AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **90(3)** (2024), Pages 378–392

Does the degree sequence of an (almost) regular graph determine if it possesses a strong vertex-magic total labeling?

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Abstract

For each positive integer $r \equiv 9 \bmod 10$, we provide a strong vertex-magic total labeling (SVMTL) for a graph with $2r$ vertices of degree 3 and $3r$ vertices of degree 2. In addition, we show that no bipartite graph with the same degree sequence can have an SVMTL. This provides the first known infinite family of connected graphs with minimum degree d and maximum degree $d+1$ each possessing an SVMTL, such that there is another family of graphs with the same respective degree sequences, but without possessing SVMTLs. (There are no previously known disconnected such families, provided we add the assumption that K_2 is not isomorphic to any of the components).

A well-known technique of Ian Gray combined with our work gives as corollary that a large range of graphs with $2r$ vertices of degree $2m+1$ and $3r$ vertices of degree $2m$ also possess SVMTLs.

There are well-known theorems and conjectures regarding the existence of SVMTLs for certain *regular* graphs of degree at least 2. Our work suggests that the regularity assumption cannot be easily weakened.

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ISSN: 2202-3518

1 Introduction

Let G be a simple graph with vertex set V and edge set E. A *labeling* λ of G is a map $\lambda : V \cup E \longrightarrow \{1, 2, \ldots, |V| + |E|\}$. The *weight* $wt_\lambda(v)$ of vertex v with incident edges $e_1, \ldots, e_t$ is given by $wt_\lambda(v) = \lambda(v) + \lambda(e_1) + \cdots + \lambda(e_t)$. The labeling λ is said to have the *magic property* if the weight of every vertex is the same. In this case we refer to λ as a *magic labeling.* If in addition the magic labeling λ is a *bijection*, then λ is called a *vertex-magic total labeling* or *VMTL* and the common weight (often denoted by h) is called the *magic constant* for the VMTL. We say that a graph G is *vertex-magic* if it has a VMTL. Otherwise, we say that it is *non-magic.*

For example, in Figure 1 there is a VMTL of C_7 having a magic constant of $h = 22$. The first paper specifically dedicated to VMTLs is [13], which appeared

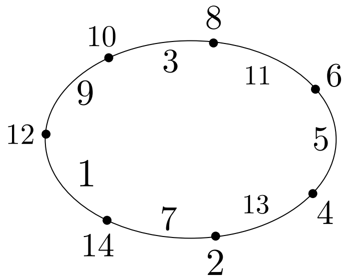

Figure 1: A VMTL for C_7 with a magic constant of $h = 22$.

at approximately the same time as the first edition of [14]. However, for 2-regular graphs, the question goes back much further, since a so-called *edge-magic total labeling* can be easily converted into a VMTL for any 2-regular graph. (See for example, [15].) In that sense, VMTLs indirectly go back at least as far as 1970, as edge-magic total labelings were called *M-valuations* in [10]. In this paper, we will only consider *vertex-magic* total labelings.

The role of regularity, or near regularity, has played a central role in the development of the literature. Most significantly, MacDougall observed that regular graphs of degree at least 2 seem to all be vertex-magic, with the exception of the disjoint union of two 3−cycles ($2C_3$). He then conjectured [11, 12] that $2C_3$ is the *only* non-magic regular graph of degree at least 2. Note that it follows immediately from the bijective property of VMTLs that any regular graph of degree 1 is non-magic; in fact, so is any graph with a component isomorphic to K_2. Thus, we view K_2 as a *forbidden component.*

To see why irregularity might provide an obstruction, consider the star $K_{1,3}$. In a total labeling, the central vertex has weight at least $1+2+3+4 = 10$; evidently not all of the other vertices could have such a large weight, with the largest available label being 7. (More impressively, it is shown in [13] that $K_{m,n}$ is never magic *whenever* $|m - n| \geq 2$.) MacDougall's Conjecture then suggests that the removal of this type of obstruction should be sufficient to ensure VMTLs for the graph, with $2C_3$ being seen as mostly a consequence of the law of small numbers. In this paper we focus on infinite families of graphs to ensure that small numbers are not the main issue.

Another perspective for considering MacDougall's conjecture is that it suggests, (certainly for regular graphs) that the *degree sequence*, and not the finer structure of the graph, determines whether it has a VMTL. Exploring this perspective, it was shown in [3] that for each $r \geq 3$ there is a vertex-magic graph with the same degree sequence as $K_{2r,2r+2}$. This provides an example of a *separating* family; that is an infinite family of degree sequences such that, for each sequence, there are two graphs sharing the degree sequence—one non-magic, and the other one vertex-magic.

To ensure that not all separating families have a bipartite graph as the non-magic partner, [3] also provides two graphs with n vertices of degree 3 and n vertices of degree 1, the vertex-magic graph being a sun graph and the non-magic graph being neither bipartite nor containing the forbidden component K_2.

It remains unknown whether there is a separating example where the difference between the maximum degree and the minimum degree is at most 1. For example, $K_{m,m+1}$ does have a VMTL for each m, so it cannot help provide the non-magic example. However, we can get results of a similar flavor if we expand our reach to include the so-called *spectrum* of magic constants.

If a graph is vertex-magic, there is a corresponding range of possible magic constants, called the *spectrum of magic constants* for the graph. For example, Figures 1 and 2 both show VMTLs of C_7, but with different magic constants. For most families of graphs, the question of determining the spectrum is a difficult one, with few general results. An exception is given in [17], where the spectrum for all odd order complete graphs is completely determined.

A special case that has generated enormous interest concerns the so-called *strong* VMTL, or *SVMTL*, defined to be a VMTL where every vertex label is bigger than every edge label. The labeling in Figure 2 is an SVMTL for C_7.

As a warning to the reader, the adjective *strong* is used inconsistently in the context of vertex-magic total labelings. We have adopted the meaning chosen by Gray (e.g. [5, 6, 7]) with some authors (e.g. [1]) using it to mean that the smallest labels are on the vertices.

SVMTLs of 2-regular graphs have enhanced importance due to the following:

Theorem 1.1 *(Gray [5]): If G is a graph of order n with a spanning subgraph H which possesses a strong VMTL and $G - E(H)$ is even regular, then G also possesses a strong VMTL.*

If when applying Gray's Theorem, H has many vertices, all of low degree, then one can imagine applying the theorem several times; adding 2-factors repeatedly and each time getting a new graph with an SVMTL. The SVMTLs we provide in Section 3 are for graphs whose vertices have either degree 2 or degree 3. The option for repeated use of Gray's Theorem suggests that SVMTLs for these low degree graphs are particularly useful. A striking application of Gray's Theorem uses a well-known SVMTL for odd order cycles. With this, Gray concludes [5, 6] that every odd order regular graph with a Hamilton cycle has an SVMTL, providing substantial evidence for MacDougall's Conjecture.

This important SVMTL for odd cycles first appears in [4] (although it is slightly disguised, as it is a paper on edge-magic total labeling). It plays a key role in our strategy in Section 3 and we refer to it as the *standard* SVMTL for C_n, for odd n. The labeling itself can be described as follows. Start by labeling an edge with 1, and then, walking around the cycle, skip an edge and label the edge after that with 2; continue in this manner until all edges are labeled. Observe that, before labeling vertices, the (partial) weights are consecutive numbers, meaning that the vertex labels $n + 1, n + 2, \dots 2n$ can easily be assigned to vertices to achieve the magic property. It is well-known that even cycles cannot have SVMTLs. (It follows from equation (3) in Section 2.) Figure 2 has the standard SVMTL for $n = 7$.

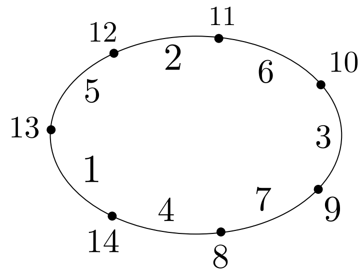

Figure 2: A Strong VMTL (SVMTL) for C_7 with a magic constant of $h = 19$.

For other (i.e. disconnected) odd-order 2-regular graphs, the story is more subtle. There are three known (vertex-magic!) examples that do not have an SVMTL: namely, the disjoint unions $C_3 \cup C_4$, $2C_3 \cup C_5$ and $3C_3 \cup C_4$. In order to ensure that there was nothing structurally prohibitive about disjoint unions of the form $(2s - 1)C_3 \cup C_4$ or $2sC_3 \cup C_5$, it was shown in [8] that all graphs of those forms actually do possess SVMTLs *except* for the three small exceptions mentioned above. Other SVMTLs for similar families of 2-regular graphs were provided in [16]. Other general constructions are given in [2] and [18].

All other 2-regular graphs of order at most 29 have been investigated in [9], and all possess SVMTLs. They also provide the numbers of distinct SMVTLs and, interestingly, this number seems to be growing tremendously quickly with the order of the graphs. This suggests that regularity is the key, rather than the structure of the graphs themselves.

Strong VMTLs play an important role in the area of vertex-magic total labelings, as well as being interesting in their own right. In this paper, we provide an infinite family of degree sequences such that, for each degree sequence, there are two graphs sharing that degree sequence; one that has an SVMTL and the other which does not. The graphs are close to being regular, as the maximum difference in degrees is 1.

In Section 2 we set up the preliminary equations and framework, similar to [13]. From there it is straightforward to show that any bipartite graph with $2r$ vertices of degree 3 and $3r$ vertices of degree 2 cannot have an SVMTL. At the same time, we will show that *any* (not necessarily bipartite) graph with that degree sequence can only have an SVMTL if $r \equiv 9 \bmod 10$. As an aside we show that no graph can have an SVMTL if it has the same degree sequence as $K_{m,m+1}$, with the exception of $K_{1,2}$ which itself has an SVMTL. Thus, we could not have used $K_{m,m+1}$ as the non-magic partner for our infinite pairing.

The heart of the paper is Section 3, where an SVMTL is constructed for a (non-bipartite!) graph having $2r$ vertices of degree 3 and $3r$ vertices of degree 2. This construction works for any positive integer $r \equiv 9 \mod 10$. Roughly speaking the strategy of construction is to begin with the standard SVMTL for C_{5r} and then, using a few numerical shifts and a permutation of the vertex labels, we end up with $3r$ vertices of the same weight and $2r$ vertices with lesser weight. We add edges with small labels to these $2r$ vertices of lesser weight, to restore the magic property. Finally, we shift the vertex labels up to ensure the labeling is a bijective map.

In Section 4, we consider a few open and hopefully tractable problems that remain.

2 Graphs without a strong VMTL

Given a graph G with VMTL λ and magic constant h, we let S_v denote the sum of vertex labels and we let S_e denote the sum of all edge labels. If G has n vertices and ϵ edges then,

$$\begin{aligned} S_v + S_e &= 1 + 2 + \cdots + (n+\epsilon) \\ &= \frac{(n+\epsilon)(n+\epsilon+1)}{2}. \end{aligned}$$

Since each edge label appears in the calculation of *two* vertex weights, but vertex labels only appear in one, we get,

$$S_v + 2S_e = nh. \tag{1}$$

Combining these gives us

$$S_e = nh - \frac{(n+\epsilon)(n+\epsilon+1)}{2}.$$

Rearranging slightly gives us

$$h = \frac{2S_e + (n+\epsilon)(n+\epsilon+1)}{2n}. \tag{2}$$

This makes precise the relationship between S_e and h, and makes clear that if there is an SVMTL, it would have the smallest possible magic constant. Note we have bounds for S_e

$$\frac{\epsilon(\epsilon+1)}{2} = 1 + 2 + \cdots + \epsilon \le S_e \le (n+1) + \cdots + (n+\epsilon) = \frac{\epsilon(2n+\epsilon+1)}{2}.$$

Combining with equation (2) we get bounds for h

$$\frac{\epsilon(\epsilon+1) + (n+\epsilon)(n+\epsilon+1)}{2n} \le h \le \frac{\epsilon(2n+\epsilon+1) + (n+\epsilon)(n+\epsilon+1)}{2n}.$$

Any integral value of h within these bounds we will call a *feasbile* value for a magic constant. Note that feasibility is a property of n and ϵ, and not the structure of the graph. If G has an SVMTL, then $S_e = \epsilon(\epsilon+1)/2$ and so

$$h = \frac{\epsilon(\epsilon+1) + (n+\epsilon)(n+\epsilon+1)}{2n}. \tag{3}$$

We summarize the part we need as follows:

Theorem 2.1 *Let λ be a strong vertex-magic total labeling of a graph consisting of exactly $5r$ vertices and $6r$ edges. Then,*

$$h = \frac{157r + 17}{10}.$$

Furthermore, $r \equiv 9 \pmod{10}$.

Proof. For the first part, use equation 3 to get

$$\begin{aligned} h &= \frac{6r(6r+1) + 11r(11r+1)}{10r} \\ &= \frac{157r+17}{10}. \end{aligned}$$

Since h must be an integer, we need $157r + 17 \equiv 0 \pmod{10}$. Reducing gives $7(r+1) \equiv 0$ or $r \equiv 9 \pmod{10}$. □

This suggests our choice of infinite family is a reasonable one. The next theorem helps to solidify that view, as it shows that we could not have used $K_{m,m+1}$ for the non-magic partner of an infinite set of separating examples.

Theorem 2.2 *Let G have m vertices of degree $m+1$ and $m+1$ vertices of degree m. If G has a strong vertex-magic total labeling then $m = 1$.*

Proof. We start with the following number theoretic fact.

Lemma 2.3 *If m and $\frac{m(m+1)(m^2+m+1)}{2m+1}$ are both positive integers then $m = 1$.*

Proof of Lemma: Since $\gcd(m, 2m+1) = 1$ and $\gcd(m+1, 2m+1) = 1$ it follows that $m(m+1)(m^2+m+1)$ can only be a multiple of $2m+1$ if m^2+m+1 is already a multiple of $2m+1$; in this case, $2(m^2+m+1)$ is also a multiple of $2m+1$. Since

$$\frac{2(m^2+m+1)}{2m+1} = m + \frac{m+2}{2m+1},$$

we conclude that $\frac{m+2}{2m+1}$ is an integer. In particular, $2m+1 \leq m+2$. This implies that $m = 1$. □

To prove the theorem, assume that G has an SVMTL with magic constant h. Since $n = 2m+1$ and $\epsilon = m^2 + m$, we see that

$$\begin{aligned} S_v &= (m^2+m+1) + (m^2+m+2) + \cdots + (m^2+3m+1) \\ &= (2m+1)(m+1)^2. \end{aligned}$$

Also,

$$\begin{aligned} 2S_e &= 2(1+2+\cdots+(m^2+m)) \\ &= m(m+1)(m^2+m+1). \end{aligned}$$

We use these, along with equation (1), to get:

$$\begin{aligned} (2m+1)h &= S_v + 2S_e \\ &= (2m+1)(m+1)^2 + m(m+1)(m^2+m+1). \end{aligned}$$

Thus,

$$h = (m+1)^2 + \frac{m(m+1)(m^2+m+1)}{2m+1}.$$

Since h must be an integer, it follows that $\frac{(m^2+m+1)m(m+1)}{2m+1}$ must also be an integer. By the lemma, this implies that $m = 1$. □

With preliminaries in place, the next result is straightforward. The key point (where the bipartite assumption gets used) is at the start of the proof, where S_e, the sum of all edge labels, occurs as a summand in two different equations.

Theorem 2.4 *Let G be a bipartite graph with vertex partite sets V_1 and V_0. Assume $|V_1| = 3r$ and $|V_0| = 2r$ and that G has exactly $6r$ edges. Then G does not possess a strong vertex-magic total labeling.*

Proof. Assume by way of contradiction that λ is an SVMTL for G with magic constant h. Let S_i denote the sum of vertex labels in V_i, for $i = 0, 1$. Summing over the weights of all vertices in V_1 we get

$$3rh = S_1 + S_e.$$

Similarly, by summing over weights of all vertices in V_0 we get

$$2rh = S_0 + S_e.$$

Combining the previous two equations gives us $rh = S_1 - S_0$. However, by Theorem 2.1 the magic constant for λ is given by $h = 15.7r + 1.7$, whence

$$S_1 - S_0 = 15.7r^2 + 1.7r. \tag{4}$$

Since λ is strong, the set of vertex labels is exactly $\{6r+1, 6r+2, \cdots, 11r\}$. Summing over the labels in this set gives us

$$\begin{aligned} S_1 + S_0 &= (6r+1) + (6r+2) + \cdots + 11r \\ &= \frac{5r(17r+1)}{2} \\ &= 42.5r^2 + 2.5r. \end{aligned} \tag{5}$$

Combining equations (4) and (5) we get

$$S_0 = 13.4r^2 + 0.4r.$$

However, $|V_0| = 2r$ and so

$$\begin{aligned} S_0 &\geq (6r+1) + (6r+2) + \cdots + 8r \\ &= 14r^2 + r. \end{aligned}$$

This contradicts the fact that $S_0 = 13.4r^2 + 0.4r$, completing the proof. □

Since any bipartite graph of order $5r$ including $2r$ vertices of degree 3 and $3r$ vertices of degree 2 has $\epsilon = 6r$ edges, we immediately have:

Corollary 2.5 *Let G be a bipartite graph of order $5r$ with one partite set consisting of $3r$ vertices of degree 2 and the other partite set consisting of $2r$ vertices of degree 3. Then G does not possess a strong vertex-magic total labeling.*

In the next section we provide, for each $r = 9, 19, \ldots$, a graph with the same degree sequence as in the corollary; we also construct a corresponding SVMTL.

3 The permutation and the SVMTL

The goal of this section is to provide an SVMTL for a graph with $3r$ vertices of degree 2 and $2r$ vertices of degree 3 for each $r \equiv 9 \mod 10$. The easiest way to proceed is in steps. To improve clarity, we provide the worked example with $r = 9$, although the reader will see that it works for every $r \equiv 9 \mod 10$.

The first step is to begin with the standard SVMTL λ for the cycle C_{5r}. Since r is odd, $5r$ is also odd, and therefore this SVMTL exists. With $5r$ vertices and $5r$ edges we set $n = \epsilon = 5r$ in equation (3) to see that the magic constant is $h = 12.5r + 1.5$. Our goal is to (eventually) add another r edges to this graph and modify the labeling to get λ^* so that the corresponding new magic constant, also given by equation (3), is $h^* = 15.7r + 1.7$.

Using λ as a starting point, we perform a series of numerical shifts. If we subtract 5r from each vertex label, then we will no longer have a *bijective* map from $V \cup E$ to $1, 2, \cdots |V| + |E|$; however, we still retain the magic property. Thus, for any vertex v in C_{5r}, we set

$$\lambda_1(v) = \lambda(v) - 5r.$$

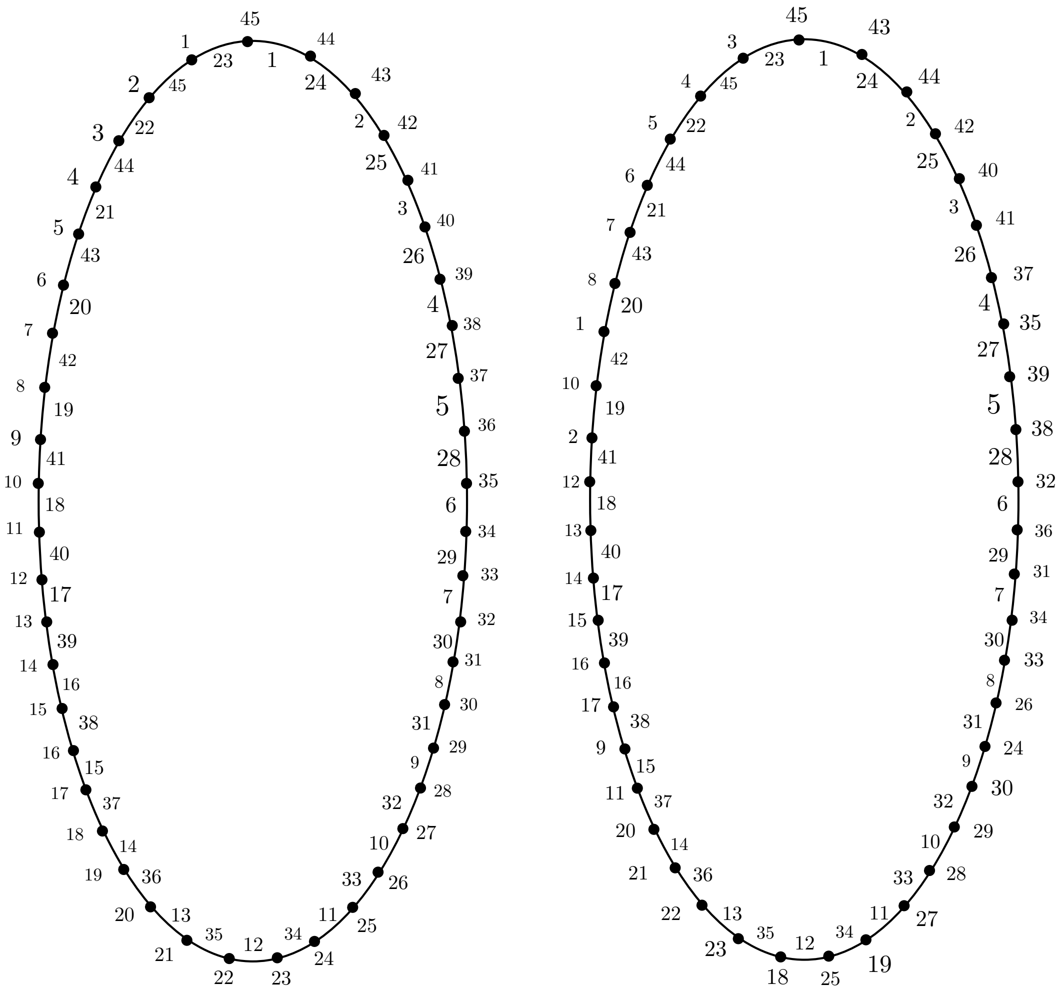

Figure 3: For $r = 9$. Left: λ_1 (magic) with $h_1 = 69$. Right: λ_2 (not magic) with $h_2 = 71$.

Furthermore, we leave the edge labels unchanged, that is $\lambda_1(e) = \lambda(e)$ for every edge e of C_{5r}. The new (common) weight is given by:

$$h_1 = 7.5r + 1.5.$$

Next, we obtain λ_2 by permuting the vertex labels $1, 2, \cdots 5r$, sending $x \mapsto f(x)$. This means that if $\lambda_1(v) = x$, then $\lambda_2(v) = f(x)$. Again, we leave the edge labels unchanged. See Figure 3 for both λ_1 and λ_2 for the case $r = 9$.

The permutation f will be defined carefully at the end of this section; for now note that the permutation will result in exactly $3r$ of the $5r$ vertices having a slight gain (of g) in weight. That is, $3r$ of the vertices will have weight:

$$h_2 = h_1 + g, \text{ where } g = \frac{r+1}{5}. \tag{6}$$

For the remaining $2r$ vertices, there will be two vertices with each of the following weights: $h_2 - 1, h_2 - 2, \ldots, h_2 - r$.

We will restore the magic property by adding, to each vertex with weight $h_2 - i$, a new edge with label i, for $i \geq 1$. If $\lambda_1(v) = x$ and $f(x) = x + g$, then v will not gain any new incident edge. This makes the quantity $x + g - f(x)$ an important one. For the worked example $r = 9$, we give here the permutation, omitting those x for which $x + g - f(x) = 0$ and noting that if $\lambda_1(v) = x$ and $x + g - f(x) > 0$, then v will acquire an edge with label $x + g - f(x)$.

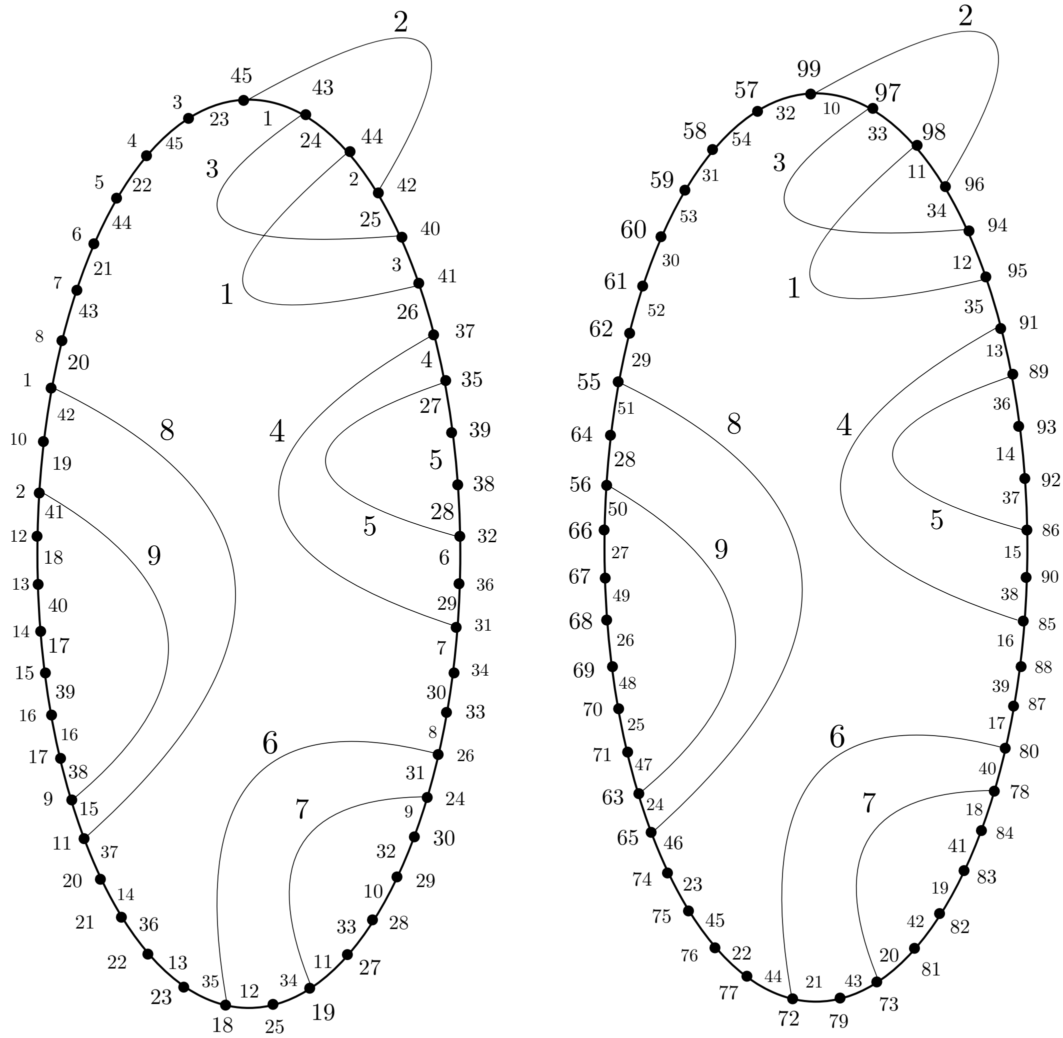

Figure 4: For $r = 9$. Left: λ_3 (magic) with $h_2 = 71$. Right: λ^* (SVMTL) with $h^* = 143$.

Thus, using the notation $\begin{pmatrix} x \\ f(x) \\ x+g-f(x) \end{pmatrix}$, the permutation f for the case $r = 9$ is given by

$$\begin{pmatrix} 7 & 9 & 16 & 17 \\ 1 & 2 & 9 & 11 \\ 8 & 9 & 9 & 8 \end{pmatrix} \begin{pmatrix} 22 & 24 & 29 & 30 \\ 18 & 19 & 24 & 26 \\ 6 & 7 & 7 & 6 \end{pmatrix} \begin{pmatrix} 33 & 35 & 38 & 39 \\ 31 & 32 & 35 & 37 \\ 4 & 5 & 5 & 4 \end{pmatrix} \begin{pmatrix} 40 & 41 & 42 & 43 & 44 & 45 \\ 41 & 40 & 42 & 44 & 43 & 45 \\ 1 & 3 & 2 & 1 & 3 & 2 \end{pmatrix}.$$

(For the general permutation, we will also have these four blocks, and they will also be denoted separately).

Now we are ready to enlarge the graph from C_{5r} by adding r new edges. For each $i \geq 1$ the two vertices of weight $h_2 - i$ will be joined by an edge that will have label i. Let λ_3 be the name of the enlarged labeling, noting that $\lambda_3(w) = \lambda_2(w)$ if w is any vertex or edge from the original cycle C_{5r}. We have added r new edges. Note that unlike λ_2 which has vertices of different weights, λ_3 has the magic property, with magic constant h_2.

The last step is to restore the bijective property to get a proper SVMTL λ^*. Thus, we add r to each *original cycle edge* label, thereby making room for the new edge labels $1, 2, \cdots r$ on the r new edges.

This numerical shift increases the weight of each vertex by $2r$. Finally we add $6r$ to each vertex label, as there are now $6r$ edges. This final shift increases the weight by an additional $6r$, and it provides an SVMTL λ^* for the required graph, with magic constant:

$$\begin{aligned} h^* &= h_2 + 2r + 6r \\ &= h_1 + g + 8r \\ &= (7.5r + 1.5) + g + 8r \\ &= 15.7r + 1.7. \end{aligned}$$

See Figure 4 for both λ_3 and λ^* for the case $r = 9$.

For describing the permutation in the general case, we opt for a notation where it is easy to see that it is, in fact a permutation. The first column has values of x and the second column has the respective values of $f(x)$, with the last column having the new edge labels. (If there is a 0 in the third column, then there will be no edge added to any of the vertices with labels from that row.)

The first block consists of a permutation of the labels $1, 2, \ldots, 9g - 1$.

x	$f(x)$	$x + g - f(x)$
$1 \leq x \leq 3g$	$g+1, g+2, \ldots, 4g$	0
$3g+1, 3g+3, \ldots, 5g-1$	$1, 2, \ldots, g$	$4g, 4g+1, \ldots, 5g-1$
$3g+2, 3g+4, \ldots, 5g-2$	$4g+2, 4g+4, \ldots, 6g-2$	0
$5g \leq x \leq 8g-1$	$6g, 6g+1, \ldots, 9g-1$	0
$8g \leq x \leq 9g-1$	$4g+1, 4g+3, \ldots, 6g-1$	$5g-1, 5g-2, \ldots, 4g$

Secondly, we permute the labels $9g, 9g + 1, \ldots, 16g - 2$.

x	$f(x)$	$x + g - f(x)$
$9g \leq x \leq 11g-1$	$10g, 10g+1, \ldots, 12g-1$	0
$11g, 11g+2, \ldots, 13g-2$	$9g, 9g+1, \ldots, 10g-1$	$3g, 3g+1, \ldots, 4g-1$
$11g+1, 11g+3, \ldots, 13g-3$	$12g+1, 12g+3, \ldots, 14g-3$	0
$13g-1 \leq x \leq 15g-2$	$14g-1, 14g, \ldots, 16g-2$	0
$15g-1 \leq x \leq 16g-2$	$12g, 12g+2, \ldots, 14g-2$	$4g-1, 4g-2, \ldots, 3g$

Next, we permute the labels, $16g-1, 16g, \ldots, 21g-3$.

x	$f(x)$	$x+g-f(x)$
$16g-1 \le x \le 17g-2$	$17g-1, 17g, \ldots, 18g-2$	0
$17g-1, 17g+1, \ldots, 19g-3$	$16g-1, 16g, \ldots, 17g-2$	$2g, 2g+1, \ldots, 3g-1$
$17g, 17g+2, \ldots, 19g-4$	$18g, 18g+2, \ldots, 20g-4$	0
$19g-2 \le x \le 20g-3$	$20g-2, 20g-1, \ldots, 21g-3$	0
$20g-2 \le x \le 21g-3$	$18g-1, 18g+1, \ldots, 20g-3$	$3g-1, 3g-2, \ldots, 2g$

Finally, we permute $21g-2, 21g-1, \ldots, 25g-5$.

x	$f(x)$	$x+g-f(x)$
$21g-2 \le x \le 22g-3$	$22g-3, 22g-4, \cdots 21g-2$	$1, 3, \cdots, 2g-1$
$22g-2 \le x \le 23g-4$	$23g-4, 23g-5, \cdots, 22g-2$	$2, 4, \cdots, 2g-2$
$23g-3 \le x \le 24g-4$	$24g-4, 24g-5, \cdots, 23g-3$	$1, 3, \cdots, 2g-1$
$24g-3 \le x \le 25g-5$	$25g-5, 25g-6, \cdots, 24g-3$	$2, 4, \cdots, 2g-2$

This description not only allows for easy verification that each block is permuted, but we also see from the third column that new edges added were not previously adjacent, and therefore we are certain that the graph we construct is a simple graph. It is also evident that there are, for each $i = 1, 2, \ldots, 5g-1$ exactly two values of x such that $x+g-f(x) = i$ and therefore there is an edge with label i to join between the two corresponding vertices. From equation (6) we see that $5g-1 = r$, and so we are indeed adding exactly r edges to C_{5r} as claimed. Thus, we have now proved the following:

Theorem 3.1 *For each positive integer $r \equiv 9$ mod 10 there is a simple graph G of order $5r$ consisting of $2r$ vertices of degree 3 and $3r$ vertices of degree 2 such that G has a strong vertex-magic total labeling.*

Let H_r be the graph with $5r$ vertices constructed in the preceding theorem ($r \equiv 9$ mod 10), and let $E(H_r)$ be its edge set. If G is any graph containing H_r as a spanning subgraph and $G - E(H_r)$ is even regular, then it immediately follows that G also possesses an SVMTL, by Theorem 1.1.

4 Concluding Remarks

Despite the fact MacDougall's Conjecture remains open, most believe it to be true. It is reasonable to attempt to make the conjecture stronger, even if just to show that it cannot be made much stronger.

Let $\delta(G)$ denote the difference between the maximum degree and minimum degree (over all vertices of G). MacDougall's Conjecture could be thought of as saying that as long as G does not have the forbidden component of K_2, and G has order at least 7, then the condition $\delta(G) = 0$ guarantees that G is vertex-magic.

This perspective suggests finding variations on MacDougall's Conjecture based on small values of δ, and, more generally, on degree sequences. We cannot guarantee that G is vertex-magic based on $\delta(G) \leq 2$, since it was shown in [13] that complete bipartite graphs with $\delta = 2$ are not. By contrast, in [3], vertex-magic graphs were found having the same degree sequence as the (non-magic) $K_{2r,2r+2}$. This prevents our ability to make overly ambitious conclusions based on the degree sequence when $\delta = 2$. This paper arose in part with an attempt to start to seriously consider the case $\delta = 1$. We ask:

Question: Is there an infinite list of degree sequences, each with $\delta \leq 1$, such that for each degree sequence there is a corresponding vertex-magic graph *and* a corresponding non-magic graph avoiding K_2 as a component?

We propose the following strengthening of MacDougall's Conjecture:

Conjecture 4.1 *Any simple graph with minimum degree at least $d \geq 2$ and maximum degree at most $d + 1$ has a vertex-magic total labeling, with at most finitely many exceptions.*

The non-magic regular graph $2C_3$ is the only exception we know of, so we are encouraging the reader to find examples of non-magic graphs where the difference between maximum degree and minimum degree is at most 1. If these graphs exist, can they be of arbitrarily large order?

Our paper shows that, even in the case where $\delta = 1$, the degree sequence does not determine all of the magic constants for the graph. However, we focused entirely on strong VMTLs.

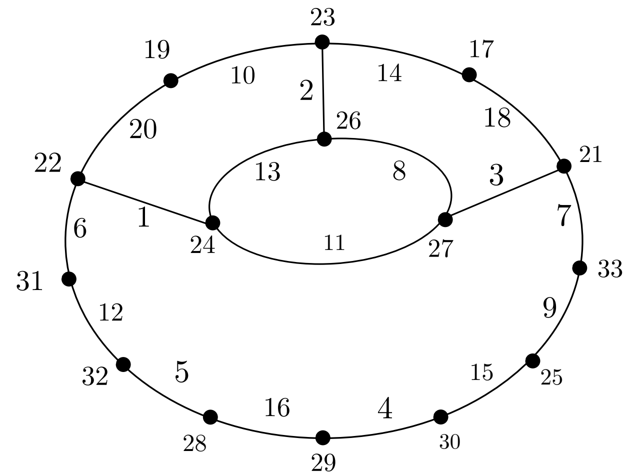

Figure 5: A VMTL with magic constant of $h = 49$.

We end by pointing out that this type of question can be extended to consider all feasible magic constants. For example, the VMTL in Figure 5 has a magic constant of $h = 49$. While this is not a strong VMTL, one can show (using methods similar to those used in the proof of Theorem 2.4), that any bipartite graph with 18 edges, and with vertex set consisting of 9 vertices in one partite set and 6 vertices in the other, cannot have a VMTL with magic constant 49. Therefore, we can also ask for the correct generalization of this fact, from the perspective of graphs with the same degree sequence having different spectra.

Acknowledgements

This project was supported by the Norwich University Undergraduate Research Program, where the first author was a Weintz Research Scholar, working under the supervision of the second author. We thank Norwich University for its support of this work.

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(Received 12 June 2024; revised 30 Oct 2024)