is called *Ramanujan* if

$$|\lambda| = n \qquad \text{or} \qquad |\lambda| \leq 2\sqrt{n-1}$$

for any eigenvalue λ of Γ. Ramanujan graphs are optimal expanders. For background on Ramanujan graphs and expanders see for instance the excellent surveys of Ram

Murty [17], Hoory, Linial and Wigderson [11] and Lubotzky [15]. There are notions of Ramanujanicity for directed graphs (digraphs). It is both of theoretical and practical interest to obtain families of Ramanujan (di)graphs.

Outline and results

The paper is organized as follows. In Section 2 we study the spectrum of GP-graphs in terms of Gaussian periods. By using period polynomials, in Section 3 we give explicit computations of $\Gamma(k,q)$ for small values of k and in Section 4 we characterize all integral GP-graphs. In Section 5 we focus on the particular case of semiprimitive GP-graphs and in Section 6 we characterize integral Ramanujan graphs for $\Gamma(k,q)$ with $1 \leq k \leq 4$ or (k,q) a semiprimitive pair. Sections 2 and 5 can be thought as a kind of survey with some extra new material or with a different exposition, while the other sections present completely new results.

Let $q = p^m$ with p prime, assume that $k \mid q-1$ and put $n = \frac{q-1}{k}$. We now summarize the main results of the paper.

In Section 2 we study the spectrum of GP-graphs. In Theorem 2.1 we show that the spectrum of $\Gamma(k,q)$ can be put in terms of the cyclotomic Gaussian periods. More precisely,

$$\mathrm{Spec}(\Gamma(k,q)) = \{[n]^{1+\mu n}, [\eta_{i_1}]^{\mu_{i_1} n}, \ldots, [\eta_{i_s}]^{\mu_{i_s} n}\}$$

where $\eta_{i_1}^{(k,q)}, \ldots, \eta_{i_s}^{(k,q)}$ are the different cyclotomic Gaussian periods and the μ_{i_j}'s are certain numbers (see (2.1) and (2.6)).

The next section is devoted to explicit computations (based on previous works of Myerson [18], Gurak [8], [9], and Hoshi [12] on period polynomials). In Theorems 3.1 and 3.2 we give the whole spectrum of $\Gamma(3,q)$ and $\Gamma(4,q)$, respectively. This, together with Examples 2.3 and 2.4 and Remark 3.6 shows that the spectrum of $\Gamma(k,q)$ with $k \mid q-1$ can be computed for every proper divisor k of 24 (i.e. $k = 1,2,3,4,6,8,12$). Moreover, we give the spectrum of $\Gamma(5,q)$ in half of the cases: the case $p \equiv 1 \pmod 5$ is given in Proposition 3.4 while the case $p \equiv -1 \pmod 5$ corresponds to the semiprimitive case and hence it is obtained by taking $k = 5$ in Theorem 5.4 (the cases $p \equiv \pm 2 \pmod 5$ remain open).

In Section 4 we study integrality of the spectrum by way of period polynomials. In Theorem 4.1, one of the main results, we show that $\mathrm{Spec}(\Gamma(k,q)) \subset \mathbb{Z}$ if and only if $k \mid \frac{q-1}{p-1}$.

In Section 5, we first recall the definition of semiprimitive GP-graphs and give some infinite families of these graphs. Then, in Subsection 5.1 we explicitly give the spectrum of semiprimitive GP-graphs by using Gauss periods (see Theorem 5.4). Previously, in [3], Brouwer, Wilson and Xiang computed the spectra of a more general family defined in terms of semiprimitive pairs by using Gauss sums. Semiprimitive GP-graphs have three different eigenvalues; hence, in the connected case, they are strongly regular graphs (and hence distance regular graphs). In Subsection 5.2 we give the parameters of the semiprimitive GP-graphs as strongly regular graphs, as distance regular graphs and as pseudo-Latin square graphs (see Theorem 5.8).

In Section 6 we study some families of Ramanujan GP-graphs. First, we characterize all semiprimitive GP-graphs which are Ramanujan. In Theorem 6.1, another main result in the paper, we prove that if $\Gamma(k,q)$ is semiprimitive, then it is Ramanujan if and only if $k=2,3,4,5$ and $q=p^m$ satisfy certain easy arithmetic conditions. In particular, we obtain eight infinite families of semiprimitive (hence integral) Ramanujan GP-graphs $\{\Gamma(k,p^{2t})\}_{t\in\mathbb{N}}$, out of which five are valid for infinite different primes p. Finally, we show that all integral GP-graphs $\Gamma(k,q)$ with $1\le k\le 4$ which are non-semiprimitive are Ramanujan.

2 The spectrum of GP-graphs via cyclotomic Gaussian periods

Here, we will express the spectra of an arbitrary GP-graph $\Gamma(k,q)$, of its complement $\bar{\Gamma}(k,q)$, and of the associated sum graph $\Gamma^+(k,q)$, in terms of cyclotomic Gaussian periods. We remark that $\bar{\Gamma}(k,q)$ is not in general a GP-graph (unless $k=2$), but a union of Cayley graphs, since

$$\bar{\Gamma}(k,q) = X(\mathbb{F}_q, R_k^c \smallsetminus \{0\}) = X(\mathbb{F}_q, C_1^{(k,q)}) \cup \cdots \cup X(\mathbb{F}_q, C_{k-1}^{(k,q)}),$$

where

$$C_i^{(k,q)} = \omega^i \langle \omega^k \rangle$$

is the coset in $\mathbb{F}_q^*$ of the subgroup $\langle \omega^k \rangle$ with ω a generator of $\mathbb{F}_q^*$. In particular, the classic Paley graphs $\Gamma(2,q)$ with $q\equiv 1 \pmod 4$ is the only GP-graph which is self-complementary.

We begin by recalling the definition and basic properties of Gaussian periods. Let p be a prime, take $q=p^m$ with $m\in\mathbb{N}$ and let $k\mid q-1$. For any $i\in\{0,1,\ldots,k-1\}$, the i-th *cyclotomic Gaussian period* is defined by

$$\eta_i^{(k,q)} = \sum_{x\in C_i^{(k,q)}} \zeta_p^{\mathrm{Tr}_{q/p}(x)} \in \mathbb{Q}(\zeta_p), \qquad \text{for } 0\le i\le k-1, \tag{2.1}$$

where $\zeta_p = e^{\frac{2\pi i}{p}}$ and $\mathrm{Tr}_{q/p}:\mathbb{F}_q\to\mathbb{F}_p$ is the trace map given by

$$\mathrm{Tr}_{q/p}(x) = x + x^p + x^{p^2} + \cdots + x^{p^{r-1}}.$$

The following relation is well-known (see for instance Proposition 1 in [18]):

$$\sum_{i=0}^{k-1} \eta_i^{(k,q)} = -1. \tag{2.2}$$

From Theorem 13 in [7] (see also [18]), if we consider

$$N = \gcd(\tfrac{q-1}{p-1}, k), \tag{2.3}$$

we have the following integrality results:

$$\eta_i^{(N,q)} \in \mathbb{Z} \qquad \text{and} \qquad N\eta_i^{(N,q)} + 1 \equiv 0 \pmod p \tag{2.4}$$

(actually, in [7] other notations are used: N and N_1 for our k and N, respectively).

The spectrum of GP*-graphs

Let $\eta_0 = \eta_0^{(k,q)}, \ldots, \eta_{k-1} = \eta_{k-1}^{(k,q)}$ be the cyclotomic Gaussian periods as in (2.1) and let

$$\eta_{i_1}, \ldots, \eta_{i_s} \tag{2.5}$$

denote the different cyclotomic Gaussian periods not equal to $n = \frac{q-1}{k}$. We define the following numbers

$$\begin{aligned} \mu &= \#\{0 \le \ell \le k-1 : \eta_\ell = n\} \ge 0, \\ \mu_{i_j} &= \#\{0 \le \ell \le k-1 : \eta_\ell = \eta_{i_j}\} \ge 1, \end{aligned} \tag{2.6}$$

for $1 \le j \le s$. For simplicity, sometimes we will need to use the notation $\mu = \mu_{i_0}$.

We now show that the spectra of both GP-graphs, their complements, and GP$^+$-graphs are determined by the Gaussian periods. We recall that $\Gamma^+(k,q) = \Gamma(k,q)$ for q even.

Theorem 2.1. *Let $q = p^m$ with p prime and $k \in \mathbb{N}$ such that $k \mid q-1$. If we put $n = \frac{q-1}{k}$ then, in the notations in (2.5) and (2.6), we have*

$$\operatorname{Spec}(\Gamma(k,q)) = \{[n]^{1+\mu n}, [\eta_{i_1}]^{\mu_{i_1} n}, \ldots, [\eta_{i_s}]^{\mu_{i_s} n}\} \tag{2.7}$$

and $\operatorname{Spec}(\bar{\Gamma}(k,q)) = \{[(k-1)n]^{1+\mu n}, [-1-\eta_{i_1}]^{\mu_{i_1} n}, \ldots, [-1-\eta_{i_s}]^{\mu_{i_s} n}\}$. *Furthermore, if q is odd and n is even then*

$$\operatorname{Spec}(\Gamma^+(k,q)) = \{[n]^{1+\mu n}, [\pm\eta_{i_1}]^{\frac{1}{2}\mu_{i_1} n}, \ldots, [\pm\eta_{i_s}]^{\frac{1}{2}\mu_{i_s} n}\}. \tag{2.8}$$

Moreover, in any case, $\Gamma(k,q)$, $\Gamma^+(k,q)$ and $\bar{\Gamma}(k,q)$ are (strongly) connected if and only if $\mu = 0$ (with $k > 1$ for $\bar{\Gamma}(k,q)$).

Proof. We first compute the eigenvalues of $\Gamma(k,q)$. It is well-known that the spectrum of a Cayley graph $X(G,S)$ is determined by the irreducible characters of G. In fact, if G is abelian, each irreducible character χ of G induces an eigenvalue λ_χ of $X(G,S)$ by the expression

$$\lambda_\chi := \chi(S) = \sum_{g \in S} \chi(g)$$

with eigenvector $v_\chi = \big(\chi(g)\big)_{g \in G}$.

For $\Gamma(k,q)$ we have $G = \mathbb{F}_q$ and $S = R_k = \{x^k : x \in \mathbb{F}_q^*\}$. The irreducible characters of $\mathbb{F}_q$ are $\{\chi_\gamma\}_{\gamma \in \mathbb{F}_q}$ where

$$\chi_\gamma(y) = \zeta_p^{\operatorname{Tr}_{q/p}(\gamma y)}$$

for $y \in \mathbb{F}_q$. Thus, since $R_k = \langle \omega^k \rangle = C_0^{(k,q)}$, the eigenvalues $\lambda_\gamma = \lambda_{\chi_\gamma}$ of $\Gamma(k,q)$ are given by

$$\lambda_\gamma = \chi_\gamma(R_k) = \sum_{y \in R_k} \chi_\gamma(y) = \sum_{y \in C_0^{(k,q)}} \zeta_p^{\operatorname{Tr}_{q/p}(\gamma y)}. \tag{2.9}$$

We have the disjoint union

$$\mathbb{F}_q = \{0\} \cup C_0^{(k,q)} \cup \cdots \cup C_{k-1}^{(k,q)}$$

and $\#C_i^{(k,q)} = \#\langle \omega^k \rangle = \frac{q-1}{k}$ for every $i = 0, \ldots, k-1$. Now, for $\gamma = 0$ we have

$$\lambda_0 = \chi_0(R_k) = |R_k| = n,$$

since χ_0 is the trivial character. This is in accordance with the fact that since $\Gamma(k,q)$ is n-regular with $n = \frac{q-1}{k}$, then n is an eigenvalue of $\Gamma(k,q)$. On the other hand, if $\gamma \in C_i^{(k,q)}$ then γy runs over $C_i^{(k,q)}$ when y runs over $C_0^{(k,q)}$ and thus, by (2.9), we have

$$\lambda_\gamma = \sum_{x \in C_i^{(k,q)}} \zeta_p^{\mathrm{Tr}_{q/p}(x)} = \eta_i^{(k,q)}$$

which does not depend on γ.

Let $\eta_{i_1}, \ldots, \eta_{i_s}$ be the different cyclotomic Gaussian periods. Notice that each $\gamma \in C_{i_\ell}^{(k,q)}$ gives the same λ_γ and that $|C_{i_\ell}^{(k,q)}| = |C_0^{(k,q)}| = n$ for $1 \le \ell \le s$. Thus, it is clear that the multiplicity of λ_γ is

$$m(\lambda_0) = 1 + \sum_{\substack{0 \le j \le k-1 \\ \eta_j = n}} |C_j^{(k,q)}|$$

and

$$m(\lambda_\gamma) = \sum_{\substack{0 \le j \le k-1 \\ \eta_{i_\ell} = \eta_j}} |C_j^{(k,q)}| \qquad (\text{for } \gamma \neq 0),$$

that is $m(n) = 1 + \mu n$ and $m(\eta_{i_\ell}) = \mu_{i_\ell} n$ for $1 \le \ell \le s$. This gives the spectrum for $\Gamma(k,q)$.

To see the spectrum of the complementary graph, if A is the adjacency matrix of $\Gamma(k,q)$ then $J - A - I$ is the adjacency matrix of $\bar{\Gamma}(k,q)$, where J stands for the all 1's matrix. Since $\Gamma(k,q)$ is n-regular with q vertices, then $\bar{\Gamma}(k,q)$ is $(q-n-1)$-regular, that is

$$\bar{\lambda}_0 = q - n - 1 = (k-1)n.$$

The remaining eigenvalues of $\bar{\Gamma}(k,q)$ are $-1-\lambda$ where λ are the non-trivial eigenvalues, and hence the result follows by (2.7).

Now consider the spectrum of $\Gamma^+(k,q)$. Since q is odd we have $\Gamma^+(k,q) \neq \Gamma(k,q)$ and, by Proposition 2.10 in [23], we get that the non-principal eigenvalues of $\Gamma^+(k,q)$ and their corresponding multiplicities are given by

$$\lambda_{\Gamma^+} = \pm \lambda_\Gamma \qquad \text{and} \qquad m(\lambda_{\Gamma^+}) = \tfrac{1}{2} m(\lambda_\Gamma),$$

where λ_Γ and $m(\lambda_\Gamma)$ (respectively λ_Γ^+ and $m(\lambda_\Gamma^+)$) are the eigenvalues and multiplicities of $\Gamma(k,q)$ (respectively $\Gamma^+(k,q)$). Thus, we get the expression for $\mathrm{Spec}(\Gamma^+(k,q))$ in (2.8) and we need n even for the multiplicities to be integers.

Finally, being n-regular, $\Gamma(k,q)$ is connected if and only if the multiplicity of n is 1, i.e. if $\mu = 0$. A similar argument applies for the graphs $\Gamma^+(k,q)$ and $\bar{\Gamma}(k,q)$. To conclude, just notice that for $k=1$ we have $\Gamma(1,q) = K_q$ and hence $\bar{\Gamma}(1,q)$ is the empty graph with q vertices which is disconnected. In this case, since $\mathrm{Spec}(K_q) = \{[q-1]^1, [-1]^{q-1}\}$ we have $\mathrm{Spec}(\bar{K}_q) = \{[0]^1, [-1-(-1)]^{q-1}\} = \{[0]^q\}$, although $\mu = 0$ (also $-1-\eta_0 = 0$ by (2.2)). □

We now make some observations on the previous theorem and point out some consequences of it for the spectrum of $\Gamma(k,q)$.

Remark 2.2. (i) If the Gaussian periods are all different, i.e. $\eta_i \neq \eta_j$ for $0 \leq i < j \leq k-1$, then $\mu = 0$ and $\mu_{i_j} = 1$ for every j since $q = kn+1$, and hence we have

$$\mathrm{Spec}(\Gamma(k,q)) = \{[n]^1, [\eta_0]^n, [\eta_1]^n, \ldots, [\eta_{k-1}]^n\}.$$

This holds, for instance, for Paley graphs $\Gamma(2,q)$ –both directed and undirected– and also for the graphs $\Gamma(3,q)$ and $\Gamma(4,q)$ in the non-semiprimitive case (i.e. $p \not\equiv -1 \pmod{k}$), as one can see in Example 2.4 and Theorems 3.1 and 3.2, respectively.

(ii) Notice that, by the theorem, $\Gamma(k,q)$ is integral if and only if $\bar{\Gamma}(k,q)$ and $\Gamma^+(k,q)$ are integral. In Section 4 we will characterize all integral GP-graphs (and hence all integral complements and all integral GP$^+$-graphs).

(iii) Theorem 2.1 allows one to compute the spectrum of families of GP-graphs in those cases where the (cyclotomic) Gaussian periods or Gaussian sums are known. The Gaussian periods $\eta_i^{(k,q)}$ for $k = 2,3,4,6,8,12$ are well-known; the cases $k = 2,3,4$ date back to Gauss (see for instance [18]) while the cases $k = 6,8,12$ are due to Gurak ([8], [9]). The case $k=5$ is partially done by Hoshi [12]. A case which is well understood is when (k,q) is a semiprimitive pair. Some of these cases will be treated in more detail in Sections 3 and 5. Other general examples of known Gaussian sums are the so-called index 2 and index 4 cases (see the literature). It would be interesting to find the spectrum of $\Gamma(k,q)$ in these cases.

To close the section we illustrate with two basic examples. We compute the spectrum of $\Gamma(k,q)$ for $k=1,2$. Using Theorem 2.1 one can also obtain the spectrum of $\bar{\Gamma}(k,q)$ and $\Gamma^+(k,q)$ for $k=1,2$ (we leave the details). More involved computations will be performed in the next section.

Example 2.3 (*Complete graphs*)**.** We have $\Gamma(1,q) = K_q$ and $\mathrm{Spec}(K_q) = \{[q-1]^1, [-1]^{q-1}\}$. Using Theorem 2.1, since $n = q-1$ and $\mu = 0$, $\mu_1 = 1$ by (2.6), we obtain that

$$\mathrm{Spec}(\Gamma(1,q)) = \{[q-1]^1, [\eta_0]^{q-1}\},$$

and $\eta_0 = -1$ by (2.2), hence recovering the known result. ◊

Example 2.4 (*Paley graphs*)**.** We recall that $\Gamma(2,q)$ with $q = p^m$ is the classic (undirected) Paley graph $P(q)$ if $q \equiv 1 \pmod 4$, hence $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$ and $m = 2t$; and it is the directed Paley graph $\vec{P}(q)$ if $q \equiv 3 \pmod 4$, hence $p \equiv 3 \pmod 4$ and $m = 2t + 1$.

If we put $n = \frac{q-1}{2}$, by Theorem 2.1 we have that

$$\mathrm{Spec}(\Gamma(2,q)) = \{[n]^{1+\mu}, [\eta_0]^{1+\mu_0 n}, [\eta_1]^{1+\mu_1 n}\}$$

where $\eta_i = \eta_i^{(2,q)}$ for $i = 0,1$. The above Gaussian periods are known, see for instance Lemma 11 in [7]. In our notations, (i.e. taking $s = 1$ in [7], $r = p^m$ is our q) we have

$$\eta_0 = \begin{cases} \frac{1}{2}\big(-1 + (-1)^{m-1}\sqrt{q}\big) & \text{if } p \equiv 1 \pmod 4, \\ \frac{1}{2}\big(-1 + (-1)^{m-1}\sqrt{-1}^{\,m}\sqrt{q}\big) & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and $\eta_1 = -1 - \eta_0$. First, notice that η_0 and hence η_1 are real if and only if $\Gamma(2,q)$ is undirected. Second, note that $\eta_0 \neq \eta_1$ and that $n \neq \eta_0, \eta_1$. These last conditions imply that $\mu = 0$ and $\mu_0 = \mu_1 = 1$ (or conversely, since $2n + 1 = q$ we must have that $\mu = 0$ and $\mu_0 = \mu_1 = 1$). Hence, we have that

$$\mathrm{Spec}(\Gamma(2,q)) = \{[n]^1, [\eta_0]^n, [\eta_1]^n\},$$

where

$$\eta_0 = \begin{cases} \frac{-1-p^t}{2} & \text{if } p \equiv 1 \pmod 4, \\ \frac{-1-(-1)^t p^t}{2} & \text{if } p \equiv 3 \pmod 4, \end{cases} \qquad \text{for } m = 2t,$$

$$\eta_0 = \begin{cases} \frac{-1+p^t\sqrt{p}}{2} & \text{if } p \equiv 1 \pmod 4, \\ \frac{-1-(-1)^t i p^t\sqrt{p}}{2} & \text{if } p \equiv 3 \pmod 4, \end{cases} \qquad \text{for } m = 2t+1,$$

and $\eta_1 = -1 - \eta_0$. Notice that $\eta_0 \in \mathbb{Z}$ for $m = 2t$ and $\eta_0 \in \mathbb{Q}(\sqrt{-p})$ for $m = 2t+1$. This coincides with the known spectrum

$$\mathrm{Spec}(P(q)) = \big\{[\tfrac{q-1}{2}]^1, [\tfrac{-1+(-1)^m\sqrt{q}}{2}]^n, [\tfrac{-1-(-1)^m\sqrt{q}}{2}]^n\big\}.$$

We also obtain

$$\mathrm{Spec}(\vec{P}(q)) = \big\{[\tfrac{q-1}{2}]^1, [\tfrac{-1+(-1)^m i^m\sqrt{q}}{2}]^n, [\tfrac{-1-(-1)^m i^m\sqrt{q}}{2}]^n\big\}. \tag{2.10}$$

Finally, the graph $\Gamma(2,q)$ is connected since the multiplicity of the regularity degree is 1. Therefore, since $\Gamma(2,q)$ is regular and connected and has exactly 3 eigenvalues it is a strongly regular graph (in the undirected case). All these facts are of course well-known. $\Diamond$

3 Explicit computations through period polynomials

Since the spectrum of GP-graphs is given in terms of Gaussian periods, we now recall the polynomial associated with them. The *period polynomial* is defined by

$$\Psi_{k,q}(x) = \prod_{i=0}^{k-1}(x - \eta_i^{(k,q)}), \tag{3.1}$$

where $\eta_i^{(k,q)}$ is the Gaussian period given in (2.1).

In some cases, the expansions of these polynomials are known, and they factor into product of polynomials of small degree, hence their roots (the Gaussian periods) can be explicitly computed. We will use some of the known cases to give the spectrum of the associated GP-graphs explicitly. In particular, the spectrum of the graphs $\Gamma(k,q)$ with $k \mid 24$ ($k \neq 24$) can be determined (although it is highly non-trivial in most of the cases). For simplicity, we will give explicitly the spectrum of the graphs $\Gamma(k,q)$ with $k = 3,4$ (the cases $k = 1,2$ were presented in Examples 2.3 and 2.4). For the remaining cases $k = 6,8,12$ we refer to the works of Gurak. Using results of Hoshi we give the spectrum of $\Gamma(5,p^{5t})$ in the case $p \equiv 1 \pmod 5$. Another well-known case is the semiprimitive one, which is delayed until Section 5.

In [24], we have computed the spectrum of $\Gamma(k,q)$ for $k \mid \frac{q-1}{p-1}$ with $k = 3,4$, where $q = p^m$. There, we used a relation that we found between the spectrum of GP-graphs $\Gamma(k,q)$ and the weight distribution of certain irreducible cyclic codes $\mathcal{C}(k,q)$, provided that $k \mid \frac{q-1}{p-1}$ (a posteriori, those GP-graphs having integral spectrum, see Section 4). Namely, we have used this relation and the fact that the weight distributions for the codes $\mathcal{C}(3,q)$ and $\mathcal{C}(4,q)$ was already known (which was computed by using Gaussian periods) in these cases.

Now, we will give the complete result, that is we give the spectrum of $\Gamma(k,q)$ for $k \mid q-1$ with $k = 3,4$, by way of explicit factorizations of the period polynomials $\Psi_{3,q}(x)$ and $\Psi_{4,q}(x)$ obtained by Myerson ([18]). It turns out that there is one extra case for $k = 3$ and two extra cases for $k = 4$ in this more general setting (i.e. $k \mid q-1$ instead of $k \mid \frac{q-1}{p-1}$).

We now give the spectrum of the GP-graphs $\Gamma(3,q)$ explicitly.

Theorem 3.1. *Let $q = p^m \geq 5$ with p prime such that $3 \mid q-1$ and put $n = \frac{q-1}{3}$. Thus, the graph $\Gamma(3,q)$ is connected and undirected with real spectrum given as follows:*

(*a*) *If $p \equiv 1 \pmod 3$ with $3 \mid m$ then*

$$\mathrm{Spec}(\Gamma(3,q)) = \left\{[n]^1, \left[\tfrac{a\sqrt[3]{q}-1}{3}\right]^n, \left[\tfrac{-\frac12(a+9b)\sqrt[3]{q}-1}{3}\right]^n, \left[\tfrac{-\frac12(a-9b)\sqrt[3]{q}-1}{3}\right]^n\right\}$$

where a,b are integers uniquely determined by

$$4\sqrt[3]{q} = a^2 + 27b^2, \qquad a \equiv 1 \pmod 3 \qquad \text{and} \qquad (a,p) = 1. \tag{3.2}$$

(*b*) *If* $p \equiv 1 \pmod 3$ *with* $3 \nmid m$ *then* $\mathrm{Spec}(\Gamma(3,q)) = \{[n]^1, [x_0]^n, [x_1]^n, [x_2]^n\}$ *where*

$$x_j = -\tfrac{1}{3}\big(1 + \omega^j W + \frac{q}{\omega^j W}\big), \qquad j \in \{0,1,2\},$$

$\omega = e^{\frac{2\pi i}{3}}$ *and* $W = \sqrt[3]{q}\sqrt[3]{\tfrac{1}{2}(-a+\sqrt{-27}b)}$, *where* a *and* b *are uniquely determined integers (*b *up to sign) satisfying*

$$4q = a^2 + 27b^2, \qquad a \equiv 1 \pmod 3 \qquad \textit{and} \qquad (a,p) = 1$$

Here, $\sqrt{\ }$ *and* $\sqrt[3]{\ }$ *denote any square and cubic root, respectively.*

(*c*) *If* $p \equiv 2 \pmod 3$ *with* m *even then*

$$\mathrm{Spec}(\Gamma(3,q)) = \begin{cases} \big\{[n]^1, \big[\frac{\sqrt{q}-1}{3}\big]^{2n}, \big[\frac{-2\sqrt{q}-1}{3}\big]^{n}\big\} & \textit{for } m \equiv 0 \pmod 4, \\ \big\{[n]^1, \big[\frac{2\sqrt{q}-1}{3}\big]^{n}, \big[\frac{-\sqrt{q}-1}{3}\big]^{2n}\big\} & \textit{for } m \equiv 2 \pmod 4. \end{cases}$$

Furthermore, the spectrum is integral in cases (*a*) *and* (*c*).

Proof. The graph $\Gamma(3,q)$ is connected and undirected (see the Introduction) since $\frac{q-1}{3}$ is a primitive divisor of $q-1$ and for q odd we have that $3 \mid \frac{q-1}{2}$ (in fact $2 \mid q-1$ and $3 \mid q-1$, hence $6 \mid q-1$, which is equivalent to $3 \mid \frac{q-1}{2}$). Thus, $\mathrm{Spec}(\Gamma(3,q)) \subset \mathbb{R}$ (the adjacency matrix of an undirected graph is symmetric and so its spectrum is real).

In Theorems 13 and 16 in [18] (see also Lemmas 7 and 8 in [7]) Myerson gave the polynomial $\Psi_{3,q}(x)$ and its factorizations over the rationals. Namely,

$$\Psi_{3,q}(x) = x^3 + x^2 - nx - d \qquad \text{with} \qquad d = \tfrac{(a+3)q-1}{27}, \tag{3.3}$$

where a and b are integers uniquely determined (b only up to sign) by $4q = a^2 + 27b^2$, with $a \equiv 1 \pmod 3$ and if $p \equiv 1 \pmod 3$ then $(a,p) = 1$. We have the following cases:

(*a*) If $p \equiv 1 \pmod 3$ and $3 \mid m$, then

$$\Psi_{3,q}(x) = \tfrac{1}{27}(3x + 1 - a\sqrt[3]{q})(3x + 1 - a\sqrt[3]{q})(3x + 1 - a\sqrt[3]{q}),$$

where a and b are as in (3.2).

(*b*) If $p \equiv 1 \pmod 3$ and $3 \nmid m$, then $\Psi_{3,q}(x)$ is irreducible over $\mathbb{Q}$.

(*c*) If $p \equiv 2 \pmod 3$ and m is even, then

$$\Psi_{3,q}(x) = \begin{cases} \frac{1}{27}(3x + 1 + 2\sqrt{q})(3x + 1 - \sqrt{q})^2 & \text{if } \frac{m}{2} \text{ is even}, \\ \frac{1}{27}(3x + 1 - 2\sqrt{q})(3x + 1 + \sqrt{q})^2 & \text{if } \frac{m}{2} \text{ is odd}. \end{cases}$$

Thus, the eigenvalues and multiplicities of $\Gamma(3,q)$ are directly obtained from these expressions in cases (a) and (c).

Now, we study case (b). We have to find the roots of $\Psi_{3,q}(x)$ in (3.3). The roots of a general cubic Ax^3+Bx^2+Cx+D are given by

$$x_j = -\tfrac{1}{3A}\big(B+\omega^j W+\tfrac{\Delta_0}{\omega^j W}\big)$$

for $j=0,1,2$, where $\omega=e^{\frac{2\pi i}{3}}$ is the primitive cubic root of 1, and

$$W=\sqrt[3]{\tfrac{1}{2}\big(\Delta_1 \pm \sqrt{\Delta_1^2-4\Delta_0^3}\big)}$$

where $\Delta_0=B^2-3AC$ and $\Delta_1=2B^3-9ABC+27A^2D$. Thus, by (3.3), we have that $A=B=1$ and we obtain that $\Delta_0=q$ and $\Delta_1=-aq$. In this way, we arrive at

$$W=\sqrt[3]{\tfrac{1}{2}\big(-aq\pm q\sqrt{a^2-4q}\big)}=\sqrt[3]{q}\sqrt[3]{\tfrac{1}{2}(-a\pm\sqrt{-27}b)},$$

where we have used that $4q=a^2+27b^2$.

Here, $\sqrt{\ }$ and $\sqrt[3]{\ }$ denote any square and any cubic root, respectively. In general, the sign $\pm$ can be randomly chosen, and if $W=0$ with one sign one has to chose the other one. It can never happen that both signs give $W=0$, since this is equivalent to $\Delta_1=\Delta_0=0$ and both Δ_0 and Δ_1 are non-zero in our case. Hence, we choose the plus sign, and part (b) is proved.

Finally, the eigenvalues in cases (a) and (c) are integers by the conditions on p and m. In case (a) we have that $a \equiv a\pm 9b \equiv 1 \pmod 3$ where $a\pm 9b$ is even since $4p^m=a^2+27b^2$ implies that a and b have the same parity. The remaining assertion is clear from the statement. □

Using the theorem one can compute, for instance, $\mathrm{Spec}(\Gamma(3,7^{3m}))$ with item (a), $\mathrm{Spec}(\Gamma(3,7^{3m+j}))$, $j=1,2$, with item (b) and $\mathrm{Spec}(\Gamma(3,5^{2m}))$ with item (c) for any $m\in\mathbb{N}$.

Next we give the spectrum of the GP-graphs $\Gamma(4,q)$ explicitly.

Theorem 3.2. *Let $q=p^m$ with p prime such that $4\mid q-1$ and put $n=\frac{q-1}{4}$. Thus, the graph $\Gamma(4,q)$ is connected (except for $q=9$) with spectrum given as follows:*

(a) *If $p\equiv 1 \pmod 4$ with $m\equiv 0 \pmod 4$ then*

$$\mathrm{Spec}(\Gamma(4,q))=\Big\{[n]^1, \big[\tfrac{\sqrt{q}+4d\sqrt[4]{q}-1}{4}\big]^n, \big[\tfrac{\sqrt{q}-4d\sqrt[4]{q}-1}{4}\big]^n, \big[\tfrac{-\sqrt{q}+2c\sqrt[4]{q}-1}{4}\big]^n, \big[\tfrac{-\sqrt{q}-2c\sqrt[4]{q}-1}{4}\big]^n\Big\}$$

where c,d are integers uniquely determined by

$$\sqrt{q}=c^2+4d^2, \qquad c\equiv 1 \pmod 4 \qquad \textit{and} \qquad (c,p)=1. \tag{3.4}$$

(b) *If* $p \equiv 1 \pmod 4$ *with* $m \equiv 2 \pmod 4$, *then*

$$\mathrm{Spec}(\Gamma(4,q)) = \Big\{[n]^1, \big[\tfrac{-(1+\sqrt{q})+\sqrt{2(q+c\sqrt{q})}}{4}\big]^n, \big[\tfrac{-(1+\sqrt{q})-\sqrt{2(q+c\sqrt{q})}}{4}\big]^n, \big[\tfrac{-(1-\sqrt{q})+\sqrt{2(q-c\sqrt{q})}}{4}\big]^n, \big[\tfrac{-(1-\sqrt{q})-\sqrt{2(q-c\sqrt{q})}}{4}\big]^n\Big\}$$

where c, d *are integers unique determined (d up to sign) by*

$$q = c^2 + 4d^2, \qquad c \equiv 1 \pmod 4 \qquad \text{and} \qquad (c,p) = 1. \tag{3.5}$$

(c) *If* $p \equiv 1 \pmod 4$ *with* m *odd and* n *odd then*

$$\mathrm{Spec}(\Gamma(4,q)) = \{[n]^1, [x_1^+]^n, [x_1^-]^n, [x_2^+]^n, [x_2^-]^n\}$$

with

$$x_1^{\pm} = \tfrac12\Big(-\sqrt{2y - \tfrac{q}{8}} \pm \sqrt{-(2y + \tfrac{q}{8}) + \tfrac{(c-1)q+1}{4\sqrt{2y-\frac{q}{8}}}}\Big) - \tfrac14,$$

$$x_2^{\pm} = \tfrac12\Big(\sqrt{2y - \tfrac{q}{8}} \pm \sqrt{-(2y + \tfrac{q}{8}) - \tfrac{(c-1)q+1}{4\sqrt{2y-\frac{q}{8}}}}\Big) - \tfrac14,$$

where $y = \frac{q}{48} + W - \frac{P}{3W}$ *and* $W = \sqrt[3]{-\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}}$ *with* $P = \frac{36-(28q-12c^2-12)q}{3\cdot 256}$, $Q = -\frac{q^2}{27\cdot 256} + \frac{q\gamma}{24} - \frac{((c-1)q+1)^2}{64\gamma}$ *and* $\gamma = \frac{(9q-4c^2-4)q-12}{256}$, *where* c, d *are integers as in* (3.5).

(d) *If* $p \equiv 1 \pmod 4$ *with* m *odd and* n *even then*

$$\mathrm{Spec}(\Gamma(4,q)) = \{[n]^1, [x_1^+]^n, [x_1^-]^n, [x_2^+]^n, [x_2^-]^n\}$$

with

$$x_1^{\pm} = \tfrac12\Big(-\sqrt{2y + \tfrac{3q}{8}} \pm \sqrt{-(2y - \tfrac{3q}{8}) + \tfrac{(3-c)q-1}{4\sqrt{2y+\frac{3q}{8}}}}\Big) - \tfrac14,$$

$$x_2^{\pm} = \tfrac12\Big(\sqrt{2y + \tfrac{3q}{8}} \pm \sqrt{-(2y - \tfrac{3q}{8}) - \tfrac{(3-c)q-1}{4\sqrt{2y+\frac{3q}{8}}}}\Big) - \tfrac14,$$

where $y = -\frac{q}{16} + \omega - \frac{P}{3\omega}$ *and* $\omega = \sqrt[3]{-\frac{Q}{2} \pm \sqrt{\frac{Q^2}{4} + \frac{P^3}{27}}}$ *with* $P = \frac{(2q+12-4c^2)q-12}{256}$, $Q = -\frac{q^2}{3\cdot 256} - \frac{q\gamma}{8} - \frac{((3-c)q-1)^2}{64\gamma}$ *and* $\gamma = \frac{(q+12-4c^2)q-12}{256}$, *where* c, d *are integers as in* (3.5).

(e) *If* $p \equiv 3 \pmod 4$ *with* m *even then* $\mathrm{Spec}(\Gamma(4,9)) = \{[2]^3, [-1]^6\}$ *and for any* $q \neq 9$ *we have*

$$\mathrm{Spec}(\Gamma(4,q)) = \begin{cases} \big\{[n]^1, \big[\frac{\sqrt{q}-1}{4}\big]^{3n}, \big[\frac{-3\sqrt{q}-1}{4}\big]^n\big\} & \text{for } m \equiv 0 \pmod 4, \\ \big\{[n]^1, \big[\frac{3\sqrt{q}-1}{4}\big]^n, \big[\frac{-\sqrt{q}-1}{4}\big]^{3n}\big\} & \text{for } m \equiv 2 \pmod 4. \end{cases}$$

Moreover, the graph $\Gamma(4,q)$ *is undirected with real spectrum in cases* (a)*,* (b)*,* (d) *and* (e)*. In particular, the spectrum is integral in cases* (a) *and* (e)*.*

Proof. The graph $\Gamma(4,q)$ is connected (except for $q=9$) since $\frac{q-1}{4}$ is a primitive divisor of $q-1$ (see the Introduction). For p odd and $m=2t$ even, i.e. in cases (a), (b) and (e), one can show that $4 \mid \frac{q-1}{2}$, and hence the graph is undirected. In fact, if $p=4t+a$, with $a=1$ or 3, then $p^2 \equiv a^2 \equiv 1 \pmod 8$. Thus, $p^{2s} \equiv 1 \pmod 8$ for every $s \in \mathbb{N}$ and $q \equiv 1 \pmod 8$. In the remaining case (d), the graph $\Gamma(4,q)$ is undirected since $n=\frac{q-1}{4}$ is even. Indeed, if $2 \mid \frac{q-1}{4}$, then $\frac{q-1}{4}=2t$ for some $t \in \mathbb{Z}$ which implies that $\frac{q-1}{2}=4t$ and therefore $4 \mid \frac{q-1}{2}$. Thus, $\mathrm{Spec}(\Gamma(4,q)) \subset \mathbb{R}$ (the adjacency matrix of an undirected graph is symmetric).

Also, the graph $\Gamma(4,9)$ is the disjoint union of three copies of K_3, and since $\mathrm{Spec}(K_3)=\{[2]^1,[-1]^2\}$ we get that the spectrum of $\Gamma(4,9)$ is as stated (notice that it is still given by the corresponding formula in (b), i.e. $\mathrm{Spec}(\Gamma(4,q))=\{[n]^1,[\frac14(3\sqrt{q}-1)]^n,[\frac14(-\sqrt{q}-1)]^{3n}\}$ for $q=9$) since, as multisets, $\{[2]^1,[2]^2,[-1]^6\}=\{[2]^3,[-1]^6\}$.

In Theorems 14 and 17 in [18] (see also Lemmas 9 and 10 in [7]) Myerson gave the polynomial $\Psi_{4,q}(x)$ and its factorizations over the rationals. Namely, we have

$$\Psi_{4,q}(x)=\begin{cases} x^4+x^3-\frac{3q-3}{8}x^2+\frac{(2c-3)q+1}{16}x+\frac{q^2-(4c^2-8c+6)q+1}{256} & \text{if } n \text{ is even,}\\ x^4+x^3+\frac{q+3}{8}x^2+\frac{(2c+1)q+1}{16}x+\frac{9q^2-(4c^2-8c-2)q+1}{256} & \text{if } n \text{ is odd,}\end{cases} \tag{3.6}$$

where c,d are integers uniquely determined (d up to sign) such that $q=c^2+4d^2$, $d \equiv 1 \pmod 4$ and if $p \equiv 1 \pmod 4$ then $(c,p)=1$. We have the following cases:

(a) If $p \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 4$ then $\Psi_{4,q}(x)$ equals

$$\tfrac{1}{64}\big((4x+1)+\sqrt{q}+2c\sqrt[4]{q}\big)\big((4x+1)+\sqrt{q}-2c\sqrt[4]{q}\big)\big((4x+1)-\sqrt{q}+2c\sqrt[4]{q}\big)\big((4x+1)-\sqrt{q}-2c\sqrt[4]{q}\big),$$

where c,d are integers uniquely determined by (3.4).

(b) If $p \equiv 1 \pmod 4$ and $m \equiv 2 \pmod 4$ then $\Psi_{4,q}(x)$ equals

$$\tfrac{1}{64}\big((4x+1)^2+2\sqrt{q}(4x+1)-q-2c\sqrt{q}\big)\big((4x+1)^2-2\sqrt{q}(4x+1)-q+2c\sqrt{q}\big)$$

where the quadratics are irreducible over $\mathbb{Q}$ and c,d are integers satisfying $q=c^2+4d^2$, $d \equiv 1 \pmod 4$ and $(c,p)=1$.

(c) If $p \equiv 1 \pmod 4$ and m odd then $\Psi_{(4,q)}(x)$ is irreducible over $\mathbb{Q}$.

(d) If $p \equiv 3 \pmod 4$ and m even then

$$\Psi_{4,q}(x)=\begin{cases} \frac{1}{64}(4x+1+3\sqrt{q})(4x+1-\sqrt{q})^3 & \text{if } \frac{m}{2} \text{ is even,}\\ \frac{1}{64}(4x+1-3\sqrt{q})(4x+1+\sqrt{q})^3 & \text{if } \frac{m}{2} \text{ is odd.}\end{cases}$$

Thus, the eigenvalues and multiplicities of $\Gamma(4,q)$ are directly obtained from these expressions in cases (a) and (d). In case (b), a routine calculation shows that the roots of these quadratics are respectively given by

$$\tfrac{-(1+\sqrt{q})\pm\sqrt{2(q+c\sqrt{q})}}{4} \qquad \text{and} \qquad \tfrac{-(1-\sqrt{q})\pm\sqrt{2(q-c\sqrt{q})}}{4},$$

from which the spectrum in this case readily follows.

We now consider cases (c) and (d). The roots of a general quartic

$$Ax^4 + Bx^2 + Cx^2 + Dx + E$$

are given by

$$\begin{aligned} x_1^{\pm} &= \tfrac{1}{2}\Big(-\sqrt{2y-\alpha} \pm \sqrt{-(2y+\alpha) + \tfrac{2\beta}{\sqrt{2y-\alpha}}}\Big) - \tfrac{1}{4}, \\ x_2^{\pm} &= \tfrac{1}{2}\Big(\sqrt{2y-\alpha} \pm \sqrt{-(2y+\alpha) - \tfrac{2\beta}{\sqrt{2y-\alpha}}}\Big) - \tfrac{1}{4}, \end{aligned} \tag{3.7}$$

where

$$y = \tfrac{\alpha}{6} + W - \tfrac{P}{3W} \qquad \text{and} \qquad W = \sqrt[3]{-\tfrac{Q}{2} \pm \sqrt{\tfrac{Q^2}{4} + \tfrac{P^3}{27}}} \tag{3.8}$$

with $P = -\frac{\alpha^2}{12} - \gamma$ and $Q = -\frac{\alpha^2}{108} + \frac{\alpha\gamma}{3} - \frac{\beta^2}{\gamma}$ where

$$\alpha = -\tfrac{3B^2}{8A^2} + \tfrac{C}{A}, \quad \beta = \tfrac{B^3}{8A^3} - \tfrac{BC}{2A^2} - \tfrac{D}{A}, \quad \text{and} \quad \gamma = -\tfrac{3B^4}{256A^4} - \tfrac{B^2C}{16A^3} - \tfrac{BD}{4A^2} + \tfrac{E}{A}.$$

If n is odd, by using the second line in (3.6), one can check that

$$\begin{gathered} \alpha = \tfrac{q}{8}, \qquad \beta = \tfrac{(c-1)q+1}{8}, \qquad \gamma = \tfrac{(9q-4c^2-4)q-12}{256}, \\ P = \tfrac{36-(28q-12c^2-12)q}{256\cdot 3}, \qquad \text{and} \qquad Q = -\tfrac{q^2}{27\cdot 256} + \tfrac{q\gamma}{24} - \tfrac{((c-1)q+1)^2}{64\gamma}; \end{gathered}$$

while if n is even, by the first line in (3.6), one has that

$$\begin{gathered} \alpha = \tfrac{-3q}{8}, \qquad \beta = \tfrac{(3-c)q-1}{8}, \qquad \gamma = \tfrac{(q+12-4c^2)q-12}{256}, \\ P = \tfrac{(2q+12-4c^2)q-12}{256}, \qquad \text{and} \qquad Q = -\tfrac{q^2}{3\cdot 256} - \tfrac{q\gamma}{8} - \tfrac{((3-c)q-1)^2}{64\gamma}. \end{gathered}$$

Putting this information in (3.7) and (3.8) we get the desired result.

Finally, the spectra in cases (a) and (e) are integral by the conditions on p and m, where in case (a) we use that $c \equiv 1 \pmod 4$). □

Using the theorem one can compute, for instance, the following spectra: $\mathrm{Spec}(\Gamma(4,5^{4m}))$ with item (a), $\mathrm{Spec}(\Gamma(4,5^{4m+2}))$ with item (b), $\mathrm{Spec}(\Gamma(4,5^{2m+1}))$ with items (c) and (d) and $\mathrm{Spec}(\Gamma(4,7^{2m}))$ with item (e), for any $m \in \mathbb{N}$.

Remark 3.3. (i) Theorems 3.1 and 3.2 extend the results obtained in Theorems 2.2 and 2.4 in [24], valid for $k \mid \frac{q-1}{p-1}$, to the general case $k \mid q-1$. As we will see in the next section, the cases considered in [24] correspond to those with integral spectrum, and this explains why we were able to obtain them via weight distribution of codes. The general case (i.e. with non-integral spectrum), however, cannot be related with weight distribution of codes.

(ii) Note that the spectra of $\Gamma(3,q)$ in Theorem 3.1 (b) and of $\Gamma(4,q)$ in Theorem 3.2 (d) are real, although this may not look so from the expressions. For instance, in the case (b) of Theorem 3.1, notice that $|W|^2 = q$ and hence $\frac{q}{\omega^j W} = \overline{\omega^j W}$. This implies that

$$\omega^j W + \tfrac{q}{\omega^j W} = 2\,\mathrm{Re}(\omega^j W)$$

for any $0 \leq j \leq 2$ and therefore $\Gamma(3,q)$ is real. However, to check that $\Gamma(4,q)$ in case (d) has real spectrum directly may be quite difficult.

Now, using a result of Hoshi [12] we can give the spectrum of $\Gamma(5,q)$ in the case $q = p^m$ with $p \equiv 1 \pmod 5$ and $5 \mid m$. In Theorem 1 in [12], for $p \equiv 1 \pmod 5$ and $m = 5s$, Hoshi obtained the factorization of the period polynomial in the reduced form

$$\Psi^*_{5,p^m}(X) = \prod_{i=1}^{5} (X - \eta_i^*),$$

where $\eta_i^* = 5\eta_i + 1$, in terms of solutions of the so-called Dickson's system of Diophantine equations:

$$\begin{cases} 16p^m = x^2 + 125w^2 + 50v^2 + 50u^2, \\ xw = v^2 - 4uv - u^2, \\ x \equiv -1 \pmod 5). \end{cases} \tag{3.9}$$

If we denote by $S(p,m)$ the set of all integer solutions of this system, it is known that $\#S(p,m) = (m+1)^2$. Moreover, the system have exactly four integer solutions satisfying $p \nmid x^2 - 125w^2$ and the set of these solutions is denoted by $S(p,m)^U$.

Proposition 3.4. *Let $q = p^{5s}$ with $s \in \mathbb{N}$, p prime of the form $p \equiv 1 \pmod 5$, and put $n = \frac{q-1}{5}$. Then, we have*

$$\mathrm{Spec}(\Gamma(5,q)) = \big\{ [n]^1, [\tfrac{\eta_0^*-1}{5}]^n, [\tfrac{\eta_1^*-1}{5}]^n, [\tfrac{\eta_2^*-1}{5}]^n, [\tfrac{\eta_3^*-1}{5}]^n, [\tfrac{\eta_4^*-1}{5}]^n \big\}$$

where

$$\eta_0^* = -\tfrac{1}{16} p^s (x^3 - 25L) \qquad \text{and} \qquad \eta_i^* = \tfrac{1}{64} p^s (x^3 - 25M)\sigma^i \quad (0 \leq i \leq 3),$$

with σ the non-singular linear transformation of order 4 given by $\sigma(x,w,v,u) = (x,-w,-u,v)$ and

$$\begin{aligned} L =& 2x(v^2+u^2) + 5w(11v^2 - 4vu - 11u^2), \\ M =& 2x^2u + 7xv^2 + 20xvu - 3xu^2 + 125w^3 + 200w^2v \\ &- 150w^2u + 5wv^2 - 20wvu - 105wu^2 - 40v^3 - 60v^2u + 120vu^2 + 20u^3, \end{aligned}$$

for (x,w,v,u) any solution of (3.9) such that $p \nmid x^2 - 125w^2$.

Proof. It follows by a direct application of Theorem 2.1 and Theorem 1 in [12]. □

Relative to the spectrum of $\Gamma(5,p^m)$, the case $p \equiv 4 \pmod 5$, i.e. $p \equiv -1 \pmod 5$, corresponds to the semiprimitive one, and hence it is obtained by taking $k=5$ in Theorem 5.4 ahead, while the cases $p \equiv 2,3 \pmod 5$ remain open in general since the Gaussian periods in these cases are unknown (except in the semiprimitive case), to our best knowledge.

Example 3.5. Here we give the spectrum of $\Gamma(5,11^5)$. By the last Example in Section 5 in [12] we have that $S(11,5)^U = \langle(-396,-100,150,-30)\rangle$, where $\langle(x,w,v,u)\rangle$ denotes the orbit of the solution (x,w,v,u) of (3.9), and that

$$\Psi^*_{5,11^5}(X) = (X+99)(X+649)(X+979)(X-451)(X-1276).$$

In this way, by Proposition 3.4 we have

$$\operatorname{Spec}(\Gamma(5,11^5)) = \{[n]^1, [255]^n, [90]^n, [-20]^n, [-130]^n, [-196]^n\}$$

where $n = \frac{11^5-1}{5} = 32210$. ◊

Remark 3.6 (*The spectrum of* $\Gamma(6,q)$, $\Gamma(8,q)$ *and* $\Gamma(12,q)$)**.** The (reduced) period polynomials

$$\Psi^*_{k,q}(x) = \prod_{i=1}^{k}(x-\eta_i^*) \qquad \text{where} \qquad \eta_i^* = k\eta_i + 1,$$

for $k=6,8,12$ and its factorizations into irreducible polynomials over $\mathbb{Z}$ were obtained by S. Gurak in two papers from 2001 and 2004. He first considered the case $q=p^2$ and gave the factorizations (see Propositions 3.1, 3.2 and 3.3 in [8]). The general case is treated in [9]. The results, which are very technical, are given in Propositions 3.2, 3.3 and 3.5 (their descriptions are out of the scope of this paper). However, since the involved irreducible polynomials are of degree ≤ 4, it is possible in principle to compute all their roots and, hence, to obtain the spectrum of $\Gamma(k,q)$ for $k=6,8,12$.

Remark 3.7. Using Theorem 2.1 one can obtain the spectrum of $\bar{\Gamma}(k,q)$ and $\Gamma^+(k,q)$ for $k=3,4,5$ enhancing Theorems 3.1 and 3.2 and Proposition 3.4. Similarly for the graphs $\Gamma(k,q)$ with $k=6,8,12$ in Remark 3.6.

We close the section with a comment on Waring numbers $g(k,q)$ over a finite field $\mathbb{F}_q$. The Waring number $g(k,q)$ is defined to be the minimal $g \in \mathbb{N}$ (if it exists) such that every element of $\mathbb{F}_q$ is a sum of a number g of k-th powers in $\mathbb{F}_q$. It is well-known that $g(k,q) \leq k$.

Remark 3.8. The Waring number $g(k,q)$ is exactly the diameter of $\Gamma(k,q)$ (see [22]). For a general graph G one has that $\operatorname{diam}(G) \leq t-1$, where t is the number of distinct non-principal eigenvalues of G (see for instance Theorem 3.13 in [5]), with equality if G is a distance regular graph (see §5.2). Hence, we have that

$$g(k,q) = \operatorname{diam}(\Gamma(k,q)) \leq s \leq k,$$

where s is the number of distinct Gaussian periods (see (2.5)). Thus, if $\Gamma(k,q)$ is a distance regular graph then $g(k,q)=s$ (this occurs for instance in the semiprimitive case or in the Hamming case).

4 All integral generalized Paley graphs

In this section, we classify those GP-graphs having integral spectrum, by way of period polynomials. The study of integral graphs is an interesting topic of research on its own, initiated by Harary and Schwenk back in the 70's (see [10]).

In 1981, Myerson proved that the period polynomial $\Psi_{k,q}(x)$ in (3.1) has integral coefficients, that is, $\Psi_{k,q}(x) \in \mathbb{Z}[x]$ (see [18, Theorem 3]). Hence, since $\Psi_{k,q}(x)$ is monic we have that all of the Gaussian periods $\eta_i^{(k,q)}$ are algebraic integers for all $i = 0, \ldots, k-1$. Also, he showed that in general the period polynomial $\Psi_{k,q}(x)$ splits over $\mathbb{Q}$ in N factors of degree $\frac{k}{N}$ (see [18, Theorem 4]). Moreover, he showed that

$$\Psi_{k,q}(x) = \prod_{i=0}^{N-1} \psi_{(k,q)}^{(i)}(x) \qquad \text{with} \qquad \psi_{(k,q)}^{(i)}(x) = \prod_{\ell=0}^{\frac{k}{N}-1} (x - \eta_{i+\ell N}^{(k,q)}) \in \mathbb{Z}[x] \tag{4.1}$$

where $\psi_{(k,q)}^{(i)}(x)$ is irreducible or a power of an irreducible polynomial over $\mathbb{Q}$. We will use these facts in the section.

As we have already mentioned in Remark 2.2, $\Gamma(k,q)$ is integral if and only if $\Gamma^+(k,q)$ is integral (or if $\bar{\Gamma}(k,q)$ is integral). By studying the period polynomial of GP-graphs we can now characterize all integral GP-graphs (and hence all integral GP$^+$-graphs).

Theorem 4.1. *Let $q = p^m$ with p prime and $m \in \mathbb{N}$ and let $k \in \mathbb{N}$ such that $k \mid q-1$. Then, the generalized Paley graph $\Gamma(k,q)$ is integral if and only if k divides $\frac{q-1}{p-1}$; i.e.*

$$\mathrm{Spec}(\Gamma(k,q)) \subset \mathbb{Z} \qquad \Leftrightarrow \qquad k \mid \tfrac{q-1}{p-1} \qquad \Leftrightarrow \qquad \eta_i^{(k,q)} \in \mathbb{Z} \quad (0 \le i \le k-1). \tag{4.2}$$

In particular, all directed GP-graphs are not integral.

Proof. Expression (2.7) gives the spectra of $\Gamma(k,q)$ in terms of the Gaussian periods $\eta_i^{(k,q)}$. By (2.3) and (2.4) we know that $\eta_i^{(N,q)} \in \mathbb{Z}$ where $N = \gcd(\frac{q-1}{p-1}, k)$. Thus, if k satisfies $k \mid \frac{q-1}{p-1}$ then $k = N$ and hence all the Gaussian periods $\eta_i^{(k,q)}$ are integers, by (2.4). This implies that $\mathrm{Spec}(\Gamma(k,q))$ is integral.

Now, assume that $\Gamma(k,q)$ is integral, then $\eta_i^{(k,q)} \in \mathbb{Z}$ for all $i = 0, \ldots, k-1$. Suppose by contradiction that $N < k$. By definition of N, there is an integer L such that $k = LN$, i.e. $L = \frac{k}{N} > 1$. Now, it is known that the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ of the cyclotomic field extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$ permutes all of the elements in the set

$$\{\eta_i^{(k,q)}, \eta_{i+N}^{(k,q)}, \eta_{i+2N}^{(k,q)}, \ldots, \eta_{i+(L-1)N}^{(k,q)}\}$$

(see Lemmas 2 and 5 in [18]). Thus, since $\eta_i^{(k,q)} \in \mathbb{Z}$, we have that $\eta_i^{(k,q)}$ is fixed by all the elements in $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ and hence we have that $\eta_{i+\ell N}^{(k,q)} = \eta_i^{(k,q)}$ for all $\ell = 0, \ldots, L-1$ and so we get

$$\psi_{(k,q)}^{(i)}(x) = (x - \eta_i^{(k,q)})^L$$

for $i = 0, \ldots, N-1$. Then, by (4.1), we obtain that

$$\Psi_{k,q}(x) = p(x)^L \qquad \text{where} \qquad p(x) = \prod_{i=0}^{N-1} (x - \eta_i^{(k,q)}) \in \mathbb{Z}[x]. \tag{4.3}$$

On the other hand, since $-\sum_{i=0}^{k-1} \eta_i^{(k,q)} = 1$ by (2.2), we obtain that the term corresponding to x^{k-1} in $\Psi_{k,q}(x)$ is 1. Finally, if b denotes the term of $p(x)$ corresponding to x^{N-1}, then the equation (4.3) implies that $1 = bL$ which is absurd since $b, L \in \mathbb{Z}$ and $L > 1$. Therefore, we must have that $k = N$, and hence $k \mid \frac{q-1}{p-1}$, as desired.

The remaining assertion is clear. Indeed, $\Gamma(k,q)$ is directed if and only if $q = p^m$ is odd with $k \nmid \frac{q-1}{2}$. Thus, if $\Gamma(k,q)$ were integral then $k \mid \frac{q-1}{p-1}$ and since $\frac{q-1}{p-1} \mid \frac{q-1}{2}$, we have that $k \mid \frac{q-1}{2}$, which is absurd. Hence, all directed GP-graphs are non-integral. $\Box$

In the next example we recap integral GP-graphs $\Gamma(k,q)$ with $k = 1, 2, 3, 4$ and check the arithmetic condition (4.2) in the theorem.

Example 4.2. (i) In Example 2.3 we saw that the graphs $\Gamma(1,q) = K_q$ are integral and the condition $1 \mid \frac{q-1}{p-1}$ is trivial.

(ii) In Example 2.4 we saw that the graphs $\Gamma(2,q)$ are integral if and only if $q \equiv 1 \pmod 4$ (hence $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$ and $m = 2t$), that is when $\Gamma(2,q)$ is the classic Paley graph. It is easy to see that $2 \mid \frac{q-1}{p-1}$ if and only if $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$ and $m = 2t$.

(iii) In Theorem 3.1 we showed that $\Gamma(3,q)$ is integral for $p \equiv 1 \pmod 3$ with $3 \mid m$ and for $p \equiv 2 \pmod 3$ and m even (and not for $p \equiv 1 \pmod 3$ with $3 \nmid m$). It is easy to see that these conditions are equivalent to $3 \mid \frac{q-1}{p-1}$.

(iv) In Theorem 3.2 we showed that $\Gamma(4,q)$ is integral for $p \equiv 1 \pmod 4$ with $m \equiv 0 \pmod 4$ or $p \equiv 3 \pmod 4$ and not for $p \equiv 1 \pmod 4$ with $m \equiv 2 \pmod 4$. One can check that $4 \mid \frac{q-1}{p-1}$ if and only if $p \equiv 1 \pmod 4$ with $m \equiv 0 \pmod 4$ or $p \equiv 3 \pmod 4$. $\Diamond$

Example 4.3. A Hamming graph $H(b,q)$ is a graph with vertex set $V = K^b$ where K is any set of size q (typically $\mathbb{F}_q$ in applications), and where two b-tuples form an edge if and only if they differ in exactly one coordinate. Notice that $H(b,q) = \Box^b K_q$ and hence, Hamming graphs are integral with spectrum given by

$$\mathrm{Spec}(H(b,q)) = \big\{ [\ell q - b]^{\binom{b}{\ell}(q-1)^{b-\ell}} : 0 \le \ell \le b \big\}.$$

Those connected GP-graphs $\Gamma(k,q)$ which are Hamming were classified by Lim and Praeger in [14]. In this case $k = \frac{p^{bm}-1}{b(p^m-1)}$ for integers $b \mid \frac{p^{bm}-1}{p^m-1}$, $q = p^{bm}$, and

$$\Gamma(\tfrac{p^{bm}-1}{b(p^m-1)}, p^{bm}) = H(b, p^m).$$

It is clear that $\frac{p^{bm}-1}{b(p^m-1)} \mid \frac{p^{bm}-1}{p-1}$ and hence Theorem 4.1 implies that $\Gamma(\frac{p^{bm}-1}{b(p^m-1)}, p^{bm})$ is an integral graph. $\Diamond$

When $\mathrm{Spec}(\Gamma(k,q))$ is not integral, the graph has at least one irrational Gaussian period.

Corollary 4.4. *Let $q = p^m$ with p prime and let $k \in \mathbb{N}$ such that $k \mid q-1$. If $k \nmid \frac{q-1}{p-1}$, then there exists at least one $j \in \{0, \ldots, k-1\}$ such that $\eta_j^{(k,q)} \notin \mathbb{Q}$.*

Proof. Since $k \nmid \frac{q-1}{p-1}$ we know that there is some $j \in \{0, \ldots, k-1\}$ such that $\eta_j^{(k,q)} \notin \mathbb{Z}$, by (4.2). Since the Gaussian periods are algebraic integers, if $\eta_j^{(k,q)} \in \mathbb{Q}$ then $\eta_j^{(k,q)} \in \mathbb{Z}$. Thus, there exists some $j \in \{0, \ldots, k-1\}$ such that $\eta_j^{(k,q)} \notin \mathbb{Q}$, as we wanted to see. $\square$

Let $q = p^m$ with p prime, let $k \in \mathbb{N}$, and consider the following condition

$$p \equiv 1 \pmod{k} \quad \text{and} \quad k \mid m \qquad \text{or} \qquad p \not\equiv 1 \pmod{k}. \tag{4.4}$$

The following corollary of Theorem 4.1 characterizes integral GP-graphs $\Gamma(k,q)$ in terms of condition (4.4).

Corollary 4.5. *Let $\Gamma(k,q)$ be a GP-graph with $q = p^m$ and p prime. If $p \equiv 1 \pmod{k}$ then $\mathrm{Spec}(\Gamma(k,q)) \subset \mathbb{Z}$ if and only if $k \mid m$. Furthermore, we have:*

(a) If $\mathrm{Spec}(\Gamma(k,q)) \subset \mathbb{Z}$ then condition (4.4) holds.

(b) If k is prime and condition (4.4) holds then $\mathrm{Spec}(\Gamma(k,q)) \subset \mathbb{Z}$.

In particular, if k is prime then $\mathrm{Spec}(\Gamma(k,q)) \subset \mathbb{Z}$ if and only if condition (4.4) holds.

Proof. By Theorem 4.1, $\mathrm{Spec}(\Gamma(k,q)) \subset \mathbb{Z}$ if and only if $k \mid \frac{q-1}{p-1}$. Notice that $k \mid \frac{q-1}{p-1}$ if and only $k \mid \Psi_m(p)$, where $\Psi_m(p) = p^{m-1} + \cdots + p^2 + p + 1$ Thus, if $p \equiv 1 \pmod{k}$ we have that $\Psi_m(p) \equiv m \pmod{k}$. That is, $k \mid \Psi_m(p)$ if and only if $k \mid m$.
(a) Suppose that $\Gamma(k,q)$ is integral. There are two possibilities: $k \mid p-1$ or not. In the first case, we known that $k \mid m$, as we wanted to show.
(b) By the first part of the statement, it is enough to check the claim for $p \not\equiv 1 \pmod{k}$ since otherwise we know that $\Gamma(k,q)$ is integral. Thus, assume that $p \not\equiv 1 \pmod{k}$. Since k is prime by hypothesis with $k \nmid p-1$ and $k \mid q-1$, then $k \mid \frac{q-1}{p-1}$. Therefore, $\Gamma(k,q)$ is integral as desired. The remaining statement is straightforward. $\square$

Example 4.6. The graphs $\Gamma(5, p^{5t})$ with $t \in \mathbb{N}$ and p prime of the form $5\ell + 1$ (studied in Proposition 3.4) have integral spectrum (notice that this is not clear at all from the expressions of the Gaussian periods in Proposition 3.4). $\lozenge$

We close the section with a result on divisibility of the energy.

Corollary 4.7. *If $\Gamma(k,q)$ is an integral GP-graph then the degree of regularity $n = \frac{q-1}{k}$ divides the energy of $\Gamma(k,q)$ and of $\Gamma^+(k,q)$, that is $n \mid E(\Gamma^*(k,q))$.*

Proof. By (2.7) and (2.8), the energy of $\Gamma^*(k,q)$ is given by

$$E(\Gamma^*(k,q)) = n(1+\mu n + \sum_{i=1}^{s} \mu_i |\eta_i|)$$

where $\mu, \mu_i \in \mathbb{N}_0$ and $\eta_i \in \mathbb{Z}$ for $i = 1, \ldots, s$ by hypothesis, and this implies the result. $\square$

We point out that for regular graphs which are not GP-graphs, the result does not hold in general. For instance, the cubic graph C_6^* which is the 6-cycle with loops, has spectrum $\{[3]^1, [2]^2, [0]^2, [-1]^1\}$ and hence energy $E(C_6^*) = 8$ and $3 \nmid 8$. For an example without loops, consider the cubic graph of six vertices numbered Γ_{51} in [6]. This graph has spectrum $\{[3]^1, [1]^1, [0]^2, [-2]^2\}$ and hence energy $E(\Gamma_{51}) = 8$ and $3 \nmid 8$ (see Table 1 in [25]).

5 Semiprimitive generalized Paley graphs

In this and the next section we focus on a particular family of GP-graphs, the semiprimitive ones. Let $\Gamma(k,q)$ with $q = p^m$ and $k \mid q-1$.

In the study of 2-weight irreducible cyclic codes, the *semiprimitive case* corresponds to -1 being a power of a prime p modulo k (see [27]). If $t \in \mathbb{N}$ is minimal such that

$$p^t \equiv -1 \pmod{k},$$

then $\mathrm{ord}_k(p) = 2t$ when $k > 2$ and so, since $q = p^m \equiv 1 \pmod{k}$, we obtain that

$$m = 2ts$$

for some positive integer s when $k > 2$. Then, we have that semiprimitiveness is equivalent to either $k = 2$ and q odd or else $k > 2$ and

$$k \mid p^t + 1 \qquad \text{for some} \qquad t \mid \tfrac{m}{2}. \tag{5.1}$$

With respect to the GP-graphs in the semiprimitive case, notice that if $k = 2$ with q odd, $\Gamma(2,q)$ is non-directed if $q \equiv 3 \pmod 4$ and directed if $q \equiv 1 \pmod 4$. On the other hand, if $k > 2$ then the graph $\Gamma(k,q)$ is always undirected, since by assumptions, if $m = 2s$ then $k \mid \frac{q-1}{2}$ since

$$\tfrac{1}{2}(q-1) = \tfrac{1}{2}(p^s - 1)(p^s + 1).$$

Furthermore, if $k = p^{\frac{m}{2}} + 1$ then $\Gamma(k,q)$ is not connected (see Proposition 4.6 in [20]). Indeed, one can prove that $\Gamma(p^{\frac{m}{2}+1}, p^m) \cong K_{p^{\frac{m}{2}}} \cup \cdots \cup K_{p^{\frac{m}{2}}}$ ($p^{\frac{m}{2}}$-times).

Definition 5.1. We say that (k,q) with $k = 2$ and $q \equiv 1 \pmod 4$ or $k > 2$ satisfying (5.1) and $k \neq p^{\frac{m}{2}} + 1$ is a *semiprimitive pair* of integers. If (k,q) is a semiprimitive pair of integers, we will refer to $\Gamma(k,q)$ as a *semiprimitive GP-graph*. Hence, every semiprimitive GP-graph $\Gamma(k,q)$ is undirected and connected.

For instance, if $p = 3$ and $m = 4$, to find the semiprimitive pairs of the form $(k, 81)$ we take $k \mid 3^2 + 1 = 2 \cdot 5$ and $k \mid 3^1 + 1 = 4$. Hence $k = 2, 4$ or 5, while $k = 10$ is not allowed since $10 = 3^{\frac{m}{2}} + 1$.

Remark 5.2. (i) Three infinite families of semiprimitive pairs, for p prime and $m = 2t \geq 2$, with $k = 2, 3, 4$ respectively are given by:

(a) the pairs $(2, p^{2t})$ with p odd;

(b) the pairs $(3, p^{2t})$ with $p \equiv 2 \pmod 3$ and $t \geq 1$ (where $t \geq 2$ if $p = 2$);

(c) the pairs $(4, p^{2t})$ with $p \equiv 3 \pmod 4$ and $t \geq 1$ (where $t \geq 2$ if $p = 3$).

The first of the three families of pairs give rise to the classical Paley graphs $\Gamma(2, p^{2t})$.
(ii) Another infinite family of semiprimitive pairs is given by $(p^\ell + 1, p^m)$ with p prime, $m \geq 2$, $\ell \mid m$ and $\frac{m}{\ell}$ even. They give the GP-graphs $\Gamma(q^\ell + 1, q^m)$, with $q = p$, considered in [20] for q a power of p. Notice that the graphs $\Gamma(3, 2^{2t})$ with $t \geq 2$ and $\Gamma(4, 3^{2t})$ with $t \geq 1$ belong to both families given in (i) and (ii). For instance, $\Gamma(3, 16)$, $\Gamma(3, 64)$, and $\Gamma(4, 81)$ are semiprimitive GP-graphs.

Using the previous definition and items (i) and (ii) in the remark, we give a list of the smallest semiprimitive pairs (k, q) with $q = p^m$ for $p = 2, 3, 5, 7$ and $m = 2, 4, 6, 8$.

Table 1: Values of k for small semiprimitive pairs (k, p^m).

	$m = 2$	$m = 4$	$m = 6$	$m = 8$
$p = 2$	–	3	3	5
$p = 3$	–	2, 4, **5**	2, 4, **7**, **14**	2, 4, **5**, **10**, **41**
$p = 5$	2, **3**	2, **3**, 6, **13**	2, **3**, 6, **7**, **9**, **14**, **18**, **21**, **42**, **63**	2, **3**, 6, **13**, 26, **313**
$p = 7$	2, **4**	2, **4**, **5**, 8, **10**, **25**	2, **4**, **5**, 8, **10**, **25**, **43**, 50, **86**, **172**	2, **4**, **5**, 8, **10**, **25**, 50, **1201**

Here we have marked in bold those k which are different from 2 and not of the type $p^\ell + 1$ for some p and ℓ, showing that in general there are much more semiprimitive graphs $\Gamma(k, q)$ than Paley graphs $\Gamma(2, q)$ or GP-graphs of the form $\Gamma(p^\ell + 1, p^m)$.

It is well-known that the Gaussian periods associated to a semiprimitive pair (k, q) are integers (see for instance [7]) and hence $\Gamma(k, q)$ is integral. We now use Theorem 4.1 to obtain the same result in an indirect but elementary way (i.e. without explicitly computing the spectrum of $\Gamma(k, q)$).

Proposition 5.3. *Every semiprimitive GP-graph $\Gamma(k, q)$ is integral.*

Proof. By Theorem 4.1, $\Gamma(k, q)$ is integral if and only if $k \mid \frac{q-1}{p-1}$, where $q = p^m$ for some m. Thus, we will show that if (k, q) is a semiprimitive pair then $k \mid \frac{q-1}{p-1}$. If t is the minimal positive integer such that $p^t \equiv -1 \pmod{k}$, then $2t = \mathrm{ord}_k(p)$ and so, since $q = p^m \equiv 1 \pmod{k}$, we obtain that $m = 2ts$ for some positive integer s.

Notice that we have the factorization

$$q - 1 = p^{2ts} - 1 = (p^t - 1)\Psi_{2s}(p^t)$$

where $\Psi_{2s}(x) = x^{2s-1} + \cdots + x^2 + x + 1$. Since $2s$ is even and $p^t \equiv -1 \pmod{k}$ we obtain that $k \mid \Psi_{2s}(p^t)$. On the other hand, $p-1 \mid p^t - 1$ trivially, and hence we have that $k \mid \frac{q-1}{p-1}$.

We have shown that $k \mid \frac{q-1}{p-1}$ for every semiprimitive pair (k,q) and therefore $\Gamma(k,q)$ is integral, as we wanted to see. $\square$

5.1 The spectrum of semiprimitive GP-graphs $\Gamma(k,q)$

In this subsection we recall the spectrum for arbitrary semiprimitive GP-graphs. In 1999, by using Gauss sums, Brouwer, Wilson and Xiang computed the spectra of a more general family defined in terms of semiprimitive pairs (see Theorem 2 in [3]). Now, for completeness, using Gaussian periods we give the spectrum of the corresponding GP*-graphs $\Gamma(k,q)$ and $\Gamma^+(k,q)$ and of the complements $\bar{\Gamma}(k,q)$.

We will need the following notation. If $q = p^m$, define the sign

$$\sigma = (-1)^{s+1} \tag{5.2}$$

where $s = \frac{m}{2t}$ and t is the least integer j such that $k \mid p^j + 1$ (hence $s \geq 1$).

Theorem 5.4. *Let (k,q) be a semiprimitive pair with $q = p^m$, m even, and put $n = \frac{q-1}{k}$. Then, the spectra of $\Gamma = \Gamma(k,q)$, $\Gamma^+ = \Gamma^+(k,q)$ and $\bar{\Gamma} = \bar{\Gamma}(k,q)$ are integral and respectively given by*

$$\begin{aligned} \mathrm{Spec}(\Gamma) &= \{[n]^1, [\lambda_1]^n, [\lambda_2]^{(k-1)n}\}, \\ \mathrm{Spec}(\bar{\Gamma}) &= \{[(k-1)n]^1, [(k-1)\lambda_2]^n, [-1-\lambda_2]^{(k-1)n}\}, \end{aligned}$$

where

$$\lambda_1 = \frac{\sigma(k-1)p^{\frac{m}{2}} - 1}{k} \qquad \text{and} \qquad \lambda_2 = -\frac{\sigma p^{\frac{m}{2}} + 1}{k} \tag{5.3}$$

with σ as given in (5.2). Furthermore, we have $\mathrm{Spec}(\Gamma^+) = \mathrm{Spec}(\Gamma)$ if q is even and $\mathrm{Spec}(\Gamma^+) = \{[n]^1, [\pm\lambda_1]^{\frac{n}{2}}, [\pm\lambda_2]^{\frac{(k-1)n}{2}}\}$ if q is odd.

Proof. We first compute the spectrum of $\Gamma = \Gamma(k,q)$, which by Theorem 2.1 is given in terms of Gaussian periods. From Lemma 13 in [7] the Gaussian periods $\eta_j^{(k,q)}$, for $j = 0, \ldots, k-1$, are given by:

(a) If p, $\alpha = \frac{p^t+1}{k}$ and s are all odd then

$$\eta_j^{(k,q)} = \begin{cases} \frac{(k-1)\sqrt{q}-1}{k} & \text{if } j = \frac{k}{2}, \\ -\frac{\sqrt{q}+1}{k} & \text{if } j \neq \frac{k}{2}. \end{cases}$$

(b) In any other case we have $\sigma = (-1)^{s+1}$ and

$$\eta_j^{(k,q)} = \begin{cases} \frac{\sigma(k-1)\sqrt{q}-1}{k} & \text{if } j=0, \\ -\frac{\sigma\sqrt{q}+1}{k} & \text{if } j \neq 0. \end{cases}$$

Thus, by Theorem 2.1, the spectrum of $\Gamma(k,q)$ is

$$\mathrm{Spec}(\Gamma(k,q)) = \{[n]^1, [\eta_{k/2}]^n, [\eta_0]^{(k-1)n}\}$$

if p, α, s are odd or $\mathrm{Spec}(\Gamma(k,q)) = \{[n]^1, [\eta_0]^n, [\eta_1]^{(k-1)n}\}$ otherwise.

Suppose we are in case (a), i.e. p, α and s are odd. Then we have

$$\lambda_1 = \eta_{k/2} = \tfrac{(k-1)p^{\frac{m}{2}}+1}{k} \qquad \text{and} \qquad \lambda_2 = \eta_j = \eta_0 = -\tfrac{p^{\frac{m}{2}}+1}{k} \quad (j \neq \tfrac{k}{2}).$$

It is clear that $\lambda_2 \neq n$ and $\lambda_2 \neq \lambda_1$. Also, $n \neq \lambda_1$ since $k \neq p^{\frac{m}{2}}+1$. Thus, all three eigenvalues are different and their corresponding multiplicities are as given in the statement.

In case (b), we have

$$\eta_0 = \tfrac{\sigma(k-1)p^{\frac{m}{2}}+1}{k} \qquad \text{and} \qquad \eta_j = -\tfrac{\sigma p^{\frac{m}{2}}-1}{k} \quad (j \neq 0).$$

Again, one checks that $\eta_0 \neq \eta_j$, $\eta_0 \neq n$ and $\eta_j \neq n$ for every $j \neq \frac{k}{2}$. Thus, the corresponding multiplicities are as stated in the proposition.

Combining cases (a) and (b) we get (5.3). Finally, the spectra of $\bar{\Gamma}(k,q)$ and $\Gamma^+(k,q)$ follow by Theorem 2.1. Just recall that for q even we have that $\Gamma^+ = \Gamma$. $\square$

Notice that for (k,q) semiprimitive with q odd, $\Gamma^+(k,q)$ has almost symmetric spectrum (see Definition 2.13 in [23]) with five different eigenvalues.

Note. Since $\lambda_1, \lambda_2 \in \mathbb{Z}$, we have that $\sigma = \pm 1$ if and only if $k \mid p^{\frac{m}{2}} \pm 1$, respectively.

Remark 5.5. The weight distribution of 2-weight irreducible cyclic codes $\mathcal{C}(k,q)$ in the semiprimitive case is known (see for instance [27]). Using this and the relation of the weight distribution of $\mathcal{C}(k,q)$ with the spectrum of GP-graphs obtained in Theorem 5.4 in [26] one can also recover the spectrum of semiprimitive GP-graphs $\Gamma(k,q)$ as in (2.7) in Theorem 2.1.

Remark 5.6. We have computed the spectrum of the GP-graphs $\Gamma_{q,m}(\ell) = \Gamma(q^\ell + 1, q^m)$ and $\bar{\Gamma}_{q,m}(\ell)$, with $\ell \mid m$ and $\frac{m}{\ell}$ even (see Theorem 3.5 and Proposition 4.3 in [20], see also [21]), by using certain sums associated with the quadratic forms

$$Q_{\gamma,\ell}(x) = \mathrm{Tr}_{p^m/p}(\gamma x^{q^\ell+1})$$

with $\gamma \in \mathbb{F}_{p^m}^*$. By (ii) in Example 5.2, the graph $\Gamma(p^\ell+1, p^m)$, i.e. with $q = p$ prime, is semiprimitive and hence its spectrum is given by Theorem 2.1. Indeed, $\mathrm{Spec}(\Gamma(p^\ell+1,p^m)) = \{[n]^1, [\lambda_1]^n, [\lambda_2]^{p^\ell n}\}$ where

$$n = \tfrac{p^m-1}{p^\ell+1}, \qquad \lambda_1 = \tfrac{\sigma p^{\frac{m}{2}+\ell}-1}{p^\ell+1}, \qquad \lambda_2 = -\tfrac{\sigma p^{\frac{m}{2}}+1}{p^\ell+1},$$

with $\sigma = (-1)^{\frac{m}{2\ell}+1}$. It is reassuring that both computations of the spectrum coincide after using these two different methods. The same happens for the complementary graphs.

We conclude the section showing that for each odd prime power p^{2m} there is only one semiprimitive GP-graph $\Gamma(k_m, p^{2m})$ which is Hamming (or equivalently, there is only one Hamming GP-graph which is semiprimitive).

Proposition 5.7. *Let $\Gamma(k,q)$ be a semiprimitive GP-graph with $q = p^t$ and p an odd prime. Then, $\Gamma(k,q)$ is Hamming if and only if $k = \frac{p^m+1}{2}$ and $t = 2m$. In this case we have*

$$\Gamma(\tfrac{p^m+1}{2}, p^{2m}) = H(2,p^m) = K_{p^m} \square K_{p^m} = L_{q,q}$$

where $L_{q,q}$ is the $q \times q$ lattice (or rook's) graph, with integral spectrum given by

$$Spec(\Gamma(\tfrac{p^m+1}{2}, p^{2m})) = \{[2(p^m-1)]^1, [p^m-2]^{2(p^m-1)}, [-2]^{(p^m-1)^2}\}.$$

Proof. In Example 4.3 we recall that Hamming GP-graphs (classified in [14]) are of the form

$$\Gamma(\tfrac{p^{bm}-1}{b(p^m-1)}, p^{bm}) = H(b,p^m) = \square^b K_{p^m}$$

with spectrum

$$\{[\ell p^m - b]^{\binom{b}{\ell}(p^m-1)^{b-\ell}}\}_{\ell=0}^b. \tag{5.4}$$

Since semiprimitive graphs have exactly three eigenvalues, we must necessarily have that $b = 2$ and we check that in this case

$$\Gamma(\tfrac{p^{2m}-1}{2(p^m-1)}, p^{2m}) = \Gamma(\tfrac{p^m+1}{2}, p^{2m})$$

is semiprimitive since $\frac{p^m+1}{2} \mid p^m+1$. It is well-known that $H(2,q)$ is the lattice graph $L_{q,q}$. The spectrum follows by taking $b = 2$ in (or one can also use Theorem 5.4). $\square$

5.2 Semiprimitive GP-graphs are strongly regular

Let Γ be a regular graph that is neither complete nor empty. Then Γ is said to be *strongly regular* with parameters $srg(v,r,e,d)$ if it is r-regular with v vertices, every pair of adjacent vertices has e common neighbours, and every pair of distinct non-adjacent vertices has d common neighbours. These parameters are tied by the relation

$$(v-r-1)d = r(r-e-1). \tag{5.5}$$

For instance, for $q \equiv 1 \pmod 4$, the classic Paley graph $P(q)$ is a strongly regular graph with parameters $srg(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$. If Γ is strongly regular with parameters $srg(v,r,e,d)$, then its complement $\bar{\Gamma}$ is also strongly regular with parameters $srg(v,\bar{r},\bar{e},\bar{d})$, where

$$\bar{r} = v-r-1, \qquad \bar{e} = v-2-2r+d \qquad \text{and} \qquad \bar{d} = v-2r+e. \tag{5.6}$$

Let $\Gamma = srg(v, r, e, d)$. If $2r - (v-1)(e-d) \neq 0$ the graph have integral different eigenvalues. On the other hand, if

$$2r - (v-1)(e-d) = 0$$

the graph is said to be a *conference graph* because of their connection with symmetric conference matrices. A conference graph has parameters

$$srg(v, \tfrac{v-1}{2}, \tfrac{v-5}{4}, \tfrac{v-1}{4}).$$

Hence, Paley graphs are conference graphs.

A strongly regular graph $srg(v, n, e, d)$ with different eigenvalues n, f, g is a *pseudo-Latin square graph* if $n = -g(f-g-1)$, where $n > f > 0 > g$. Equivalently, it is denoted $PL_\delta(w)$ and has parameters

$$PL_\delta(w) = srg(w^2, \delta(w-1), \delta^2 - 3\delta + w, \delta(\delta-1)), \tag{5.7}$$

where $w = f - g$ and $\delta = -g$. A graph with the parameters as above, changing δ and w by $-\delta$ and $-w$ is called a *negative Latin square graph*. It is denoted by $NL_\delta(w)$ and has the parameters

$$NL_\delta(w) = srg(w^2, \delta(w+1), \delta^2 + 3\delta - w, \delta(\delta+1)). \tag{5.8}$$

See for instance Chapter 8 in [4] for the definitions of pseudo-Latin and negative Latin square graphs.

A regular graph is called *distance regular* if for any two vertices v and w, the number of vertices at distance j from v and at distance k from w depends only upon j, k, and the distance $d(v, w)$ between v and w. A connected strongly regular graph Γ, being a distance regular graph of diameter $\delta = 2$, have intersection array of the form

$$\mathcal{A}(\Gamma) = \{b_0, b_1, b_2; c_0, c_1, c_2\}.$$

For every $i = 0, 1, 2$ and every pair of vertices x, y at distance i, the intersection numbers are defined by

$$b_i = \#\{z \in N(y) : d(x,z) = i+1\} \qquad \text{and} \qquad c_i = \#\{z \in N(y) : d(x,z) = i-1\},$$

where $N(y)$ denotes the set of neighbours of y. Since we trivially have $b_2 = c_0 = 0$, we will simply write $\mathcal{A}(\Gamma) = \{b_0, b_1; c_1, c_2\}$, as usual. See [2] for an introduction and general examples about strongly regular graphs.

We now give some structural properties of the graphs $\Gamma(k, q)$ throughout the spectrum.

Theorem 5.8. *Let (k, q) be a semiprimitive pair with $q = p^m$, $m = 2ts$ where t is the least integer satisfying $k \mid p^t + 1$ and put $n = \frac{q-1}{k}$. Then we have:*

(*a*) *$\Gamma(k, q)$ and $\bar{\Gamma}(k, q)$ are primitive, non-bipartite, integral, strongly regular graphs with corresponding parameters $srg(q, n, e, d)$ and $srg(q, (k-1)n, e', d')$ given by*

$$e = d + (\sigma p^{\frac{m}{2}} + 2\lambda_2), \quad d = n + (p^{\frac{m}{2}} + \lambda_2)\lambda_2), \quad e' = q - 2 - 2n + d, \quad d' = q - 2n + e.$$

(*b*) *$\Gamma(k,q)$ and $\bar{\Gamma}(k,q)$ are distance regular graphs of diameter 2 with intersection arrays*

$$\mathcal{A} = \{n, n-e-1; 1, d\} \qquad \textit{and} \qquad \bar{\mathcal{A}} = \{(k-1)q, n-d; 1, q-2n+e\}.$$

(*c*) *If s is odd then $\Gamma(k,q)$ and $\bar{\Gamma}(k,q)$ are pseudo-Latin square graphs with parameters*

$$PL_\delta(w) = srg(w^2, \delta(w-1), \delta^2 - 3\delta + w, \delta(\delta-1)),$$

where $w = f-g$, $\delta = -g$ and $f > 0 > g$ are the non-trivial eigenvalues of $\Gamma(k,q)$ or $\bar{\Gamma}(k,q)$.

Proof. We will use the spectral information from Theorem 5.4. We prove first the results for $\Gamma(k,q)$.

(*a*) Since the multiplicity of the degree of regularity n is 1, the graph is connected. Also, one can check that $-n$ is not an eigenvalue of $\Gamma(k,q)$ and hence the graph is non-bipartite. Now, since $k \mid p^t + 1$ then $k \mid p^{ts} + 1$ if s is odd and $k \mid p^{ts} - 1$ if s is even, hence $\beta = \frac{p^{\frac{m}{2}} + \sigma}{k}$ is an integer. Thus, since $\lambda_1 = \sigma(p^{ts} - \beta)$ and $\lambda_2 = -\sigma\beta$, by (5.3), the eigenvalues are all integers (we also know this from Theorem 2.1).

Finally, since the graph is connected, n-regular with q-vertices and has exactly three eigenvalues, it is a strongly regular graph with parameters $srg(q,n,e,d)$. We now compute e and d. It is known that the non-trivial eigenvalues of an strongly regular graph are of the form

$$\lambda^{\pm} = \tfrac{1}{2}\{(e-d) \pm \Delta\} \qquad \text{where} \qquad \Delta = \sqrt{(e-d)^2 + 4(n-d)}.$$

Thus, $d = n + \lambda^+\lambda^-$ and $e = d + \lambda^+ + \lambda^-$. From this and (5.3) the result follows.

(*b*) We know that $\Gamma = srg(q,n,e,d)$ is primitive for (k,q) a semiprimitive pair. Since Γ is connected with diameter $\delta = 2$, its intersection array is $\{n, n-e-1; 1, d\}$. In fact, it is clear that $b_0 = n$ and $c_1 = 1$. Let x, y be vertices of Γ. Thus, if $d(x,y) = 1$, then

$$b_1 = \#(N(y) \smallsetminus \{x\}) - \#N(x) = n - 1 - e.$$

If $d(x,y) = 2$, then $c_2 = \#(N(x) \cap N(y))$. Since $\bar{\Gamma}$ is also connected with diameter 2, its intersection array is $\{\bar{n}, \bar{n} - \bar{e} - 1; 1, \bar{d}\}$. Now, since

$$q - 2n + e = (q - n + 1) - (n - e + 1),$$

by using (5.5) and (5.6) we get the desired result.

(*c*) Note that the regularity degree of $\Gamma = \Gamma(k,q)$ equals the multiplicity of a non-trivial eigenvalue by Theorem 5.4. Thus, Proposition 8.14 in [4], we have that Γ is of pseudo-Latin square type graph (PL), of negative Latin square type (NL) or is a conference graph. By definition, a conference graph satisfy

$$2n + (q-1)(e-d) = 0.$$

It is easy to check that this condition holds for $\Gamma(k,q)$ if and only if $\Gamma(1,4)=K_4$, and hence Γ is not a conference graph. Now, put $w=f-g$, where f,g are the non-trivial eigenvalues with $f>0>g$ and $\delta=-g$. Then, Γ is a pseudo-Latin square graph with parameters as in (5.7) or a negative Latin square graph with parameters as in (5.8). It is clear that

$$n=\delta(w-1)$$

if and only if s is odd and that for s even $n\neq\delta(w+1)$. For $\bar{\Gamma}$ one proceeds similarly. Hence the only possibility for Γ and $\bar{\Gamma}$ is to be pseudo-Latin square graphs.

Now, it is easy to see that $\bar{\Gamma}(k,q)$ is also a primitive non-bipartite integral strongly regular graph with parameters and intersection array as stated. The proof that $\bar{\Gamma}(k,q)$ is a pseudo-Latin square if s is odd is analogous to the previous one for $\Gamma(k,q)$ and we omit the details. Finally, since $\Gamma(k,q)$ is a pseudo-Latin square graph $PL_\delta(w)$ then $\bar{\Gamma}(k,q)$ is a pseudo-Latin square graph $PL_{\delta'}(w)$ with $\delta'=u+1-\delta$ (see [4]). □

Remark 5.9. Notice that if we take $h=\min\{|f|,|g|\}$, then for s even (in the previous notations), $\Gamma(k,q)$ satisfy the same parameters as in (5.8) with δ replaced by h, that is $\Gamma(k,q)$ is a strongly regular graph with parameters, in terms of the eigenvalues, given by

$$\widetilde{NL}=srg(w^2,h(w+1),h^2+3h-w,h(h+1)).$$

Example 5.10. From Theorem 5.4 and Proposition 5.8 we obtain Table 2 below. Here $s=\frac{m}{2t}$ where t is the least integer such that $k\mid p^t+1$.

We have marked in bold those graphs $\Gamma(k,p^m)$ with $k\neq p^\ell+1$ for some $\ell\mid\frac{m}{2}$. We point out that, for instance, the graphs with $q=7^4$ do not appear in the Brouwer's lists ([1]) of strongly regular graphs. ♢

6 Ramanujan GP-graphs

If Γ is an n-regular graph, then n is the greatest eigenvalue of Γ. Recall that a connected n-regular undirected graph is Ramanujan if

$$\lambda(\Gamma)\leq 2\sqrt{n-1}, \tag{6.1}$$

where $\lambda(\Gamma)$ is the maximum absolute value of the non-principal eigenvalues of Γ

$$\lambda(\Gamma)=\max_{\lambda\in\mathrm{Spec}(\Gamma)}\{|\lambda|:|\lambda|\neq n\}. \tag{6.2}$$

Here we are interested in Ramanujan generalized Paley graphs: we will first classify all semiprimitive GP-graphs which are Ramanujan and then show that all GP-graphs $\Gamma(k,q)$ with $1\leq k\leq 4$ are indeed Ramanujan.

Table 2: Smallest semiprimitive graphs: srg parameters and spectra

graph	srg parameters	spectrum	t	s	pseudo-latin square
$\Gamma(3,2^4)$	$(16,5,0,2)$	$\{[5]^1,[1]^{10},[-3]^5\}$	1	2	no
$\bar{\Gamma}(3,2^4)$	$(16,10,6,6)$	$\{[10]^1,[2]^5,[-2]^{10}\}$	1	2	no
$\Gamma(3,2^6)$	$(64,21,8,6)$	$\{[21]^1,[5]^{21},[-3]^{42}\}$	1	3	$PL_3(8)$
$\bar{\Gamma}(3,2^6)$	$(64,42,26,30)$	$\{[42]^1,[2]^{42},[-6]^{21}\}$	1	3	$PL_6(8)$
$\Gamma(\mathbf{3},\mathbf{5^2})$	$(25,8,3,2)$	$\{[8]^1,[3]^8,[-2]^{16}\}$	1	1	$PL_2(5)$
$\bar{\Gamma}(\mathbf{3},\mathbf{5^2})$	$(25,16,9,12)$	$\{[16]^1,[1]^{16},[-4]^8\}$	1	1	$PL_4(5)$
$\Gamma(\mathbf{3},\mathbf{5^4})$	$(625,208,63,72)$	$\{[208]^1,[8]^{416},[-17]^{208}\}$	1	2	no
$\bar{\Gamma}(\mathbf{3},\mathbf{5^4})$	$(625,416,279,272)$	$\{[416]^1,[16]^{208},[-9]^{416}\}$	1	2	no
$\Gamma(4,3^4)$	$(81,20,1,6)$	$\{[20]^1,[2]^{60},[-7]^{20},\}$	1	2	no
$\bar{\Gamma}(4,3^4)$	$(81,60,45,42)$	$\{[60]^1,[6]^{20},[-3]^{60},\}$	1	2	no
$\Gamma(4,3^6)$	$(729,182,55,42)$	$\{[182]^1,[20]^{182},[-7]^{546}\}$	1	3	$PL_7(27)$
$\bar{\Gamma}(4,3^6)$	$(729,546,405,420)$	$\{[546]^1,[6]^{546},[-21]^{182}\}$	1	3	$PL_{21}(27)$
$\Gamma(\mathbf{4},\mathbf{7^2})$	$(49,12,5,2)$	$\{[12]^1,[5]^{12},[-2]^{36}\}$	1	1	$PL_2(7)$
$\bar{\Gamma}(\mathbf{4},\mathbf{7^2})$	$(49,36,25,30)$	$\{[36]^1,[1]^{36},[-6]^{12}\}$	1	1	$PL_6(7)$
$\Gamma(\mathbf{4},\mathbf{7^4})$	$(2401,600,131,156)$	$\{[600]^1,[12]^{1800},[-37]^{600}\}$	1	2	no
$\bar{\Gamma}(\mathbf{4},\mathbf{7^4})$	$(2401,1800,1332,1355)$	$\{[1800]^1,[36]^{600},[-13]^{1800}\}$	1	2	no
$\Gamma(\mathbf{5},\mathbf{3^4})$	$(81,16,7,2)$	$\{[16]^1,[7]^{16},[-2]^{64}\}$	2	1	$PL_2(9)$
$\bar{\Gamma}(\mathbf{5},\mathbf{3^4})$	$(81,64,49,56)$	$\{[64]^1,[1]^{64},[-8]^{16}\}$	2	1	$PL_8(9)$
$\Gamma(\mathbf{5},\mathbf{7^4})$	$(2401,480,119,90)$	$\{[480]^1,[39]^{480},[-10]^{1920}\}$	2	1	$PL_{10}(49)$
$\bar{\Gamma}(\mathbf{5},\mathbf{7^4})$	$(2401,1920,1560,1529)$	$\{[1920]^1,[9]^{1920},[-40]^{480}\}$	2	1	$PL_{40}(49)$

6.1 All Ramanujan semiprimitive GP-graphs

We recall that semiprimitive graphs are integral and undirected. We now give a complete characterization of the semiprimitive generalized Paley graphs which are Ramanujan. In particular, we will show that if a semiprimitive GP-graph $\Gamma(k,q)$ is Ramanujan then $k \in \{2,3,4,5\}$.

Theorem 6.1. *Let $q = p^m$ with p prime and let (k,q) be a semiprimitive pair. Then, the graph $\Gamma = \Gamma(k,p^m)$ is Ramanujan if and only if $\Gamma^+ = \Gamma^+(k,p^m)$ is Ramanujan and this happens if and only if*

(*a*) *Γ is the classic Paley graph $\Gamma(2,q)$, with $q \equiv 1 \pmod 4$,*

or m is even and k,p,m are as in one of the following cases:

(*b*) *$k=3$, $p=2$ and $m \geq 4$.*

(*c*) *$k=3$, $p \neq 2$ with $p \equiv 2 \pmod 3$ and $m \geq 2$.*

(*d*) *$k=4$, $p=3$ and $m \geq 4$.*

(*e*) *$k=4$, $p \neq 3$ with $p \equiv 3 \pmod 4$ and $m \geq 2$.*

(*f*) *$k=5$, $p=2$ and $m \geq 8$ with $4 \mid m$.*

(*g*) $k=5$, $p \neq 2$ *with* $p \equiv 2,3 \pmod 5$ *and* $m \geq 4$ *with* $4 \mid m$.

(*h*) $k=5$, $p \equiv 4 \pmod 5$ *and* $m \geq 2$ *even.*

Moreover, $\bar{\Gamma}(k,q)$ *is Ramanujan for every semiprimitive pair* (k,q).

Proof. We begin by noticing that, by Theorem 5.4, the graphs $\Gamma(k,q)$, $\bar{\Gamma}(k,q)$ and $\Gamma^+(k,q)$ are connected for any semiprimitive pair (k,q), since the multiplicity of the principal eigenvalue is one. Also, that Γ is Ramanujan if and only if Γ^+ is Ramanujan follows directly from the fact that

$$\lambda(\Gamma^+(k,q)) = \lambda(\Gamma(k,q))$$

by Theorem 2.1, and hence (6.1) holds for both or for none of the graphs.

Now, note that $k=1$ is excluded since $(1,q)$ is not a semiprimitive pair and that $k=2$ corresponds to the classic Paley graph $\Gamma(2,q)$, with $q \equiv 1 \pmod 4$, which is well-known to be Ramanujan (hence (*a*)). So it is enough to consider semiprimitive pairs (k,p^m) with $k>2$.

We divide the proof of the characterization of semiprimitive Ramanujan GP-graphs into three steps: in steps 1 and 2 we prove the statement for the graphs $\Gamma(k,q)$, and in step 3 we prove it for the complements $\bar{\Gamma}(k,q)$.

Step 1. Here we prove that if Γ is Ramanujan with (k,p^m) a semiprimitive pair with $k \neq 2$, then $3 \leq k \leq 5$.

Note that for $k \geq 3$ we have $\lambda(\Gamma) = |\lambda_1|$ (see (5.3) in Theorem 5.4). Since Γ is Ramanujan and undirected, (6.1) reads

$$\tfrac{1}{k}|\sigma(k-1)p^{\frac{m}{2}} - 1| \leq 2\sqrt{\tfrac{p^m-(k+1)}{k}}.$$

This inequality is equivalent to $(k-1)^2p^m - 2\sigma(k-1)p^{\frac{m}{2}} + 1 \leq 4k(p^m-(k+1))$ which holds if and only if

$$4k(k+1)+1 \leq p^m(4k-(k-1)^2) + 2(k-1)\sigma p^{\frac{m}{2}}. \tag{6.3}$$

Assume first that $\sigma=-1$. Then, (6.3) takes the form

$$2(k-1)p^{\frac{m}{2}} + 4k(k+1)+1 \leq p^m(4k-(k-1)^2).$$

Since the left hand side of this inequality is positive, we have that $4k-(k-1)^2>0$, and this can only happen if $k \leq 5$.

Now, let $\sigma=1$. In this case, inequality (6.3) is equivalent to

$$0 \leq (4k-(k-1)^2)p^m + 2(k-1)p^{\frac{m}{2}} - (2k+1)^2. \tag{6.4}$$

Suppose $k>5$ and consider the quadratic polynomial

$$P_k(x) = (4k-(k-1)^2)x^2 + 2(k-1)x - (2k+1)^2.$$

Hence, $P_k(x)$ has negative leading coefficient and its discriminant is given by

$$\Delta(k) = 4\big((k-1)^2 - ((k-3)^2 - 8)(2k+1)^2\big).$$

Since $(2k+1)^2 > (k-1)^2$, the sign of $\Delta(k)$ depends on $(k-3)^2 - 8$. Since $k > 5$, we have that $(k-3)^2 - 8 > 0$ and thus $\Delta(k) < 0$. So, the quadratic polynomial P_k has no real roots and since $P_k(0) = -(2k+1)^2 < 0$, we obtain that $P_k(x) < 0$ for all $x \in \mathbb{R}$, in particular $P_k(p^{\frac{m}{2}}) < 0$ for all p and m, contradicting (6.4). Therefore, if Γ is Ramanujan then $k \le 5$, as we wanted to show.

Step 2. We now show that the pair (k, q) semiprimitive with $k \le 5$ can only happen as stated in the theorem; and, in these cases, $\Gamma(k, q)$ is Ramanujan.

As mentioned at the beginning, the case $k = 1$ is excluded and $k = 2$ corresponds to the classic Paley graph, which is Ramanujan. If $k = 3$, then necessarily $p \equiv 2 \pmod 3$ and m is even, for if not the pair (k, p^m) is not semiprimitive. In this case, (6.3) is given by $49 \le 8p^m + 4\sigma p^{\frac{m}{2}}$. The worst possibility is when $\sigma = -1$, and in this case the previous inequality reads

$$12 + \tfrac{1}{4} \le p^{\frac{m}{2}}(2p^{\frac{m}{2}} - 1).$$

This clearly holds if and only if p is odd and $m \ge 2$ or $p = 2$ and $m \ge 4$, and thus $\Gamma(3, p^m)$ is Ramanujan in these cases. This proves (b) and (c).

If $k = 4$, then we must have $p \equiv 3 \pmod 4$ and m is even, for if not the pair (k, p^m) is not semiprimitive. In this case, (6.3) is given by $81 \le 7p^m + 6\sigma p^{\frac{m}{2}}$. As before, the worst case is when $\sigma = -1$, and thus the inequality is equivalent to

$$11 + \tfrac{4}{7} \le p^{\frac{m}{2}}(p^{\frac{m}{2}} - \tfrac{6}{7}).$$

This holds if and only if $p > 3$ and $m \ge 2$ or $p = 3$ and $m \ge 4$ and hence $\Gamma(4, p^m)$ is Ramanujan in these cases, thus showing (d) and (e).

In the last case, if $k = 5$, then $(5, p^m)$ is semiprimitive if and only if $p \equiv 2, 3 \pmod 5$ and $4 \mid m$ or else $p \equiv 4 \pmod 5$ and $m \ge 2$ even. On the other hand, in this case (6.3) is given by $121 \le 4p^m + 8\sigma p^{\frac{m}{2}}$, which is equivalent to

$$30 + \tfrac{1}{4} \le p^{\frac{m}{2}}(p^{\frac{m}{2}} + 2\sigma).$$

If $p = 2$, then necessarily $m \ge 8$ since $4 \mid m$ and $m = 4$ does not satisfy the above inequality. Clearly, the inequality holds for $p \equiv 2, 3 \pmod 5$ with $p \ne 2$ and $4 \mid m$. Finally, notice that the right hand side of the inequality increases when p increases. The first prime p with $p \equiv 4 \pmod 5$ is $p = 19$ that clearly satisfies the inequality for $m \ge 2$ even, so we obtain that the inequality holds for all of primes $p \equiv 4 \pmod 5$ with $m \ge 2$ even. In this way we have shown that $\Gamma(5, p^m)$ is Ramanujan in all the cases in the statement, proving items (f)–(h).

Step 3. Now, we consider the complementary graphs $\bar{\Gamma} = \bar{\Gamma}(k, q)$. We have that

$$\lambda(\bar{\Gamma}) = |\bar{\lambda}_1| = (k-1)\frac{p^{\frac{m}{2}} + \sigma}{k}$$

and the regularity degree of $\bar{\Gamma}$ is $n(k-1)$. Notice that we can assume that $k > 2$, since $k = 2$ correspond to the classic Paley graph which is self-complementary, and hence Ramanujan. Also, without loss of generality we can assume that $\sigma = 1$. Inequality (6.1) becomes

$$(k-1)\tfrac{p^{\frac{m}{2}}+1}{k} \le 2\sqrt{\tfrac{(p^m-1)(k-1)-k}{k}} \tag{6.5}$$

which is equivalent to $(k-1)^2p^m + 2(k-1)^2p^{\frac{m}{2}} + (k-1)^2 \le 4k(p^m(k-1) - (2k-1))$ and therefore we have

$$2(k-1)^2p^{\frac{m}{2}} + (k-1)^2 + 4k(2k-1) \le p^m(4k(k-1) - (k-1)^2).$$

Notice that $(k-1)^2+4k(2k-1) = (3k-1)^2$ and $4k(k-1)-(k-1)^2 = (k-1)(3k+1)$. Let us consider the quadratic polynomial

$$Q_k(x) = x^2 - b_kx - c_k \qquad \text{where} \qquad b_k = \tfrac{2(k-1)}{(3k+1)} \quad \text{and} \quad c_k = \tfrac{(3k-1)^2}{(k-1)(3k+1)}.$$

Hence, $\bar{\Gamma}(k, q)$ is Ramanujan if and only (6.5) holds, that is if and only if $Q_k(p^{\frac{m}{2}}) > 0$.

Clearly $b_k < 1$ and $4c_k < 15$, this implies that the greatest real root r of Q_k satisfies

$$r = \tfrac{b_k}{2} + \tfrac{1}{2}\sqrt{b_k^2 + 4c_k} < \tfrac{1}{2} + 2 < 3.$$

Since (k, p^m) is a semiprimitive pair and $k > 2$, we have that $p^{\frac{m}{2}} \ge 3$. This implies that $Q_k(p^{\frac{m}{2}}) > 0$ since Q_k has a positive leading coefficient. Therefore $\bar{\Gamma}(k, p^m)$ is Ramanujan for all semiprimitive pair (k, p^m) with $\sigma = 1$. The case $\sigma = -1$ can be proved analogously. □

The previous result gives the following eight infinite families of Ramanujan semiprimitive GP-graphs:

(a) $\{\Gamma(2, q)\}$ with $q \equiv 1 \pmod 4$, i.e. the classic Paley graphs,

(b) $\{\Gamma(3, 4^t)\}_{t\ge2}$,

(c) $\{\Gamma(3, p^{2t})\}_{t\ge1}$ with $p \equiv 2 \pmod 3$ and $p \ne 2$,

(d) $\{\Gamma(4, 9^t)\}_{t\ge2}$,

(e) $\{\Gamma(4, p^{2t})\}_{t\ge1}$ with $p \equiv 3 \pmod 4$ and $p \ne 3$,

(f) $\{\Gamma(5, 16^t)\}_{t\ge2}$,

(g) $\{\Gamma(5, p^{4t})\}_{t\ge1}$, with $p \equiv 2, 3 \pmod 5$ and $p \ne 2$,

(h) $\{\Gamma(5, p^{2t})\}_{t\ge1}$ with $p \equiv 4 \pmod 5$.

Note that five of them are valid for an infinite number of primes. The smallest graphs in each family are:

$$\Gamma(2,5),\ \Gamma(3,16),\ \Gamma(3,49),\ \Gamma(4,81),\ \Gamma(4,49),\ \Gamma(5,256),\ \Gamma(5,81),\ \text{and}\ \Gamma(5,361),$$

respectively. Also, notice that all the graphs in Table 2 are Ramanujan, corresponding to the families (c), (e) and (g).

Remark 6.2. (i) The Ramanujan GP-graphs $\Gamma(k,q)$ with $k=p^\ell+1$ are characterized in Theorem 8.1 in [20]. There, we proved that

$$\Gamma_{q,m}(\ell)=\Gamma(p^\ell+1,p^m),$$

with $\ell \mid m$ such that m_ℓ even and $\ell \neq \frac{m}{2}$, is Ramanujan if and only if $q=2,3,4$ with $\ell=1$ and $m\geq 4$ even. This says that $\Gamma(p^\ell+1,p^m)$ is Ramanujan only in the cases (b), (d) and (f), giving the infinite families

$$\{\Gamma(3,4^t)\}_{t\geq 2}, \qquad \{\Gamma(4,9^t)\}_{t\geq 2} \qquad \text{and} \qquad \{\Gamma(5,16^t)\}_{t\geq 2}$$

of Ramanujan graphs. The first two families coincide with those in (b) and (d), while the third one gives just half the graphs in (f), precisely those with t even in (f). Thus, the last proposition extends this characterization of Ramanujan GP-graphs $\Gamma(q^\ell+1,q^m)$ to all semiprimitive pairs (k,p^m), that is in the case $q=p$.

(ii) When $p=2$, the last proposition gives nothing new, since the possible values of $k\in\{2,3,4,5\}$ such that $(k,2^m)$ is a semiprimitive pair reduces to $k=3,5$, which corresponds to the cases $p=2$ with $\ell=1,2$ in (i) above.

Example 6.3. From Theorem 6.1, the following GP-graphs are Ramanujan:

$p=2$	$\Gamma(3,16)$, $\Gamma(3,64)$, $\Gamma(3,256)$, $\Gamma(5,256)$,
$p=3$	$\Gamma(2,81)$, $\Gamma(4,81)$, $\Gamma(\mathbf{5},\mathbf{81})$, $\Gamma(2,729)$, $\Gamma(4,729)$, $\Gamma(2,6.561)$, $\Gamma(4,6.561)$, $\Gamma(\mathbf{5},\mathbf{6.561})$
$p=5$	$\Gamma(2,25)$, $\Gamma(\mathbf{3},\mathbf{25})$, $\Gamma(2,625)$, $\Gamma(\mathbf{3},\mathbf{625})$, $\Gamma(2,15.625)$, $\Gamma(\mathbf{3},\mathbf{15.625})$, $\Gamma(2,390.625)$, $\Gamma(\mathbf{3},\mathbf{390.625})$
$p=7$	$\Gamma(\mathbf{4},\mathbf{49})$, $\Gamma(\mathbf{4},\mathbf{2.401})$, $\Gamma(\mathbf{5},\mathbf{2.401})$, $\Gamma(\mathbf{4},\mathbf{117.649})$, $\Gamma(\mathbf{4},\mathbf{5.764.801})$, $\Gamma(\mathbf{5},\mathbf{5.764.801})$

where $6.561=3^8$, $15.625=5^6$, $390.625=5^8$, $7^4=2.401$, $7^6=117.649$ and $7^8=5.764.801$. We have marked in bold those graphs $\Gamma(k,p^m)$ with $k>2$ and $k\neq p^\ell+1$ for some $\ell \mid \frac{m}{2}$. $\lozenge$

6.2 Ramanujan graphs $\Gamma(k,q)$ with $1\leq k\leq 4$

It is well-known that the complete graphs K_n and the classic Paley graphs $P(q)$ with $q\equiv 1 \pmod 4$ are Ramanujan (it is immediate to check it from (6.1) and Examples 2.3 and 2.4).

In general, $\Gamma(k,q)$ can be directed. There are (to the authors knowledge) two notions of Ramanujan n-regular digraphs. We recall that a directed graph is n-regular if its in-degree and out-degree are both equal to n. A connected n-regular undirected graph is *Ramanujan* if it satisfies (6.1), that is

$$\lambda(\Gamma) \leq 2\sqrt{n-1},$$

where $\lambda(\Gamma)$ is the maximum absolute value of the non-principal eigenvalues of Γ. An n-regular connected directed graph Γ is Ramanujan if it satisfies (6.1) and also its adjacency matrix can be diagonalized by a unitary matrix, see for instance [13]. A more recent definition due to Lubotzky and Parzanchevski (see [16], [19]) is that an n-regular connected digraph Γ is Ramanujan if

$$\lambda(\Gamma) \leq \sqrt{n}.$$

One can check from the spectrum given in (2.10) that the directed Paley graphs $\vec{P}(q)$, i.e. those $\Gamma(2,q)$ with $q \equiv 3 \pmod 4$, are Ramanujan under the two notions (in the second sense, it is mentioned in §13.3.3 in [19] for $q=p$ prime). In this way we have that the graphs $\Gamma(1,q)$ and $\Gamma(2,q)$ are Ramanujan for any q.

We will now study which GP-graphs $\Gamma(3,q)$ and $\Gamma(4,q)$ are Ramanujan. In Theorem 6.1 we have found those semiprimitive GP-graphs $\Gamma(3,q)$ and $\Gamma(4,q)$ which are Ramanujan (they are the graphs given in (b)–(e) in the previous list). For this reason, we now assume that $\Gamma(k,q)$ is non-semiprimitive for $k=3,4$, where $q=p^m$, that is to say $p \equiv 1 \pmod k$.

To study the Ramanujanicity of the graphs $\Gamma(3,q)$ we will need a lemma. So, we first fix some notations. Denote by $\lambda_0, \lambda_1, \lambda_2$ the non-principal eigenvalues of $\Gamma(3,q)$, that is (see (a) in Theorem 3.1)

$$\lambda_0 = \tfrac{a\sqrt[3]{q}-1}{3}, \qquad \lambda_1 = \tfrac{-\frac{1}{2}(a+9b)\sqrt[3]{q}-1}{3} \qquad \text{and} \qquad \lambda_2 = \tfrac{-\frac{1}{2}(a-9b)\sqrt[3]{q}-1}{3}, \tag{6.6}$$

where a,b are integers uniquely determined by the conditions (3.2). Notice that $\lambda(\Gamma(3,q))$ can be realized by any of the three non-principal eigenvalues of $\Gamma(3,q)$.

Lemma 6.4. *Let $q=p^{3t}$ for some p prime such that $p \equiv 1 \pmod 3$ and let λ_1 and λ_2 be as in (6.6). Then, we have:*

(a) If $|\lambda_1| > |\lambda_2|$, then a and b have the same sign.

(b) If $|\lambda_2| > |\lambda_1|$, then a and b have different signs.

Proof. (a) In this case, $|\lambda_1| > |\lambda_2|$ is equivalent to $\frac{1}{2}(a+9b)p^t+1 > \frac{1}{2}(a-9b)p^t+1$. This implies that

$$\big((a+9b)p^t+2\big)^2 > \big((a-9b)p^t+2\big)^2,$$

from which after some computations we obtain that

$$b(2+ap^t) > 0.$$

Hence, b and $2 + ap^t$ have the same sign. Taking into account that $2 + ap^t$ has the same sign as ap^t, since $p \geq 7$, and so the same sign of a, we see that a and b have the same sign.

(b) This case can be proved in the same way as (a), by noticing that $|\lambda_1| < |\lambda_2|$ implies that $b(2 + ap^t) < 0$. □

We now show that any non-semiprimitive GP-graph $\Gamma(3, q)$ and any non-semiprimitive GP-graph $\Gamma(4, q)$ with q a square are Ramanujan.

Theorem 6.5. *Let $k \in \{3, 4\}$ and $q = p^m$ for some prime p and $m \in \mathbb{N}$. If $p \equiv 1 \pmod{k}$, with m even if $k = 4$, then $\Gamma(k, q)$ is Ramanujan.*

Proof. Notice that the hypothesis $p \equiv 1 \pmod{k}$ implies that $\Gamma(k, q)$ is well defined and it is non-semiprimitive. We divide the proof in two parts, one for $\Gamma(3, q)$ and one for $\Gamma(4, q)$. By Theorems 3.1 and 3.2 we know that $\Gamma(3, q)$ and $\Gamma(4, q)$ are both undirected graphs in all the cases. Thus, by (6.1), Γ is Ramanujan if and only if $\lambda(\Gamma) \leq 2\sqrt{n-1}$.

• *The graphs* $\Gamma(3, q)$. We begin by showing that $\Gamma(3, q)$ in the non-semiprimitive case is Ramanujan. Let $p \equiv 1 \pmod 3$, $k = 3$, and $n = \frac{q-1}{3}$ where $q = p^m$.

(i) Suppose that $m = 3t$ for some $t \in \mathbb{N}$ (so we are in case (a) of Theorem 3.1). Assume first that

$$\lambda(\Gamma(3, q)) = |\lambda_0| = |\tfrac{ap^t - 1}{3}|.$$

Notice that $\lambda(\Gamma(3, q)) \leq 2(\frac{q-1}{3} - 1)^{\frac{1}{2}}$ is equivalent to

$$|ap^t - 1| \leq 2\sqrt{3}\sqrt{p^{3t} - 4}.$$

Since $|ap^t - 1| = |a|p^t \pm 1$, after some computations, we have that the above inequality is equivalent to

$$(|a|p^t \pm 1)^2 \leq 12(p^{3t} - 4).$$

By taking into account that $a^2 \leq 4p^t$, we have that

$$(|a|p^t \pm 1)^2 \leq 4p^{3t} + 2|a|p^t + 1.$$

Finally, since $p \equiv 1 \pmod 3$ then $p \geq 7$ and so we have that $4p^{3t} + 2|a|p^t + 1 \leq 12(p^{3t} - 4)$ is always true which implies that $\Gamma(3, q)$ is Ramanujan in this case.

Now assume that

$$\lambda(\Gamma(3, q)) = |\lambda_1| = |\tfrac{-\frac{1}{2}(a+9b)p^t - 1}{3}|.$$

In this case, $\lambda(\Gamma(3, q)) \leq 2(\frac{q-1}{3} - 1)^{\frac{1}{2}}$ is equivalent to

$$|\tfrac{-(a+9b)p^t - 2}{6}| \leq \tfrac{2\sqrt{3}}{3}\sqrt{p^{3t} - 4}.$$

Since $a + 9b \neq 0$ and $p \geq 7$, we have that $|-(a + 9b)p^t - 2| = |a + 9b|p^t \pm 2$, and so the above inequality is equivalent to

$$(|a + 9b|p^t \pm 2)^2 \leq 48(p^{3t} - 4).$$

By Lemma 6.4, the integers a and b have the same sign and so $(a+9b)^2 = a^2 + 18|a||b| + 81b^2$, which implies that

$$(|a+9b|p^t \pm 2)^2 \leq (a^2 + 18|a||b| + 81b^2)p^{2t} + 4|a+9b|p^t + 4.$$

By taking into account that $4p^t = a^2 + 27b^2$ we have that

$$|a| \leq 2p^{\frac{t}{2}} \quad \text{and} \quad |b| \leq \tfrac{2\sqrt{3}}{9}p^{\frac{t}{2}}$$

which implies that

$$a^2 + 18|a||b| + 81b^2 \leq 12p^{\frac{t}{2}} + 8\sqrt{3}p^{\frac{t}{2}} = 4(3+2\sqrt{3})p^{\frac{t}{2}} \leq 28p^{\frac{t}{2}}.$$

Now, since a and b have the same sign and $4p^t = a^2 + 27b^2$, we have that

$$|a+9b| = |a| + 9|b| \leq 2(1+\sqrt{3})p^{\frac{t}{2}} \leq 6p^{\frac{t}{2}}.$$

In this way we obtain that

$$(a^2 + 18|a||b| + 81b^2)p^{2t} + 4|a+9b|p^t + 4 \leq 28p^{3t} + 24p^{\frac{3t}{2}} + 4.$$

Finally, since $p \geq 7$, we have that $28p^{3t} + 24p^{\frac{3t}{2}} + 4 \leq 48(p^{3t} - 4)$, which implies that $\Gamma(3,q)$ is Ramanujan.

The remaining case, $\lambda(\Gamma(3,q)) = |\lambda_2|$, can be proved in a similar way, using item (b) instead of (a) of the above Lemma.

(ii) Now assume that $3 \nmid m$ (so we are in case (b) of Theorem 3.1). In this case, the eigenvalues of $\Gamma(3,q)$ are given by

$$x_j = -\tfrac{1}{3}\big(1 + \omega^j C + \tfrac{q}{\omega^j C}\big) \qquad \text{with} \qquad C = \sqrt[3]{q}\sqrt[3]{\tfrac{1}{2}(-a_0 + i3\sqrt{3}b_0)}$$

for $j = 0,1,2$ where $\omega = e^{\frac{2\pi i}{3}}$ and a_0, b_0 are integers satisfying $4q = a_0^2 + 27b_0^2$, $a \equiv 1 \pmod 3$ and $(a_0, p) = 1$. Notice that $|C| = \sqrt{q}$. In fact,

$$|C| = \sqrt[3]{q}\,\big|\tfrac{a_0 + i\sqrt{27}b_0}{2}\big|^{\frac{1}{3}} = \sqrt[3]{q}\,\Big(\sqrt{\tfrac{a_0^2 + 27b_0^2}{4}}\Big)^{\frac{1}{3}} = \sqrt[3]{q}\sqrt[3]{\sqrt{q}} = \sqrt{q}.$$

Thus, for any $j \in \{0,1,2\}$ we have that $|x_j| \leq \frac{1+2\sqrt{q}}{3}$. It is straightforward to see that

$$\tfrac{1+2\sqrt{q}}{3} \leq 2\sqrt{n-1}$$

holds for any $q \geq 7$, since it is equivalent to $8q - 4\sqrt{q} - 49 \geq 0$. Therefore, (6.1) holds, and $\Gamma(3,q)$ is Ramanujan in this case.

Thus, we have seen that any non-semiprimitive GP-graph $\Gamma(3,p^m)$ with $m \in \mathbb{N}$ (i.e. with $p \equiv 1 \pmod 3$) is Ramanujan.

• *The graphs* $\Gamma(4,q)$. We now look at the non-semiprimitive GP-graphs $\Gamma(4,q)$. So, let $k = 4$, $n = \frac{q-1}{4}$ and $q = p^m$ for some $m \in \mathbb{N}$ even.

(i) Suppose first that $4 \mid m$. By (a) in Theorem 3.2, the spectrum of $\Gamma(4, p^{4t})$ is given by

$$\left\{[n]^1, \left[\tfrac{p^{2t}+4dp^t-1}{4}\right]^n, \left[\tfrac{p^{2t}-4dp^t-1}{4}\right]^n, \left[\tfrac{-p^{2t}+2cp^t-1}{4}\right]^n, \left[\tfrac{-p^{2t}-2cp^t-1}{4}\right]^n\right\}$$

where $n = \frac{p^{4t}-1}{4}$ and c, d are integers uniquely determined by $p^{2t} = c^2 + 4d^2$, $c \equiv 1 \pmod 4$ and $(c, p) = 1$. In particular we have that

$$|c| \le p^t \qquad \text{and} \qquad |d| \le \tfrac{1}{2}p^t. \tag{6.7}$$

By a simple comparison of the eigenvalues, we have that

$$\lambda(\Gamma(4, p^{4t})) = \max\left\{\frac{p^{2t} + 4|d|p^t - 1}{4}, \frac{p^{2t} - 2|c|p^t + 1}{4}\right\}.$$

In this case the inequality $\lambda(\Gamma(4, p^{4t})) \le 2\sqrt{n-1}$ is equivalent to

$$\lambda(\Gamma(4, p^{4t}))^2 \le p^{4t} - 5. \tag{6.8}$$

It is enough to see what happens in any possible case.

Suppose first that $\lambda(\Gamma(4, p^{4t})) = \frac{1}{4}(p^{2t} + 4|d|p^t - 1)$, in this case the equation (6.8) turns into

$$(p^{2t} + 4|d|p^t - 1)^2 \le 16p^{4t} - 80.$$

Clearly,

$$(p^{2t} + 4|d|p^t - 1)^2 \le (p^{2t} + 4|d|p^t)^2 + 1 = p^{4t} + 8|d|p^{3t} + 16d^2p^{2t} + 1.$$

By (6.7) we have that $8|d|p^{3t} + 16d^2p^{2t} \le 4p^{4t} + 4p^{4t} = 8p^{4t}$ and hence

$$(p^{2t} + 4|d|p^t - 1)^2 \le 9p^{4t} + 1,$$

since $p \ge 5$, we have that $9p^{4t} + 1 \le 16p^{4t} - 80$ is true and so $\Gamma(4, p^{4t})$ is Ramanujan as desired.

On the other hand, when $\lambda(\Gamma(4, p^{4t})) = \frac{1}{4}(p^{2t} - 2|c|p^t + 1)$, by (6.7) we have that

$$(p^{2t} - 2|c|p^t - 1)^2 \le (p^{2t} + 1)^2 + 4c^2p^{2t} \le 4p^{4t} + 4p^{4t} = 8p^{4t} \le 16p^{4t} - 80.$$

A similar argument as above allows us to conclude that $\Gamma(4, p^{4t})$ is Ramanujan as asserted.

(ii) Now, if $m \equiv 2 \pmod 4$, proceeding similarly as in the previous case, by (b) in Theorem 3.2 we have that

$$\lambda(\Gamma(4, p^{4t+2})) = \max\left\{\tfrac{p^{2t+1}+1+\sqrt{2(p^{4t+2}+cp^{2t+1})}}{4}, \tfrac{p^{2t+1}-1+\sqrt{2(p^{4t+2}-cp^{2t+1})}}{4}\right\}$$

where $n = \frac{p^{4t+2}-1}{4}$ and $c, d \in \mathbb{Z}$ are uniquely determined by $p^{2t+1} = c^2 + 4d^2$, $c \equiv 1 \pmod 4$ and $(c, p) = 1$.

A similar computation as in the case $m \equiv 0 \pmod 4$ shows that $\Gamma(4, p^{4t+2})$ is Ramanujan (we leave the details to the reader), and the result is thus proved. $\square$

To conclude, we now summarize the results of this section on the Ramanujanicity of the GP-graphs $\Gamma(k,q)$ with $1 \le k \le 5$.

Remark 6.6. Consider $\Gamma(k,q)$ where $1 \le k \le 5$ and $k \mid q-1$ with $q=p^m$ with p prime. Then we have the following:

(a) The graphs $\Gamma(1,q)$ and $\Gamma(2,q)$ are all Ramanujan.

(b) In the semiprimitive case, the graphs $\Gamma(k,q)$ with $k \in \{3,4\}$ are Ramanujan if and only if $\Gamma(k,p^m)$ with $p \equiv -1 \pmod k$ and $m \ge 2$, where $m \ge 4$ if $p = k-1$ (see Theorem 6.1).

(c) In the non-semiprimitive case, the graphs $\Gamma(3,q)$ with q arbitrary and $\Gamma(4,q)$ with q a square are all Ramanujan (see Theorem 6.5).

(d) The graph $\Gamma(5,q)$ in the semiprimitive case is Ramanujan if and only if m is even and either $p \equiv 2,3 \pmod 5$, where $m = 4t$ ($t \ge 2$ if $p=2$), or else $p \equiv 4 \pmod 5$, by items (f)–(h) in Theorem 6.1.

(e) In all the previous cases (a)–(d), if $\Gamma(k,q)$ is Ramanujan with $1 \le k \le 5$, then $\Gamma^+(k,q)$ is also Ramanujan, since $\lambda(\Gamma^+) = \lambda(\Gamma)$, see (6.2), by Theorem 2.1 and hence (6.1) holds.

Relative to the GP-graphs with $k=5$, the Ramanujanicity of the non-semiprimitive case is open. On the one hand, the spectrum of $\Gamma(5,q)$ for $p \equiv 2,3 \pmod 5$ is still unknown (when the graph is not semiprimitive). For $p \equiv 1 \pmod 5$ with $q = p^{5t}$, the spectrum $\mathrm{Spec}(\Gamma(5,q))$ is given in Proposition 3.4. For instance, it is immediate to check that the graph $\Gamma(5,11^5)$ in Example 3.5 is Ramanujan. In general, we leave the following question: *which graphs* $\Gamma(5,p^{5t})$ *with* $p \equiv 1 \pmod 5$ *are Ramanujan?*

References

[1] A. E. Brouwer, Strongly regular graphs' page, `www.win.tue.nl/~aeb/graphs/srg/srgtab.html`.

[2] A. E. Brouwer and H. Van Maldeghem, Strongly regular graphs, *Cambridge University Press* Vol. 182, 2022.

[3] A. E. Brouwer, R. M. Wilson and Q. Xiang, Cyclotomy and Strongly Regular Graphs, *J. Algebr. Comb.* **10** (1999), 25–28.

[4] P. J. Cameron and J. H. van Lint, Designs, graphs, codes and their links, *Cambridge University Press*, LMSST 22, 1991.

[5] D. Cvetkovič, M. Doob and H. Sachs, Spectra of Graphs — Theory and Application, *Academic Press, New York*, 1980.

[6] D. Cvetkovič and M. Petrič, A table of connected graphs of six vertices, *Discrete Math.* **50** (1984), 37–49.

[7] C. Ding and J. Yang, Hamming weights in irreducible cyclic codes, *Discrete Math.* **313**(4) (2013), 434–446.

[8] S. Gurak, Period polynomials for $\mathbb{F}_{q^2}$ of fixed small degree, In: "Finite Fields and Applications", Proc. Fifth Int. Conf. Finite Fields and Applic. F_{q^5}, University of Augsburg, Germany, August 2-6, 1999; Berlin: Springer, 196–207 (2001).

[9] S. Gurak, Period polynomials for $\mathbb{F}_q$ of fixed small degree, *CRM Proc. Lect. Notes* **36** (2004), 127–145.

[10] F. Harary and A. J. Schwenk, Which graphs have integral spectra?, *Graphs and Combin., Proc. Capital Conf., Washington D.C. 1973, Lec. Notes Math.* **406** (1974), 45–51.

[11] S. Hoory, N. Linial and A. Wigderson, Expander graphs and their applications, *Bull. Amer. Math. Soc.* **43**(4) (2006), 439–561.

[12] A. Hoshi, Explicit lifts of quintic Jacobi sums and period polynomials for $\mathbb{F}_q$, *Proc. Japan Acad.* **82**(7) Ser. A (2006), 87–92.

[13] K. Feng and W. W. Li, Character sums and abelian Ramanujan graphs, *J. Number Theory* **41**(2) (1992), 199–217.

[14] T. K. Lim and C. Praeger, On Generalised Paley Graphs and their automorphism groups, *Michigan Math. J.* **58** (2009), 294–308.

[15] A. Lubotzky, Expander graphs in pure and applied mathematics, *Bull. Amer. Math. Soc.* **49** (2012), 113–162.

[16] A. Lubotzky and O. Parzanchevski, Ramanujan graphs to Ramanujan complexes, *Phil. Trans. Roy. Soc. A, Math. Phys. Eng. Sci.* **378**(2163), Article ID 20180445, 9 pp.

[17] M. Ram Murty, Ramanujan Graphs, *J. Ramanujan Math. Soc.* **18**(1) (2003), 1–20.

[18] G. Myerson, Period polynomials and Gauss sums, *Acta Arith.* **39**(3) (1981), 251–264.

[19] O. Parzanchevski, Ramanujan graphs and digraphs, *London Math. Soc. Lec. Note Ser.* **461**, 344–367.

[20] R. A. Podestá and D. E. Videla, Spectral properties of generalized Paley graphs of $(q^\ell + 1)$-th powers and applications, *Discrete Math. Algorithms Appl.* (2024), online first: https://doi.org/10.1142/S1793830924500563.

[21] R. A. Podestá and D. E. Videla, Weight distribution of cyclic codes defined by quadratic forms and related curves, *Rev. Unión Mat Argent.* **62**(1) (2021), 219–242.

[22] R. A. Podestá and D. E. Videla, The Waring's problem over finite fields through generalized Paley graphs, *Discrete Math.* **344** (2021), 112324.

[23] R. A. Podestá and D. E. Videla, Integral equienergetic non-isospectral unitary Cayley graphs, *Linear Algebra Appl.* **612** (2021), 42–74.

[24] R. A. Podestá and D. E. Videla, Generalized Paley graphs equienergetic with their complements, *Linear Multilin. Algebra* **72**(3) (2024), 488–515.

[25] R. A. Podestá and D. E. Videla. On regular graphs equienergetic with their complements. *Linear Multilin. Algebra* **71**(3) (2023), 422–456.

[26] R. A. Podestá and D. E. Videla, Spectral properties of generalized Paley graphs and their associated irreducible cyclic codes, in progress.

[27] B. Schmidt and C. White, All two weight irreducible cyclic codes?, *Finite Fields Appl.* **8** (2002), 1–17.

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