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Forbidden caterpillars for 3-connected graphs with girth at least five

Yoshimi Egawa

Department of Applied Mathematics
Tokyo University of Science
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan

Abstract

For a family $\mathcal{F}$ of graphs, a graph G is said to be $\mathcal{F}$-free if G contains no member of $\mathcal{F}$ as an induced subgraph. We let $\mathcal{G}_3(\mathcal{F})$ be the family of 3-connected $\mathcal{F}$-free graphs. Let P_n and C_n denote the path and the cycle of order n, respectively. Let $S_9(\{5\}, \emptyset)$ be the tree obtained from P_9 by adding a vertex and joining it to the central vertex of P_9, and $S_9(\{2\}, \emptyset)$ be the tree obtained by adding a vertex and joining it to a vertex adjacent to an endvertex of P_9. We show that $\mathcal{G}_3(\{C_3, C_4, S_9(\{5\}, \emptyset)\})$ and $\mathcal{G}_3(\{C_3, C_4, S_9(\{2\}, \emptyset)\})$ are finite families.

1 Introduction

By a graph, we mean a finite, simple, undirected graph. Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $u \in V(G)$, we let $N_G(u)$ and $\deg_G(u)$ denote the neighborhood and the degree of G, respectively; thus $\deg_G(u) = |N_G(u)|$. We let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and the maximum degree of G, respectively. For $U \subseteq V(G)$, we set $N_G(U) = \cup_{u \in U} N_G(u)$, and let $G[U]$ denote the subgraph of G induced by U. For $U, U' \subseteq V(G)$ with $U \cap U' = \emptyset$, we let $E_G(U, U')$ be the set of edges of G joining a vertex in U and a vertex in U'. When G is connected, for $u,\ v \in V(G)$, we let $\mathrm{dist}_G(u, v)$ denote the distance of u and v in G, and let $\mathrm{diam}(G)$ denote the maximum of $\mathrm{dist}_G(u, v)$ as u and v range over $V(G)$. We let C_n and K_n denote the cycle and the complete graph of order n, respectively. We let K_{m_1,m_2} denote the complete bipartite graph with partite sets having cardinalities m_1 and m_2, respectively. For terms and symbols not defined here, we refer the reader to [1].

Let $n \geq 5$ be an integer, and let I, J be subsets of $\{2, 3, \ldots, n-1\}$ with $J \subseteq \{3, \ldots, n-2\}$ and $I \cap J = \emptyset$. We let $S_n(I, J)$ denote the tree obtained from a path $u_1 u_2 \cdots u_n$ of order n by adding vertices v_i $(i \in I \cup J)$ and v'_i $(i \in J)$ and edges $u_i v_i$ $(i \in I \cup J)$ and $v_i v'_i$ $(i \in J)$. Also we let S^* denote the tree obtained from a path $u_1 u_2 \cdots u_7$ of order 7 by adding vertices v_4, v'_4, v''_4 and edges $u_4 v_4, v_4 v'_4, v_4 v''_4$ (see

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Figure 1). A tree T is a caterpillar if there exists a path P of T such that $T - V(P)$ has no edges. Note that $S_n(I, \emptyset)$ is a caterpillar.

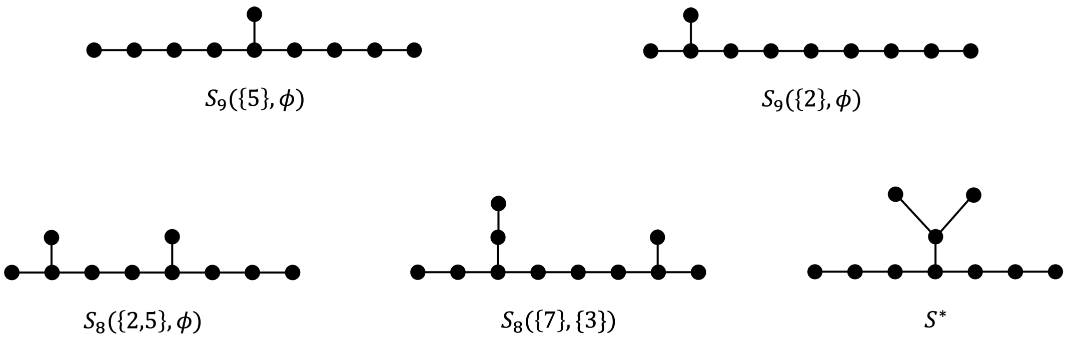

Figure 1: Trees $S_n(I, J)$ and S^*.

For two graphs G and H, we say that G is H-free if G does not contain an induced copy of H. For a family $\mathcal{F}$ of connected graphs, a graph G is said to be *$\mathcal{F}$-free* if G is H-free for every $H \in \mathcal{F}$. For an integer $k \geq 2$ and a family $\mathcal{F}$ of connected graphs, let $\mathcal{G}_k(\mathcal{F})$ denote the family of k-connected *$\mathcal{F}$-free* graphs. In this context, members of $\mathcal{F}$ are often referred to as forbidden subgraphs. Note that $\mathcal{G}_k(\{C_3, C_4\})$ is the family of k-connected graphs with girth at least five.

Let $k \geq 2$ be an integer. In this paper, we consider families $\mathcal{F}$ of connected graphs such that

$$\mathcal{G}_k(\mathcal{F}) \text{ is a finite family.} \tag{1.1}$$

Note that if a family $\mathcal{F}$ satisfies (1.1), then for any property P on graphs, although the proposition that all k-connected *$\mathcal{F}$-free* graphs satisfy P with finite exceptions holds, the proposition gives no information about P. Thus it is important to identify families $\mathcal{F}$ satisfying (1.1) in advance. With such a motivation, studies of $\mathcal{F}$ satisfying (1.1) have been started by Fujisawa, Plummer and Saito in [7]. In particular, it is known that if a finite family $\mathcal{F}$ of connected graphs satisfies (1.1), then $\mathcal{F}$ contains a complete graph, a complete bipartite graph and a tree. Based on this result, families $\mathcal{F}$ satisfying (1.1) which can be written in the form $\mathcal{F} = \{K_n, K_{m_1,m_2}, T\}$ where $n \geq 3$, $2 \leq m_1 \leq m_2$ and T is a tree, have intensively been studied (for a result concerning the case where $|\mathcal{F}| = 4$, we refer the reader to [8]). For $k = 2$, such families are completely characterized in [7]. For $k = 3$, such families are characterized expect for the case where $n = 3$ and $m_1 = m_2 = 2$ (see [2, 4, 6]). This paper is concerned with the case where $n = 3$ and $m_1 = m_2 = 2$ (note that $K_3 = C_3$ and $K_{2,2} = C_4$).

The following conjecture is proposed in [5].

Conjecture 1.1 *Let T be a tree. Then $\mathcal{G}_3(\{C_3, C_4, T\})$ is finite if and only if T is a subgraph of one of $S_9(\{5\}, \emptyset)$, $S_9(\{2\}, \emptyset)$, $S_9(\emptyset, \{3\})$, $S_8(\{2,5\}, \emptyset)$, $S_8(\{7\}, \{3\})$, $S_8(\{4,5,6\}, \{3\})$, $S_8(\{4\}, \{3,6\})$, $S_7(\{2\}, \{4\})$, $S_7(\{3\}, \{4\})$, $S_7(\{4\}, \{3,5\})$, S^* and $S_6(\emptyset, \{3,4\})$.*

The "only if" part of the conjecture is proved in [5]. In this paper, we prove the following theorem as a partial solution of the "if" part.

Theorem 1.2 *The families* $\mathcal{G}_3(\{C_3, C_4, S_9(\{5\}, \emptyset)\})$*,* $\mathcal{G}_3(\{C_3, C_4, S_9(\{2\}, \emptyset)\})$ *and* $\mathcal{G}_3(\{C_3, C_4, S_8(\{2, 5\}, \emptyset)\})$ *are finite families.*

The following lemma is well-known (for a proof, see for example Lemma 1.6 in [2]).

Lemma 1.3 *Let* $m \geq 2$ *and* $k \geq 3$ *be integer, and let* G *be a graph with* $\Delta(G) \leq m$ *and* $\mathrm{diam}(G) \leq k$*. Then* $|V(G)| \leq m^k$*.*

In view of Lemma 1.3, Theorem 1.2 follows from the following four propositions.

Proposition 1.4 *Let* G *be a 3-connected* $\{C_3, C_4, T\}$*-free graph, where* $T = S_9(\{5\}, \emptyset)$*,* $S_9(\{2\}, \emptyset)$ *or* $S_8(\{2, 5\}, \emptyset)$*. Then* $\mathrm{diam}(G) \leq 7$*.*

Proposition 1.5 *Let* G *be a 3-connected* $\{C_3, C_4, S_9(\{5\}, \emptyset)\}$*-free graph. Then* $\Delta(G) < 2 \cdot 10^{41}$*.*

Proposition 1.6 *Let* G *be a 3-connected* $\{C_3, C_4, S_9(\{2\}, \emptyset)\}$*-free graph. Then* $\Delta(G) < 5.5 \cdot 10^{35}$*.*

Proposition 1.7 *Let* G *be a 3-connected* $\{C_3, C_4, S_8(\{2, 5\}, \emptyset)\}$*-free graph. Then* $\Delta(G) \leq 1220$*.*

We remark that it is known that $\mathcal{G}_3(\{C_3, C_4, S_9(\emptyset, \{3\})\})$ and $\mathcal{G}_3(\{C_3, C_4, S_8(\{7\}, \{3\})\})$ are finite families (see [3, 5]). Thus Conjecture 1.1 is reduced to the following conjecture.

Conjecture 1.8 *Let* T *be a tree isomorphic to* $S_8(\{4, 5, 6\}, \{3\})$*,* $S_8(\{4\}, \{3, 6\})$*,* $S_7(\{2\}, \{4\})$*,* $S_7(\{3\}, \{4\})$*,* $S_7(\{4\}, \{3, 5\})$*,* S^* *or* $S_6(\emptyset, \{3, 4\})$*. Then* $\mathcal{G}_3(\{C_3, C_4, T\})$ *is a finite family.*

We prove Proposition 1.4 in Section 2. After preparing auxiliary lemmas in Sections 3 and 4, we prove Propositions 1.5–1.7 in Sections 5–7. In Section 3, we make use of the fact that $R(3, 3) = 6$ and $R(3, 6) = 18$, where $R(s, t)$ demotes the usual Ramsey number, i.e., the minimum positive integer R such that any graph of order at least R contains a complete subgraph of order s or an independent set of cardinality t.

We conclude this section by stating a corollary of a famous theorem of Turán [9]. Let n, k be integers with $n \geq k \geq 1$, and write $n = kq + r$, where q, r are integers and $0 \leq r \leq k - 1$. Turán's theorem shows that if H is a graph of order n and k is the maximum order of a complete subgraph of the complement of H, then $|E(H)| \geq r|E(K_{q+1})| + (k - r)|E(K_q)|$ (see Section 7.1 of [1]). Note that k is the maximum cardinality of an independent set of H. Note also that since $2r|E(K_{q+1})| + 2(k-r)|E(K_q)| = kq^2 + (2r-k)q \geq (kq+r)(q+r/k-1) = n(n/k-1)$, the average degree d of H satisfies $d \geq n/k - 1$, i.e., $(d+1)k \geq n$. Thus we have the following lemma, which we use in Section 3.

Lemma 1.9 *Let H be a graph with average degree d. Then H contains an independent set with cardinality greater than or equal to $n/(d+1)$.*

2 Diameter

For simplicity, we hereafter let $S_9(\{5\}, \emptyset)$, $S_9(\{2\}, \emptyset)$ and $S_8(\{2,5\}, \emptyset)$ be denoted by $T_{9,1}$, $T_{9,2}$ and T_8.

In this section, we prove Proposition 1.4. Thus let $T = T_{9,1}, T_{9.2}$ or T_8, and let G be a 3-connected $\{C_3, C_4, T\}$-free graph and, by way of contradiction, suppose that $\mathrm{diam}(G) \geq 8$. Take $u, v \in V(G)$ with $\mathrm{dist}_G(u, v) = 8$, and let $P = u_1u_2 \cdots u_9$ be a shortest $u - v$ path. For each $i \in \{2, 5, 8\}$, take $a_i \in N_G(u_i) - \{u_{i-1}, u_{i+1}\}$. Assume first that $T = T_{9,1}$. Note that $a_5u_i \notin E(G)$ for each $i \in \{3, 4, 6, 7\}$ because G is $\{C_3, C_4\}$-free. Since P is a shortest $u - v$ path, we also have $a_5u_i \notin E(G)$ for each $i \in \{1, 2, 8, 9\}$. Hence $\{u_1, \dots, u_9, a_5\}$ induces a copy of $T_{9,1}$, which contradicts the assumption that G is $T_{9,1}$-free. In the case where $T = T_{9,2}$, we can similarly get a contradiction because $\{u_1, \dots, u_9, a_2\}$ induces a copy of $T_{9,2}$.

We now assume that $T = T_8$. Since P is a shortest $u - v$ path, we have $a_2a_5 \notin E(G)$ or $a_5a_8 \notin E(G)$. By symmetry, we may assume that $a_2a_5 \notin E(G)$. then arguing as above, we see that $\{u_1, \dots, u_8, a_2, a_5\}$ induces a copy of T_8, which is a contradiction.

3 Paths of order four

Throughout the rest of this paper, we fix a 3-connected $\{C_3,\ C_4\}$-free graph G and, for $u \in V(G)$ and $U \subseteq V(G)$, we write $N(u)$ and $N(U)$ for $N_G(u)$ and $N_G(U)$.

In this section and the following section, we study the relation between induced paths joining two given vertices and the existence of an induced tree. For an integer $k \geq 4$ and two nonadjacent vertices v, w of G, we let

$$M_k^w(v) = \{x \in N(v) \mid \text{there exists an induced } v - w \text{ path } P \text{ of order } k \text{ such that } N_P(v) = \{x\}\}.$$

In the remainder of this section and the following section, we let v, w be nonadjacent vertices of G. In this section, we deal with the case where $M_4^w(v)$ is large. We first consider $T_{9,1}$.

Lemma 3.1 *Suppose that $|M_4^w(v)| \geq 52$. Then G contains an induced copy of $T_{9,1}$.*

Proof. Take $a_1, \dots, a_{52} \in M_4^w(v)$. For each $i \in \{1, \dots, 52\}$, let va_ib_iw be an induced $v - w$ path. Since G is $\{C_3, C_4\}$-free, $\{a_i, b_i\} \cap \{a_j, b_j\} = \emptyset$ for any i, j with $i \neq j$, and

$$E(G[\{v, w\} \cup \{a_i, b_i | 1 \leq i \leq 52\}]) = \{va_i, a_ib_i, b_iw | 1 \leq i \leq 52\}. \tag{3.1}$$

For each $i \in \{1, \dots, 52\}$, take $x_i \in N(a_i) - \{v, b_i\}$. By (3.1), $\{x_i | 1 \le i \le 52\} \cap (\{v, w\} \cup \{a_i, b_i | 1 \le i \le 52\}) = \emptyset$. Since G is $\{C_3, C_4\}$-free, $x_1, \dots, x_{52}$ are distinct,

$$x_i a_j \notin E(G) \text{ for any } i, j \in \{1, \dots, 52\} \text{ with } i \ne j, \tag{3.2}$$

and

$$x_i v, x_i w, x_i b_i \notin E(G) \text{ for every } i \in \{1, \dots, 52\}. \tag{3.3}$$

Now let D be the digraph on $\{1, \dots, 52\}$ obtained by joining i to j ($i \ne j$) if and only if $x_i b_j \in E(G)$, and let H be the (simple) graph obtained by ignoring the direction of the edges of D. Since G is $\{C_3, C_4\}$-free, each $i \in \{1, \dots, 52\}$ has outdegree at most one in D. Hence $|E(H)| \le |V(H)|$, which means that the average degree of H is at most two. In view of Lemma 1.9, we may assume that $\{1, \dots, 18\}$ is independent in H. Then by (3.2) and (3.3),

$$E_G(\{x_i \mid 1 \le i \le 18\},\ \{v, w\} \cup \{a_i, b_i \mid 1 \le i \le 18\}) = \{x_i a_i \mid 1 \le i \le 18\}. \tag{3.4}$$

Note that $R(3, 6) = 18$. Since G is C_3-free, we may assume that

$$\{x_i \mid 1 \le i \le 6\} \text{ is independent.} \tag{3.5}$$

For each $i \in \{1, \dots, 6\}$, since G is $\{C_3, C_4\}$-free, we can take $z_i \in N(x_i) - \{a_i\}$ so that $z_i w \notin E(G)$. By (3.4) and (3.5),

$$\{z_i \mid 1 \le i \le 6\} \cap (\{v, w\} \cup \{a_i, b_i \mid a \le i \le 18\} \cup \{x_i \mid 1 \le i \le 6\}) = \emptyset.$$

Since G is $\{C_3, C_4\}$-free, we have

$$z_i v, z_i w, z_i a_i \notin E(G) \text{ for every } i \in \{1, \dots, 6\}. \tag{3.6}$$

Assume for the moment that some two indices in $\{1, \dots, 6\}$, say 1 and 2, satisfy $z_1 x_2 \in E(G)$. Since G is $\{C_3, C_4\}$-free, $|N(z_1) \cap \{a_7, \dots, a_{10}\}| \le 1$ and $|N(z_1) \cap \{b_7, \dots, b_{10}\}| \le 1$. We may assume that $z_1 a_7, z_1 b_7, z_1 a_8, z_1 b_9 \notin E(G)$. Then by (3.1), (3.4), (3.5) and (3.6), $\{x_2, z_1, x_1, a_1, v, a_8, a_7, b_7, w, b_9\}$ induces a copy of $T_{9,1}$, as desired (see Figure 2).

Thus we may assume that

$$z_i x_j \notin E(G) \text{ for any } i, j \in \{1, \dots, 6\} \text{ with } i \ne j. \tag{3.7}$$

This in particular implies that $z_1, \dots, z_6$ are distinct. Since G is $\{C_3, C_4\}$-free, $|N(z_1) \cap \{a_3, \dots, a_6\}| \le 1$, $|N(z_1) \cap \{b_3, \dots, b_6\}| \le 1$, $|N(z_2) \cap \{a_3, \dots a_6\}| \le 1$ and $|N(z_2) \cap \{b_3, \dots, b_6\}| \le 1$. We may assume that

$$z_1 a_3, z_1 b_3, z_2 a_4, z_2 b_4 \notin E(G). \tag{3.8}$$

Since G is $\{C_3, C_4\}$-free, for each $u \in \{z_1, x_1, z_2, x_2, x_3, x_4\}$, we have $|N(u) \cap \{b_{19}, \dots, b_{50}\}| \le 1$. We may assume that

$$E_G(\{z_1, x_1, z_2, x_2, x_3, x_4\}, \{b_{19}, \dots, b_{44}\}) = \emptyset. \tag{3.9}$$

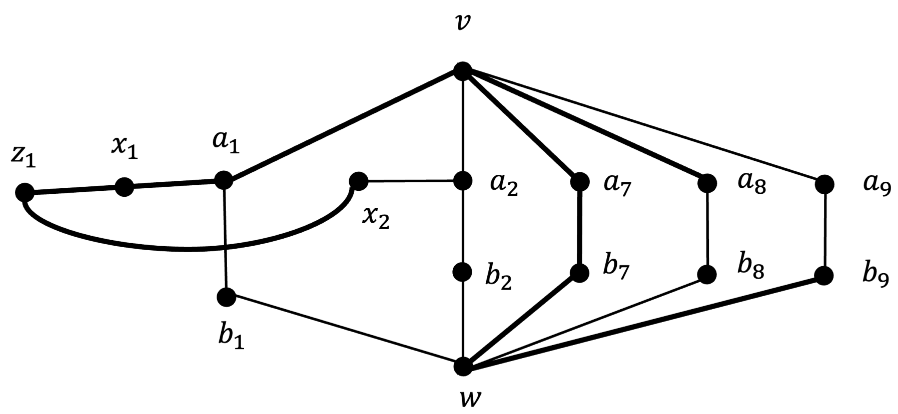

Figure 2: $T_{9,1}$ in Lemma 3.1 (first case).

For each $j \in \{19, \ldots, 44\}$, take $y_j \in N(b_j) - \{w, a_j\}$. By (3.1) and (3.9),

$$\{y_{19}, \ldots, y_{44}\} \cap (\{v, w, z_1, x_1, z_2, x_2, x_3, x_4\} \cup \{a_i, b_i \mid 1 \le i \le 44\}) = \emptyset.$$

Since G is $\{C_3, C_4\}$-free, $y_{19}, \ldots, y_{44}$ are distinct,

$$y_j b_i \notin E(G) \text{ for any } j \in \{19, \ldots, 44\} \text{ and } i \in \{1, \ldots, 44\} \text{ with } i \neq j, \tag{3.10}$$

and

$$y_j v, y_j w \notin E(G) \text{ for every } j \in \{19, \ldots, 44\}. \tag{3.11}$$

If $N(\{z_1, x_1, a_1, x_3, a_3\}) \cap N(\{z_2, x_2, a_2, x_4, a_4\}) \supseteq \{y_{19}, \ldots, y_{44}\}$, then there exist $u \in \{z_1, x_1, a_1, x_3, a_3\}$ and $u' \in \{z_2, x_2, a_2, x_4, a_4\}$ such that $|N(u) \cap N(u') \cap \{y_{19}, \ldots, y_{44}\}| \ge 2$, which contradicts the assumption that G is $\{C_3, C_4\}$-free. Thus

$$N(\{z_1, x_1, a_1, x_3, a_3\}) \cap N(\{z_2, x_2, a_2, x_4, a_4\}) \not\supseteq \{y_{19}, \ldots, y_{44}\}.$$

We may assume that $N(\{z_1, x_1, a_1, x_3, a_3\}) \not\supseteq \{y_{19}, \ldots, y_{44}\}$. We may also assume that $y_{44} \notin N(\{z_1, x_1, a_1, x_3, a_3\})$. It now follows from (3.1) and (3.4) through (3.11) that $\{z_1, x_1, a_1, v, a_3, x_3, b_3, w, b_{44}, y_{44}\}$ induces a copy of $T_{9,1}$ (see Figure 3).

This completes the proof of Lemma 3.1. □

Next we consider $T_{9,2}$.

Lemma 3.2 *Suppose that $|M_4^w(v)| \ge 16$. Then G contains an induced copy of $T_{9,2}$.*

Proof. Take $a_1, \ldots, a_{16} \in M_4^w(v)$. For each $i \in \{1, \ldots, 16\}$, let va_ib_iw be an induced $v - w$ path, and take $x_i \in N(a_i) - \{v_i, b_i\}$. As in the proof of Lemma 3.1, we see that $\{x_i | \ 1 \le i \le 16\} \cap (\{v, w\} \cup \{a_i, \ b_i \mid 1 \le i \le 16\}) = \emptyset$, $x_1, \ldots, x_{16}$ are distinct,

$$E(G[\{v, w\} \cup \{a_i, b_i \mid 1 \le i \le 16\}]) = \{va_i, a_ib_i, b_iw \mid 1 \le i \le 16\}, \tag{3.12}$$

$$x_i a_j \notin E(G) \text{ for any } i, j \in \{1, \ldots, 16\} \text{ with } i \neq j, \tag{3.13}$$

and

$$x_i v, x_i w \notin E(G) \text{ for every } i \in \{1, \ldots, 16\}. \tag{3.14}$$

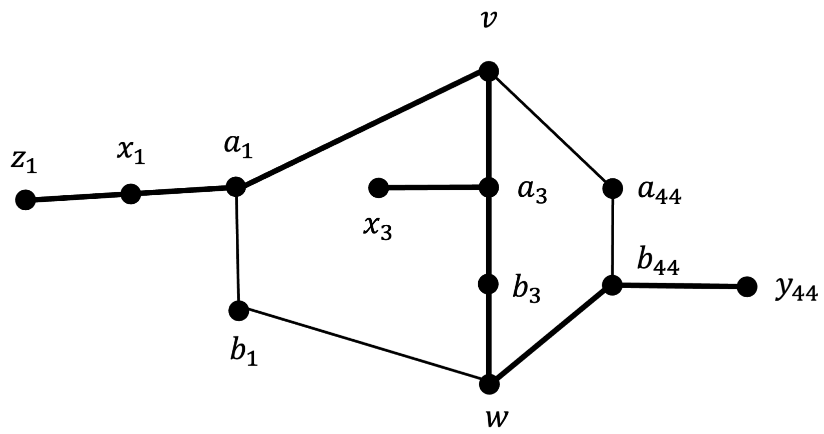

Figure 3: $T_{9,1}$ in Lemma 3.1 (second case).

Claim 3.2.1 *Let $1 \leq i \leq 16$, and suppose that there exist $z \in N(x_i) - \{a_i\}$ and $u \in N(z) - \{x_i\}$ such that $zw, uv, uw \notin E(G)$. Then G contains an induced copy of $T_{9,2}$.*

Proof. We may assume that $i = 1$. By (3.13) and (3.14), $z \notin \{v, w\} \cup \{a_1, \dots, a_{16}\}$. Since $zw, uv, uw \notin E(G)$, we also have $z, u \notin \{b_1, \dots, b_{16}\}$ and $u \notin \{a_1, \dots, a_{16}\}$. Since G is $\{C_3, C_4\}$-free, $zv, za_1, ua_1, ux_1 \notin E(G)$. From $zv, zw \notin E(G)$, we get $u \notin \{v, w\}$. Since G is $\{C_3, C_4\}$-free, $|N(t) \cap \{b_i \mid 2 \leq i \leq 7\}| \leq 1$ for each $t \in \{x_1, z, u\}$. We may assume that $E_G(\{x_1, z, u\}, \{b_2, b_3, b_4\}) = \emptyset$. Similarly $|N(t) \cap \{a_2, a_3, a_4\}| \leq 1$ for each $t \in \{z, u\}$. We may assume that $za_2, ua_2 \notin E(G)$. We now see from (3.12), (3.13) and (3.14) that $\{b_4, w, b_3, b_2, a_2, v, a_1, x_1, z, u\}$ induces a copy of $T_{9,2}$ (see Figure 4).

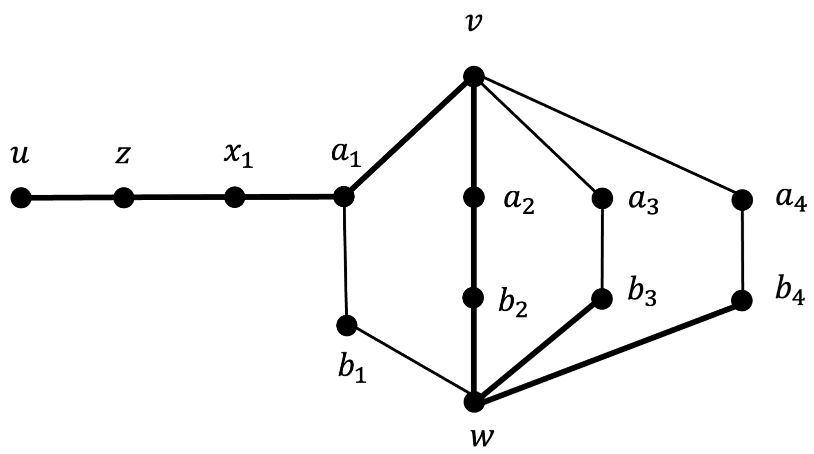

Figure 4: $T_{9,2}$ in Claim 3.2.1.

We return to the proof of the lemma. Since $R(3,3) = 6$, we may assume that $\{x_1, x_2, x_3\}$ is independent. For each $i \in \{1, 2, 3\}$, since G is $\{C_3, C_4\}$-free, we can take $z_i \in N(x_i) - \{a_i\}$ so that $z_i w \notin E(G)$. Then $\{z_1, z_2, z_3\} \cap \{b_1, \dots, b_{16}\} = \emptyset$. By (3.13) and (3.14), $\{z_1, z_2, z_3\} \cap (\{v, w\} \cup \{a_1, \dots, a_{16}\}) = \emptyset$. Since $\{x_1, x_2, x_3\}$ is independent, we also have $\{z_1, z_2, z_3\} \cap \{x_1, x_2, x_3\} = \emptyset$. Since G is $\{C_3, C_4\}$-free, $z_i v \notin E(G)$ for every $i \in \{1, 2, 3\}$. Now if there exist $i, j \in \{1, 2, 3\}$ with $i \neq j$ such that $z_i x_j \in E(G)$, then by (3.14), we can apply claim 3.2.1 with $z = z_i$ and $u = x_j$ to see that G contains an induced copy of $T_{9,2}$. Thus we may assume that $z_i x_j \notin E(G)$ for any $i, j \in \{1, 2, 3\}$ with $i \neq j$ which, in particular, implies that z_1, z_2, z_3 are distinct. Since G is C_3-free, we may assume that $z_1 z_2 \notin E(G)$, and we thus have

$$\{x_1x_2, z_1x_2, z_2x_1, z_1z_2, z_1v, z_2v, z_1w, z_2w, z_1a_1, z_2a_2\} \cap E(G) = \emptyset. \tag{3.15}$$

For each $i \in \{1, 2\}$, take $u_i, u_i' \in N(z_i) - \{x_i\}$ with $u_i \neq u_i'$. By (3.15), $\{u_1, u_1', u_2, u_2'\} \cap \{v, w\} = \emptyset$. By Claim 3.2.1, we may assume that $N(u_i) \cap \{v, w\} \neq \emptyset$ and $N(u_i') \cap \{v, w\} \neq \emptyset$ for each $i \in \{1, 2\}$. Since G is $\{C_3, C_4\}$-free, we also have $|N(v) \cap \{u_i, u_i'\}| \leq 1$ and $|N(w) \cap \{u_i, u_i'\}| \leq 1$ for each $i \in \{1, 2\}$. Thus we may assume that

$$u_i v \notin E(G) \text{ and } u_i w \in E(G) \text{ for each } i \in \{1, 2\}. \tag{3.16}$$

By (3.14), (3.15) and (3.16), $\{u_1, u_2\} \cap \{x_1, x_2, z_1, z_2\} = \emptyset$ (it is possible that $u_1 = u_2$). By (3.16), $\{u_1, u_2\} \cap \{a_1, \dots, a_{16}\} = \emptyset$. Also we may clearly assume that $\{u_1, u_2\} \cap \{b_3, b_4, \dots, b_{14}\} = \emptyset$. Since $u_1 w, u_2 w \in E(G)$ by (3.16), from the assumption that G is $\{C_3, C_4\}$-free, it follows that

$$E_G(\{z_1, z_2\},\ \{b_j \mid 3 \leq j \leq 14\}) = \emptyset \tag{3.17}$$

and

$$E_G(\{u_1, u_2\},\ \{a_j, b_j \mid 3 \leq j \leq 14\}) = \emptyset. \tag{3.18}$$

Also since G is $\{C_3, C_4\}$-free, $|N(x_i) \cap \{b_j \mid 3 \leq j \leq 14\}| \leq 1$ for each $i \in \{1, 2\}$. We may assume that

$$E_G(\{x_1, x_2\},\ \{b_j \mid 3 \leq j \leq 12\}) = \emptyset. \tag{3.19}$$

For each $j \in \{3, \dots, 12\}$, take $y_j \in N(b_j) - \{w, a_j\}$. We have $\{y_j \mid 3 \leq j \leq 12\} \cap (\{v, w\} \cup \{a_i, b_i \mid 1 \leq i \leq 12\}) = \emptyset$. By (3.17), (3.18) and (3.19), $\{y_j \mid 3 \leq j \leq 12\} \cap \{x_1, z_1, u_1, x_2, z_2, u_2\} = \emptyset$. Since G is $\{C_3, C_4\}$-free, $y_3, \dots, y_{12}$ are distinct, and we see from (3.16) that

$$y_j u_1, y_j u_2, y_j v, y_j w \notin E(G) \text{ for every } j \in \{3, \dots, 12\}. \tag{3.20}$$

If $N(\{a_1, x_1, z_1\}) \cap N(\{a_2, x_2, z_2\}) \supseteq \{y_3, \dots, y_{12}\}$. Then there exist $t \in \{a_1, x_1, z_1\}$ and $t' \in \{a_2, x_2, z_2\}$ such that $|N(t) \cap N(t') \cap \{y_3, \dots, y_{12}\}| \geq 2$, which contradicts the assumption that G is $\{C_3, C_4\}$-free. Thus one of $\{a_1, x_1, z_1\}$ and $\{a_2, x_2, z_2\}$, say $\{a_1, x_1, z_1\}$, satisfies $N(\{a_1, x_1, z_1\}) \not\supseteq \{y_3, \dots, y_{12}\}$. We may assume that

$$y_{12} \notin N(\{a_1, x_1, z_1\}). \tag{3.21}$$

Since G is $\{C_3, C_4\}$-free, $|N(z_1)\cap\{a_8, a_9, a_{10}, a_{11}\}| \leq 1$ and $|N(y_{12})\cap\{a_8, a_9, a_{10}, a_{11}\}| \leq 1$. We may assume that $z_1a_{10}, y_{12}a_{10}, z_1a_{11}, y_{12}a_{11} \notin E(G)$. We now see from (3.11) through (3.21) that $\{a_{11}, v, a_{10}, a_1, x_1, z_1, u_1, w, b_{12}, y_{12}\}$ induces a copy of $T_{9,2}$ (see Figure 5).

This completes the proof of Lemma 3.2. □

Finally we consider T_8.

Lemma 3.3 *Suppose that $|M_4^w(v)| \geq 6$. Then G contains a copy of T_8.*

Proof. Take $a_1, \ldots, a_6 \in M_4^w(v)$. For each $i \in \{1, \ldots, 6\}$, let va_ib_iw be an induced $v{-}w$ path. Take $x_1 \in N(a_1){-}\{v, b_1\}$, and take $z_1 \in N(x_1){-}\{a_1\}$ so that $z_1w \notin E(G)$. As in Lemma 3.2, we get $E(G[\{v, w\} \cup \{a_i, b_i|\ 1 \leq i \leq 6\}]) = \{va_i, a_ib_i, b_iw \mid 1 \leq i \leq 6\}, x_1, z_1 \notin \{v, w\} \cup \{a_i, b_i \mid 1 \leq i \leq 6\}$, $x_1a_i \notin E(G)$ for every $i \in \{2, \ldots, 6\}$, and $x_1v, x_1w, z_1v, z_1a_1 \notin E(G)$. Since G is $\{C_3, C_4\}$-free, $|N(x_1) \cap \{b_2. \ldots, b_6\}| \leq 1, |N(z_1) \cap \{a_2, \ldots, a_6\} \leq 1$ and $|N(z_1) \cap \{b_2, \ldots, b_6\}| \leq 1$. Hence $|\{i \in \{2, \ldots, 6\} \mid x_1b_i, z_1a_i, z_1b_i \notin E(G)\}| \geq 2$. We may assume that $x_1b_2, z_1a_2, z_1b_2 \notin E(G)$. Similarly $|\{i \in \{3, \ldots, 6\} \mid x_1b_i, z_1b_i \notin E(G)\}| \geq 2$. We may assume that $x_1b_3, z_1b_3, x_1b_4, z_1b_4 \notin E(G)$. Finally $|\{i \in \{5,\ 6\} \mid z_1a_i \notin E(G)\}| \geq 1$. We may assume that $z_1a_5 \notin E(G)$. It follows that $\{b_4, w, b_3, b_2, a_2, v, a_5, a_1, x_1, z_1\}$ induces a copy of T_8 (see Figure 6). □

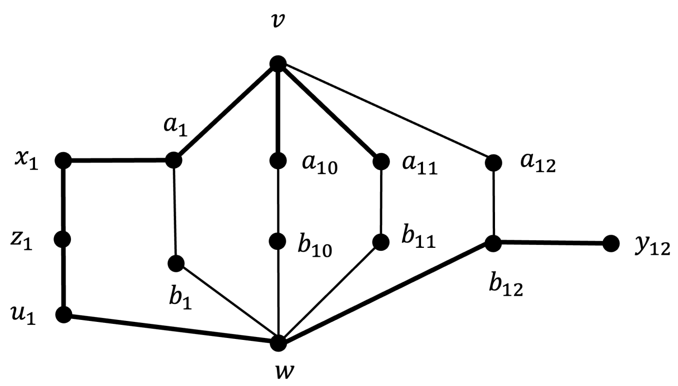

Figure 5: $T_{9,2}$ in Lemma 3.2.

4 Paths of order five

We continue with the notation of the preceding section. In this section, we deal with the case where $M_5^w(v)$ is large.

Lemma 4.1 *Suppose that $|M_4^w(v) \cup M_5^w(v)| \geq 5458$. Then G contains an induced copy of $T_{9,1}$.*

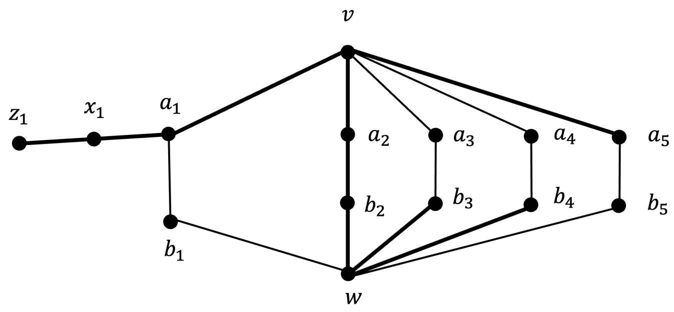

Figure 6: T_8 in Lemma 3.3.

Proof. In view of Lemma 3.1, we may assume that $|M_4^w(v)| \leq 51$. Then $|M_5^w(v) - M_4^w(v)| \geq 5407$. Take $a_1, \dots, a_{5407} \in M_5^w(v) - M_4^w(v)$. For each $i \in \{1, \dots, 5407\}$, let $va_ib_ic_iw$ be an induced $v - w$ path. If there exists $c \in \{c_1, \dots, c_{5407}\}$ such that $|\{i \in \{1, \dots, 5407\} \mid c_i = c\}| \geq 52$, then $|M_4^c(v)| \geq 52$, and hence the desired conclusion follows from Lemma 3.1. Thus we may assume that $|\{i \in \{1, \dots, 5407\} \mid c_i = c\}| \leq 51$ for each $c \in \{c_1, \dots, c_{5407}\}$. Then $|\{c_1, \dots, c_{5407}\}| \geq \lceil 5407/51 \rceil = 107$. We may assume that $c_1, \dots, c_{107}$ are distinct. Since G is $\{C_3, C_4\}$-free, we see that $\{a_i, b_i, c_i\} \cap \{a_j, b_j, c_j\} = \emptyset$ for any $i, j \in \{1, \dots, 107\}$ with $i \neq j$. Since $a_1, \dots, a_{107} \notin M_4^w(v)$, $a_ic_j \notin E(G)$ for any $i, j \in \{1, \dots, 107\}$ with $i \neq j$. Since G is $\{C_3, C_4\}$-free, it follows that

$$\begin{aligned} &E(G[\{v, w\} \cup \{a_i, b_i, c_i \mid 1 \leq i \leq 107\}]) - \{va_i, a_ib_i, b_ic_i, c_iw \mid 1 \leq i \leq 107\} \\ &\quad = E(G[\{b_i \mid 1 \leq i \leq 107\}]). \end{aligned} \tag{4.1}$$

Note that $N(b_1) \cap \{b_i \mid 2 \leq i \leq 107\} \in M_4^w(b_1)$. Thus by Lemma 3.1, we may assume that $|N(b_1) \cap \{b_i \mid 2 \leq i \leq 107\}| \leq 51$. We may assume that

$$b_1b_i \notin E(G) \text{ for every } i \in \{2, \dots, 56\}. \tag{4.2}$$

Take $y_1 \in N(b_1) - \{a_1, c_1\}$. By (4.1) and (4.2), $y_1 \notin \{v, w\} \cup \{a_i, b_i, c_i| \ 1 \leq i \leq 56\}$. Since G is $\{C_3, C_4\}$-free,

$$y_1v, y_1w, y_1a_1, y_1c_1 \notin E(G). \tag{4.3}$$

Note that $b_1 \in M_4^w(y_1)$ and $N(y_1) \cap \{b_i \mid 2 \leq i \leq 56\} \subseteq M_4^w(y_1)$. Thus by Lemma 3.1, we may assume that $|N(y_1) \cap \{b_i \mid 2 \leq i \leq 56\}| \leq 50$. We may assume that

$$y_1b_i \notin E(G) \text{ for every } i \in \{2, \dots, 6\}. \tag{4.4}$$

Since G is $\{C_3, C_4\}$-free, $|N(y_1) \cap \{a_2, \dots, a_6\}| \leq 1$ and $|N(y_1) \cap \{c_2, \dots c_6\}| \leq 1$. We may assume that $y_1a_2, y_1a_3 \notin E(G)$ and $y_1c_4, y_1c_5 \notin E(G)$. Since G is $\{C_3, C_4\}$-free, one of b_2 and b_3, say b_2, and one of b_4 and b_5, say b_4, are nonadjacent. It now follows from (4.1) through (4.4) that $\{b_2, a_2, v, a_1, b_1, y_1, c_1, w, c_4, b_4\}$ induces a copy of $T_{9,1}$, as desired (see Figure 7). □

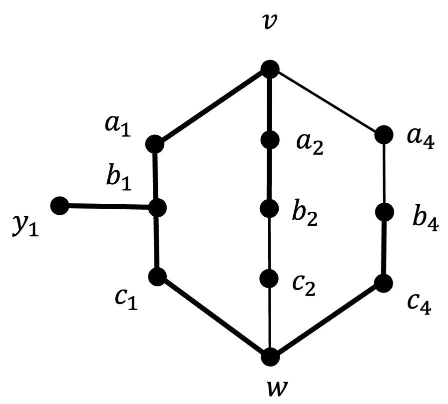

Figure 7: $T_{9,1}$ in Lemma 4.1.

Lemma 4.2 *Suppose that* $|M_4^w(v) \cup M_5^w(v)| \geq 526$. *Then* G *contains an induced copy of* $T_{9,2}$.

Proof. By Lemma 3.2, we may assume that $|M_5^w(v) - M_4^w(v)| \geq 526 - 15 = 511$. Take $a_1, \dots, a_{511} \in M_5^w(v) - M_4^w(v)$. For each $i \in \{1, \dots, 511\}$, let $va_ib_ic_iw$ be an induced $v - w$ path. By Lemma 3.2, we may assume that $|\{c_1, \dots, c_{511}\}| \geq \lceil 511/15 \rceil = 35$. We may assume that $c_1, \dots, c_{35}$ are distinct. As in the proof of Lemma 4.1, we get

$$E(G[\{v, w\} \cup \{a_i, b_i, c_i \mid 1 \leq i \leq 35\}]) - \{va_i, a_ib_i, b_ic_i, c_iw \mid 1 \leq i \leq 35\} = E(G[\{b_i \mid 1 \leq i \leq 35\}]). \qquad (4.5)$$

Take $x_1 \in N(a_1) - \{v, b_1\}$. By (4.5), $x_1 \notin \{v, w\} \cup \{a_i, b_i, c_i \mid 1 \leq i \leq 35\}$. Since $a_1 \notin M_4^w(v)$,

$$x_1w \notin E(G). \qquad (4.6)$$

Since G is $\{C_3, C_4\}$-free,

$$x_1a_i \notin E(G) \text{ for every } i \in \{2, \dots, 35\} \text{ and } x_1v, x_1b_1 \notin E(G). \qquad (4.7)$$

In view of Lemma 3.2, we may assume that $|N(b_1) \cap \{b_2, \dots, b_{35}\}| \leq 15$ and $|N(x_1) \cap \{b_2, \dots, b_{35}\}| \leq 15$. We may assume that

$$b_1b_i, x_1b_i \notin E(G) \text{ for every } i \in \{2, 3, 4, 5\}. \qquad (4.8)$$

Since G is $\{C_3, C_4\}$-free, $|N(x_1) \cap \{c_2, c_3, c_4, c_5\}| \leq 1$. We may assume that x_1c_2, x_1c_3, $x_1c_4 \notin E(G)$. Since G is C_3-free, we may assume that $b_2b_3 \notin E(G)$. It now follows from (4.5) through (4.8) that $\{b_1, a_1, x_1, v, a_2, b_2, c_2, w, c_3, b_3\}$ induces a copy of $T_{9,2}$ (see Figure 8). □

Lemma 4.3 *Suppose that* $|M_4^w(v) \cup M_5^w(v)| \geq 61$. *Then* G *contains an induced copy of* T_8.

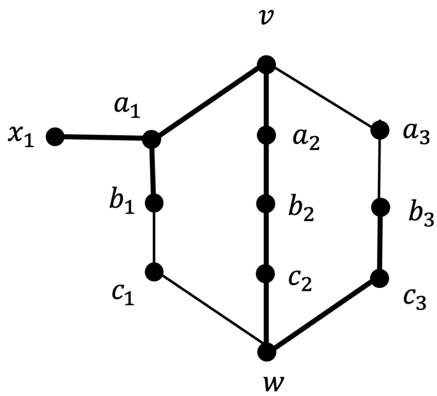

Figure 8: $T_{9,2}$ in Lemma 4.2.

Proof. By Lemma 3.3, we may assume that $|M_5^w(v) - M_4^w(v)| \geq 61 - 5 = 56$. Take $a_1, \ldots, a_{56} \in M_5^w(v) - M_4^w(v)$. For each $i \in \{1, \ldots, 56\}$, let $va_ib_ic_iw$ be an induced $v - w$ path. By Lemma 3.3, we may assume that $|\{c_1, \ldots, c_{56}\}| \geq \lceil 56/5 \rceil = 12$. We may assume that $c_1, \ldots, c_{12}$ are distinct. As in Lemma 4.1, we get $E(G[\{v, w\} \cup \{a_i, b_i, c_i \mid 1 \leq i \leq 12\}]) - \{va_i, a_ib_i, b_ic_i, c_iq \mid 1 \leq i \leq 12\} = E(G[\{b_i| \; 1 \leq i \leq 12\}])$. Take $x_1 \in N(a_1) - \{v, b_1\}$. As in Lemma 4.2, $x \notin \{v, w\} \cup \{a_i, b_i, c_i| \; 1 \leq i \leq 12\}$, $x_1w, x_1v, x_1b_1, x_1c_1 \notin E(G)$, and $x_1a_i \notin E(G)$ for every $i \in \{2, \ldots, 12\}$. In view of Lemma 3.3, we may assume that $|N(b_1) \cap \{b_i \mid 2 \leq i \leq 12\}| \leq 5$ and $|N(x_1) \cap \{b_i \mid 2 \leq i \leq 12\}| \leq 5$. We may assume that $b_1b_2, x_1b_2 \notin E(G)$. Since G is $\{C_3, C_4\}$-free, $|N(x_1) \cap \{c_3, c_4, c_5\}| \leq 1$. We may assume that $x_1c_3, x_1c_4 \notin E(G)$. It follows that $\{c_4, w, c_3, c_1, b_1, a_1, x_1, v, a_2, b_2\}$ induces a copy of T_8 (see Figure 9). □

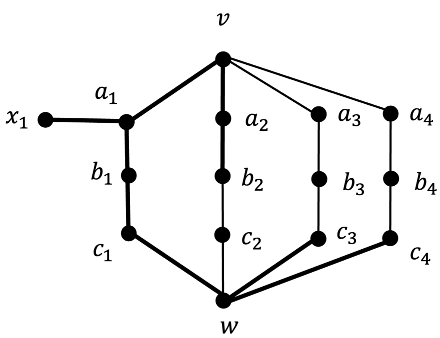

Figure 9: T_8 in Lemma 4.3.

5 Proof of Proposition 1.5

Recall that G is a 3-connected $\{C_3, C_4\}$-free graph. Throughout the rest of this paper, we fix a vertex $w \in V(G)$ with $\deg_G(w) = \Delta(G)$.

For a vertex $u \in V(G)$ and a nonnegative integer d, let $N_d(u)$ be the set of vertices of G such that $\mathrm{dist}_G(u,\ v) = d$, and let $N_{\leq d}(u) = \cup_{0\leq i\leq d}\ N_i(u)$ and $N_{\geq d}(u) = \cup_{i\geq d}\ N_i(u)$; thus $N_0(u) = \{u\}$ and $N_1(u) = N(u)$. Clearly $N(w)$ is independent, $|N(x) \cap N_2(w)| \geq 2$ for every $x \in N(w)$, and

$$|N(y) \cap N(w)| \leq 1 \text{ for every } y \in N_{\geq 2}(w). \tag{5.1}$$

We have $\delta(G[N_{\geq 2}(w)]) \geq 2$ by (5.1). As in [3], for $U \subseteq V(G)$, we let $L(U)$ denote the set of those vertices $v \in N_2(w) \cup N_3(w)$ for which there exists a $v - w$ path of order four avoiding U. The following two lemmas are proved in Section 5 of [3].

Lemma 5.1 *Let $X \subseteq N_{\geq 2}(w)$, and set $Y_1 = (X \cup N(X)) \cap N_2(w)$, $Y_2 = N(Y_1) \cap N(w)$, $Z_1 = N(X) \cap L(X)$, $Z_2 = (X \cup N(X \cup Z_1)) \cap N_2(w)$ and $Z_3 = N(Z_2) \cap N(w)$. Then the following hold.*

(i) If $a \in N(w) - Y_2$, then $E_G(X,\ N_{\leq 1}(a)) = \emptyset$.

(ii) If $a \in N(w) - Z_3$, then $E_G(X,\ N_{\leq 2}(a) - Z_3) = \emptyset$.

Lemma 5.2 *Let $X \subseteq N_{\geq 2}(w)$. Then $N(X) \cap N_2(w) \subseteq \cup_{u\in X} M_4^w(u)$ and $N(X) \cap L(X) \subseteq \cup_{u\in X} M_5^w(u)$.*

The following lemma follows from Lemma 5.2 and (5.1), and is virtually the same as Lemma 5.3 in [3].

Lemma 5.3 *Let $X \subseteq N_{\geq 2}(w)$, and let Y_1, Y_2, Z_1, Z_2, Z_3 be as in Lemma 5.2. Then*

(i) $|Y_2| \leq |Y_1| \leq |X - N(X)| + \Sigma_{u\in X}|M_4^w(u)|$,

(ii) $|Z_1| \leq \Sigma_{u\in X}|M_5^w(u)|$, *and*

(iii) $|Z_3| \leq |Z_2| \leq |X - N(X \cup Z_1)| + \Sigma_{u\in X\cup Z_1}|M_4^w(u)|$.

In this section, we prove Proposition 1.5. By way of contradiction, suppose that G is $T_{9,1}$-free and $\Delta(G) \geq 2 \cdot 10^{41}$. By Lemmas 3.1 and 4.1,

$$|M_4^w(u)| \leq 51 \text{ and } |M_5^w(u)| \leq |M_4^w(u) \cup M_5^w(u)| \leq 5457 \text{ for every } u \in N_{\geq 2}(w). \tag{5.2}$$

We derive a contradiction by proving several claims. The main claim is Claim 5.13, in which we show that $\Delta(G[N_{\geq 2}(w)]) < 5.5 \cdot 10^3$. We start with two claims concerning paths in $G - w$.

Claim 5.4 *Let $a_6a_5a_4a_3a_2a_3'$ be a path in $G[N_{\geq 2}(w)]$ with $a_2 \in N_2(w)$, and write $N(a_2) \cap N(w) = \{a_1\}$. Then $\{a_6a_2, a_6a_3', a_6a_1, a_5a_3', a_5a_1\} \cap E(G) \neq \emptyset$.*

Proof. Set $X = \{a_2, \dots, a_6, a'_3\}$. We have $X - N(X) = \emptyset$. Suppose that $\{a_6a_2, a_6a'_3, a_6a_1, a_5a'_3, a_5a_1\} \cap E(G) = \emptyset$. Then $E(G[X \cup \{a_1\}]) = \{a_6a_5, a_5a_4, a_4a_3, a_3a_2, a_2a'_3, a_2a_1\}$. Let Y_2 be as in Lemma 5.1. By Lemma 5.3 (i) and (5.2), $|Y_2| \leq 6 \cdot 51 < |N(w)|$. Take $b_1 \in N(w) - Y_2$ and $b_2 \in N(b_1) \cap N_{\geq 2}(w)$ (see Figure 10). Since G is $\{C_3, C_4\}$-free, $E(G[\{a_1, w, b_1, b_2\}]) = \{a_1w, wb_1, b_1b_2\}$. Since $E_G(X, \{w,\ b_1,\ b_2\}) = \emptyset$ by Lemma 5.1 (i), it follows that $X \cup \{a_1, w, b_1\ b_2\}$ induces a copy of $T_{9,1}$, which contradicts the assumption that G is $T_{9,1}$-free. □

Claim 5.5 *Let $a_5a_4a_3a_2a_1a'_2$ be a path in $G - w$ with $a_1 \in N(w)$ and $a_2, a_3, a_4, a_5, a'_2 \in N_{\geq 2}(w)$. Then $\{a_5a_1, a_5a'_2, a_4a'_2\} \cap E(G) \neq \emptyset$*

Proof. Set $X = \{a_2, \dots, a_5, a'_2\}$, and let Z_1, Z_3 be as in Lemma 5.1. We have $X - N(X \cup Z_1) \subseteq X - N(X) \subseteq \{a'_2\}$. Suppose that $\{a_5a_1, a_5a'_2, a_4a'_2\} \cap E(G) = \emptyset$. Then $E(G[X \cup \{a_1\}]) = \{a_5a_4, a_4a_3, a_3a_2, a_2a_1, a_1a'_2\}$. By Lemma 5.3 (ii), (iii) and (5.2), $|Z_1| \leq 5 \cdot 5457 = 27285$ and $|Z_3| \leq 1 + 27290 \cdot 51 < |N(w)|$. Take $b_1 \in$

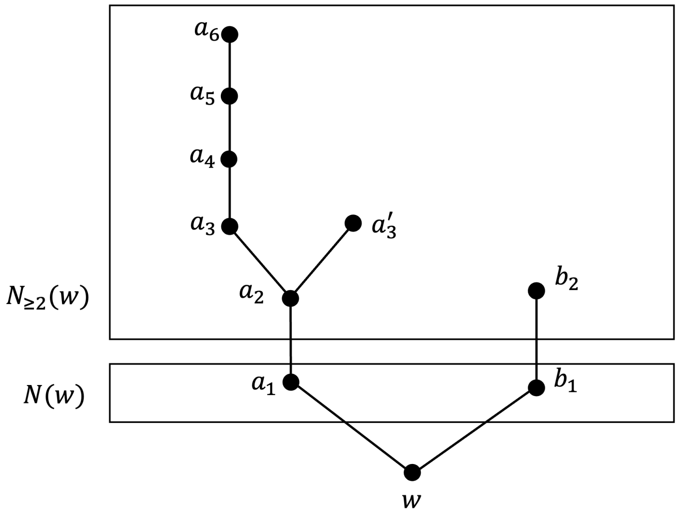

Figure 10: $T_{9,1}$ in Claim 5.4.

$N(w) - Z_3$ and $b_2 \in N(b_1) \cap N_{\geq 2}(w)$. Since $|N(b_2) \cap N_{\geq 2}(w)| \geq 2$, we can take $b_3 \in N(b_2) \cap N_{\geq 2}(w)$ so that $a_1b_3 \notin E(G)$ (see Figure 11). Then $E(G[\{a_1, w, b_1, b_2, b_3\}]) = \{a_1w, wb_1, b_1b_2, b_2b_3\}$. Since $E_G(X, \{w, b_1, b_2, b_3\}) = \emptyset$ by Lemma 5.1 (ii), it follows that $X \cup \{a_1, w, b_1, b_2, b_3\}$ induces a copy of $T_{9,1}$, a contradiction. □

Claim 5.6 *We have $N_{\geq 4}(w) = \emptyset$.*

Proof. Suppose that $N_{\geq 4}(w) \neq \emptyset$. Take $u \in N_4(w)$, and take $z, z', z'' \in N(u)$ so that $z \in N_3(w)$. Take $y \in N(z) \cap N_2(w)$, write $N(y) \cap N(w) = \{x\}$, and take $y' \in (N(x) \cap N_2(w)) - \{y\}$. Since $z', z'' \in N_{\geq 3}(w)$ and $u \in N_4(w)$, we have $z'x, z''x \notin E(G)$ and $uy' \notin E(G)$. Since $z', z'' \in N(u)$, $|N(y') \cap \{z', z''\}| \leq 1$.

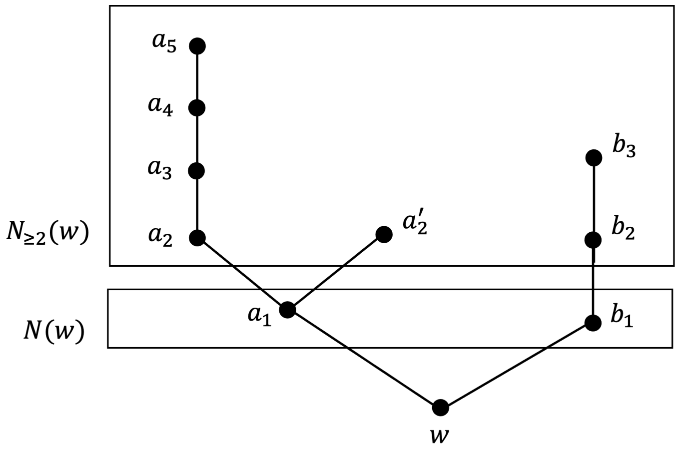

Figure 11: $T_{9,1}$ in Claim 5.5.

Consequently we get a contradiction by applying Claim 5.5 to $z'uzyxy'$ or $z''uzyxy'$. □

Set

$$R_1 = \{z \in N_3(w) \mid |N(z) \cap N_2(w)| = 1\},$$
$$R_2 = \{z \in N_3(w) \mid |N(z) \cap N_2(w)| \geq 2\}.$$

For the purpose of showing that $\Delta(G[N_{\geq 2}(w)]) < 5.5 \cdot 10^3$, we aim at bounding $|N(y) \cap R_1|$ for $y \in N_2(w)$ (Claim 5.11). We here prove a technical claim.

Claim 5.7 *Let $z \in R_1$ and $z' \in N_3(w)$ with $zz' \in E(G)$. Write $N(z) \cap N_2(w) = \{y\}$, let $y' \in N(z') \cap N_2(w)$, and write $N(y) \cap N(w) = \{x\}$ and $N(y') \cap N(w) = \{x'\}$. Then $x = x'$.*

Proof. Since G is C_3-free, $y \neq y'$. Suppose that $x \neq x'$. Then $yx' \notin E(G)$. Take $y'' \in (N(x') \cap N_2(w)) - \{y'\}$. Since $yx' \notin E(G)$, $y'' \neq y$. Since $z \in R_1$, $zy'' \notin E(G)$. Since $z \in R_1$, we can take $z'' \in N(z) \cap N_3(w)$ with $z'' \neq z'$. Since $x' \in N(w)$, $z''x' \notin E(G)$. Consequently $\{z''x', yx', zy''\} \cap E(G) = \emptyset$. Since $y, z'' \in N(z)$, $|N(y'') \cap \{y, z''\}| \leq 1$. Therefore we get a contradiction by applying Claim 5.5 to $yzz'y'x'y''$ or $z''zz'y'x'y''$. □

Using Claim 5.7, we obtain the following three claims.

Claim 5.8 *Let H be a component of $G[R_1]$. Then $N(V(H)) \cap N_2(w)$ is independent and $|N(N(V(H)) \cap N_2(w)) \cap N(w)| = 1$.*

Proof. Since H is connected, it follows from Claim 5.7 that $N(y) \cap N(w) = N(y') \cap N(w)$ for all $y, y' \in N(V(H)) \cap N_2(w)$, which implies that $|N(N(V(H)) \cap N_2(w)) \cap N(w)| = 1$. Since G is C_3-free, it follows that $N(V(H)) \cap N_2(w)$ is independent. □

Claim 5.9 *We have* $\Delta(G[R_1]) \leq 2$.

Proof. Suppose that there exists $z_0 \in R_1$ with $\deg_{G[R_1]}(z_0) \geq 3$, and take three distinct vertices z, z', z'' in $N(z_0) \cap R_1$. Write $N(z) \cap N_2(w) = \{y\}$, $N(z') \cap N_2(w) = \{y'\}$ and $N(y) \cap N(w) = \{x\}$. Since G is $\{C_3, C_4\}$-free, $y \neq y'$. By Claim 5.8, $y'x \in E(G)$. Since $z'' \in N_3(w)$ and $x \in N(w)$, $z''x \notin E(G)$. Since G is $\{C_3, C_4\}$-free, $z''y', z_0y' \notin E(G)$. Consequently we get a contradiction by applying Claim 5.5 to $z''z_0zyxy'$. □

Claim 5.10 *We have* $E_G(R_1, R_2) = \emptyset$.

Proof. Suppose that there exist $z \in R_1$ and $z' \in R_2$ with $zz' \in E(G)$. Write $N(z) \cap N_2(w) = \{y\}$ and $N(y) \cap N(w) = \{x\}$, and take $y', y'' \in N(z') \cap N_2(w)$ with $y' \neq y''$. By Claim 5.7, $y'x, y''x \in E(G)$, which contradicts the assumption that G is $\{C_3, C_4\}$-free. □

Since $\deg_G(z) \geq 3$ for all $z \in R_1$, it follows from Claims 5.6, 5.9 and 5.10 that each component of $G[R_1]$ is a cycle, and is a component of $G[N_3(w)] = G[N_{\geq 3}(w)]$. The following claim is a key result in bounding $\Delta(G[N_{\geq 2}(w)])$.

Claim 5.11 *For every* $y \in N_2(w)$, $|N(y) \cap R_1| \leq 6$.

Proof. Suppose that there exists $y_1 \in N_2(w)$ such that $|N(y_1) \cap R_1| \geq 7$. We derive a contradiction by proving several subclaims concerning components of $G[R_1]$.

Subclaim 5.11.1 *Let H be a component of $G[R_1]$ with $N(y_1) \cap V(H) \neq \emptyset$. Then* $|V(H)| \equiv 0 \pmod 3$, *and we can write $H = z_1z_2 \dots z_{|V(H)|}z_1$ so that $N(z_j) \cap N_2(w) = \{y_1\}$ for every j with* $j \equiv 1 \pmod 3$.

Proof. Take $u_1 \in N(y_1) \cap V(H)$ and $u_2 \in N(u_1) \cap V(H)$, and write $(N(u_2) \cap V(H)) - \{u_1\} = \{u_3\}$ and $(N(u_3) \cap V(H)) - \{u_2\} = \{u_4\}$. Suppose that $u_4y_1 \notin E(G)$. Since H is a cycle and $|(N(y_1) \cap R_1) - \{u_1\}| \geq 7 - 1 \geq 2$, there exists $z \in (N(y_1) \cap R_1) - \{u_1\}$ with $u_3z, u_4z \notin E(G)$. Write $N(y_1) \cap N(w) = \{x\}$. Since $u_3, u_4 \in N_3(w)$ and $x \in N(w)$, $u_3x, u_4x \notin E(G)$. Consequently we get a contradiction by applying Claim 5.4 to $u_4u_3u_2u_1y_1z$. Thus $u_4y_1 \in E(G)$. Since $u_1 \in N(y_1) \cap V(H)$ is arbitrary, this implies the desired conclusion. □

Subclaim 5.11.2 *Let H be a component of $G[R_1]$ with $N(y_1) \cap V(H) \neq \emptyset$, and write $|V(H)| = 3m$. Then $N(V(H)) \cap N_2(w)$ is an independent set of cardinality 3, and we can write $H = z_1z_2 \dots z_{3m}z_1$ and $N(V(H)) \cap N_2(w) = \{y_1, y_2, y_3\}$ so that for each $k \in \{1, 2, 3\}$, $N(z_j) \cap N_2(w) = \{y_k\}$ for every j with* $j \equiv k \pmod 3$.

Proof. By Subclaim 5.11.1, we can write $H = z_1z_2 \dots z_{3m}z_1$ so that $N(z_{3i+1}) \cap N_2(w) = \{y_1\}$ for every $i \in \{0, 1, \dots, m-1\}$. Take $h \in \{0, 1, \dots m-1\}$, and write $N(z_{3h+2}) \cap N_2(w) = \{y_2\}$. Then $y_2 \neq y_1$. Suppose that $z_{3h-1}y_2 \notin E(G)$ (indices of z are to be read modulo $3m$). Take $u \in (N(y_2) \cap N_{\geq 2}(w)) - \{z_{3h+2}\}$. We show that

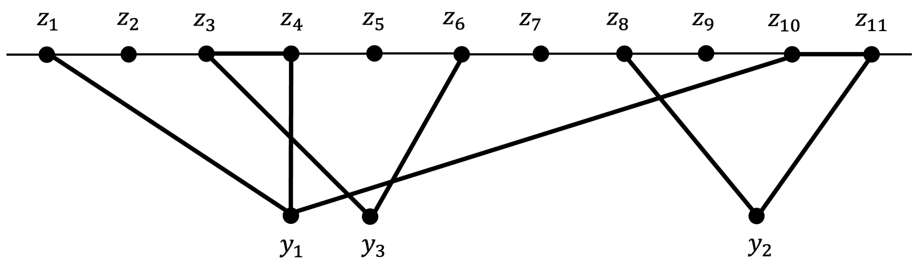

Figure 12: $T_{9,1}$ in Subclaim 5.11.3.

$z_{3h}u, z_{3h-1}u \notin E(G)$. Suppose that $\{z_{3h}u, z_{3h-1}u\} \cap E(G) \neq \emptyset$. If $u \in N_2(w)$, then $u, y_2 \in N(V(H)) \cap N_2(w)$, and hence $uy_2 \notin E(G)$ by Claim 5.8, which contradicts our choice of u. Thus $u \in N_3(w)$. Since $\{z_{3h}u, z_{3h-1}u\} \cap E(G) \neq \emptyset$, this forces $u \in \{z_{3h+1}, z_{3h}, z_{3h-1}, z_{3h-2}\}$. Since G is $\{C_3, C_4\}$-free and $z_{3h-1}y_2 \notin E(G)$, it follows that $u = z_{3h-2}$, which contradicts the fact that $N(z_{3h-2}) \cap N_2(w) = \{y_1\}$. Consequently $z_{3h}u, z_{3h-1}u \notin E(G)$. Write $N(y_2) \cap N(w) = \{x\}$. Since $z_{3h}, z_{3h-1} \in N_3(w)$ and $x \in N(w)$, $z_{3h}x, z_{3h-1}x \notin E(G)$. Therefore we get a contradiction by applying Claim 5.4 to $z_{3h-1}z_{3h}z_{3h+1}z_{3h+2}y_2u$. Thus $z_{3h-1}y_2 \in E(G)$. Since h is arbitrary, this implies that $N(z_{3i+2}) \cap N_2(w) = \{y_2\}$ for every $i \in \{0, 1, \ldots, m-1\}$. Similarly if we write $N(z_3) \cap N_2(w) = \{y_3\}$, then $N(z_{3i+3}) \cap N_2(w) = \{y_3\}$ for every $i \in \{0, 1, \ldots, m-1\}$. Hence $N(V(H)) \cap N_2(w) = \{y_1, y_2, y_3\}$, and $\{y_1, y_2, y_3\}$ is independent by Claim 5.8. □

Subclaim 5.11.3 *Let H be a component of $G[R_1]$ with $N(y_1) \cap R_1 \neq \emptyset$. Then $|V(H)| \leq 9$.*

Proof. Write $|V(H)| = 3m$, and let $H = z_1z_2 \ldots z_{3m}z_1$ and $N(V(H)) \cap W_2(z) = \{y_1, y_2, y_3\}$ be as in Subclaim 5.11.2. Suppose that $|V(H)| \geq 10$. Then $m \geq 4$. Since $\{y_1, y_2, y_3\}$ is independent by Subclaim 5.11.2. $\{z_6, y_3, z_3, z_4, y_1, z_1, z_{10}, z_{11}, y_2, z_8\}$ induces a copy of $T_{9,1}$, a contradiction (see Figure 12). □

We can now complete the proof of Claim 5.11. Since $|N(y_1) \cap R_1| \geq 7$, it follows from Subclaim 5.11.3 that there exist three distinct components of $G[R_1]$ intersecting with $N(y_1)$. Let H, H', H'' be such components. Write $|V(H)| = 3m$ and $|V(H')| = 3m'$ ($m, m' \in \{2, 3\}$). Let $H = z_1z_2 \ldots z_{3m}z_1$ and $N(V(H)) \cap N_2(w) = \{y_1, y_2, y_3\}$ be as in Subclaim 5.11.2, and also let $H' = z'_1z'_2 \ldots z'_{3m'}z'_1$ and $N(V(H')) \cap N_2(w) = \{y_1, y'_2, y'_3\}$ be as in Subclaim 5.11.2 (it is possible that $\{y_2, y_3\} \cap \{y'_2, y'_3\} \neq \emptyset$). Write $N(y_1) \cap N(w) = \{x\}$. Applying Claim 5.8 to H and H', we see that $\{y_1, y_2, y_3, y'_2, y'_3\} \subseteq N(x)$. Hence $\{y_1, y_2, y_3, y'_2, y'_3\}$ is independent. We have $y_2 \neq y'_2$ or $y_2 \neq y'_3$. Replacing y'_2 by y'_3 and $z'_1z'_2 \ldots z'_{3m'}z'_1$ by $z'_1z'_{3m'}z'_{3m'-1} \ldots z'_2z'_1$ if necessary, we may assume that $y_2 \neq y'_2$. Take $z \in N(y_1) \cap V(H'')$. We now see that $\{z_5, y_2, z_2, z_1, y_1, z, z'_1, z'_2, y'_2, z'_5\}$ induces a copy of $T_{9,1}$ (see Figure 13). This is a

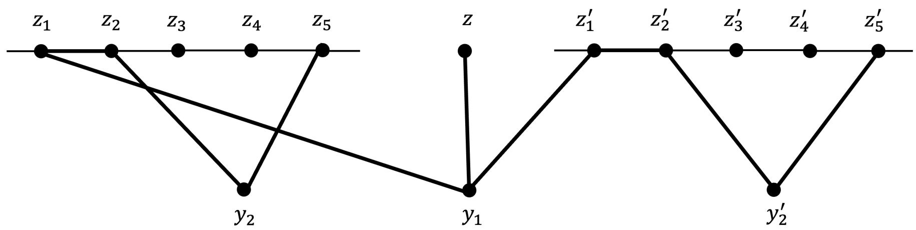

Figure 13: $T_{9,1}$ in Claim 5.11.

contradiction, which completes the proof of Claim 5.11. □

Recall that for $y \in N_{\geq 2}(w)$, $L(\{y\})$ denotes the set of those vertices $v \in N_2(w) \cup N_3(w)$ for which there exists a $v - w$ path of order four avoiding y.

Claim 5.12 *For every $y \in N_2(w)$, $|N(y) \cap (N_2(w) \cup R_2)| \leq 5457$.*

Proof. Let $y \in N_2(w)$. By the definition of $L(\{y\})$ and R_2, $R_2 \subseteq L(\{y\})$. Hence $N(y) \cap N_2(w) \subseteq M_4^w(y)$ and $N(y) \cap R_2 \subseteq M_5^w(y)$ by Lemma 5.2. Consequently $|N(y) \cap (N_2(w) \cup R_2)| \leq |M_4^w(y) \cup M_5^w(y)| \leq 5457$ by (5.2). □

Claim 5.13 *We have $\Delta(G[N_{\geq 2}(w)]) < 5.5 \cdot 10^3$.*

Proof. Let $y \in N_{\geq 2}(w)$. If $y \in N_2(w)$, then it follows from Claims 5.11 and 5.12 that $\deg_{G[N_{\geq 2}(w)]}(y) = |N(y) \cap R_1| + |N(y) \cap (N_2(w) \cup R_2)| \leq 6 + 5457 < 5.5 \cdot 10^3$; if $y \in N_3(w)$, then $N(y) \cap N_3(w) \subseteq N(y) \cap L(\{y\})$ by the definition of $L(\{y\})$, and hence $\deg_{G[N_{\geq 2}(w)]}(y) = |N(y) \cap (N_2(w) \cup N_3(w))| \leq |M_4^w(y) \cup M_5^w(y)| \leq 5457 < 5.5 \cdot 10^3$ by Lemma 5.2 and (5.2). Since $y \in N_{\geq 2}(w)$ is arbitrary, we obtain $\Delta(G[N_{\geq 2}(w)]) < 5.5 \cdot 10^3$, as desired. □

We proceed to consider components of $G[N_{\geq 2}(w)]$.

Claim 5.14 *If H and H' are components of $G[N_{\geq 2}(w)]$, then we have $N(V(H)) \cap N(w) \subseteq N(V(H')) \cap N(w)$ or $N(V(H')) \cap N(w) \subseteq N(V(H)) \cap N(w)$.*

Proof. Suppose that there exist components H, H' of $G[N_{\geq 2}(w)]$ such that $(N(V(H)) - N(V(H'))) \cap N(w) \neq \emptyset$ and $(N(V(H')) - N(V(H))) \cap N(w) \neq \emptyset$, and let $a_1 \in (N(V(H)) - N(V(H'))) \cap N(w)$ and $b_1 \in (N(V(H')) - N(V(H))) \cap N(w)$. Take $a_2 \in N(a_1) \cap V(H)$ and $b_2 \in N(b_1) \cap V(H')$. Since $\delta(G[N_{\geq 2}(w)]) \geq 2$, there exist $a_3, a_4 \in V(H) - \{a_2\}$ with $a_3 \neq a_4$ such that $a_2a_3, a_3a_4 \in E(G)$. Since G is $\{C_3, C_4\}$-free and $a_4 \in N_{\geq 2}(w)$, it follows that $wa_1a_2a_3a_4$ is an induced path. Similarly there exist b_3, $b_4 \in V(H')$ such that $wb_1b_2b_3b_4$ is an induced path. We have $|N(\{a_2, a_3, a_4, b_2, b_3, b_4\}) \cap N(w)| \leq 6$ by (5.1). Take $c \in N(w) - N(\{a_2a_3, a_4, b_2, b_3, b_4\})$. Then $\{a_4, a_3, a_2, a_1, w, c, b_1, b_2, b_3, b_4\}$ induces a copy of $T_{9,1}$, a contradiction. □

Claim 5.15 *There exists a component H of $G[N_{\geq 2}(w)]$ such that $|V(H)| \geq 2 \cdot 10^{41}$.*

Proof. Since $N(N_2(w)) \supseteq N(w)$, we see from Claim 5.14 that there exists a component H of $G[N_{\geq 2}(w)]$ such that $N(V(H)) \supseteq N(w)$. By (5.1), it follows that $|V(H)| \geq |N(w)| \geq 2 \cdot 10^{41}$. □

Let H be as in Claim 5.15. In view of Lemma 1.3, it follows from Claim 5.13 that if $\mathrm{diam}(H) \leq 11$, then $|V((H)| < (5.5 \cdot 10^3)^{11} < 2 \cdot 10^{41}$, a contradiction. Thus $\mathrm{diam}(H) \geq 12$. Take $y, y' \in V(H)$ with $\mathrm{dist}_H(y, y') = 12$, and let $P = y_1 y_2 \ldots y_{13}$ be a shortest $y - y'$ path in H.

Claim 5.16 *We have $y_5, y_6, y_7 \in N_2(w)$.*

Proof. Suppose that $y_i \in N_3(w)$ for some $i \in \{5, 6, 7\}$, and take $z \in N(y_i) - \{y_{i-1}, y_{i+1}\}$. Then $z \in N_{\geq 2}(w)$. Since G is $\{C_3, C_4\}$-free and P is a shortest $y - y'$path in $G[N_{\geq 2}(w)]$, it follows that $\{y_{i-4}, y_{i-3}, \ldots, y_{i+4}, z\}$ induces a copy of $T_{9,1}$, a contradiction. □

Claim 5.17 *Let $5 \leq i \leq 9$. Suppose that $y_i \in N_2(w)$, and write $N(y_i) \cap N(w) = \{x\}$. Then $y_{i-3}, y_{i+3} \in N(x)$.*

Proof. If $N(x) \cap \{y_{i-4}, y_{i-3}, y_{i+3}, y_{i+4}\} = \emptyset$, then $\{y_{i-4}, y_{i-3}, \ldots, y_{i+4}, x\}$ induces a copy of $T_{9,1}$, a contradiction, Thus $N(x) \cap \{y_{i-4}, y_{i-3}, y_{i+3}, y_{i+4}\} \neq \emptyset$. Suppose that $N(x) \cap \{y_{i-3}, y_{i+3}\} = \emptyset$. Then $N(x) \cap \{y_{i-4}, y_{i+4}\} \neq \emptyset$. We may assume that $xy_{i+4} \in E(G)$. Since P is an induced path, $y_{i-3}y_{i+4}, y_{i-2}y_{i+4} \notin E(G)$. Hence we get a contradiction by applying Claim 5.5 to $y_{i-3}y_{i-2}y_{i-1}y_i x y_{i+4}$. Thus $N(x) \cap \{y_{i-3}, y_{i+3}\} \neq \emptyset$. Now if $xy_{i-3} \notin E(G)$, then $xy_{i+3} \in E(G)$ and, since $y_{i-3}y_{i+3}, y_{i-2}y_{i+3} \notin E(G)$, we get a contradiction by applying Claim 5.5 to $y_{i-3}y_{i-2}y_{i-1}y_i x y_{i+3}$. Thus $xy_{i-3} \in E(G)$. By symmetry, we also obtain $xy_{i+3} \in E(G)$, as desired. □

We are now in a position to complete the proof of Proposition 1.5. Having Claim 5.16 in mind, write $N(y_5) \cap N(w) = \{x_2\}$, $N(y_6) \cap N(w) = \{x_3\}$ and $N(y_7) \cap N(w) = \{x_1\}$. Then it follows from Claim 5.17 that for each $k \in \{0, 1, 2\}$, $N(y_i) \cap N(w) = \{x_k\}$ for every $i \in \{2, 3, \ldots, 12\}$ with $i \equiv k (\mathrm{mod}\ 3)$. Since $N(w)$ is independent and P is an induced path, it follows that $\{y_7, x_1, y_4, y_5, x_2, y_2, y_{11}, y_{12}, x_3, y_9\}$ induces a copy of $T_{9,1}$ (see Figure 14). This contradicts the assumption that G is $T_{9,1}$-free.

This completes the proof of Proposition 1.5.

6 Proof of Proposition 1.6

Recall that w is a vertex of a 3-connected $\{C_3, C_4\}$-free graph G with $\deg_G(w) = \Delta(G)$. In this section, we prove Proposition 1.6. By way of contradiction, suppose that G is $T_{9,2}$-free and $\Delta(G) \geq 5.5 \cdot 10^{35}$. We argue as in Section 5. The main claim

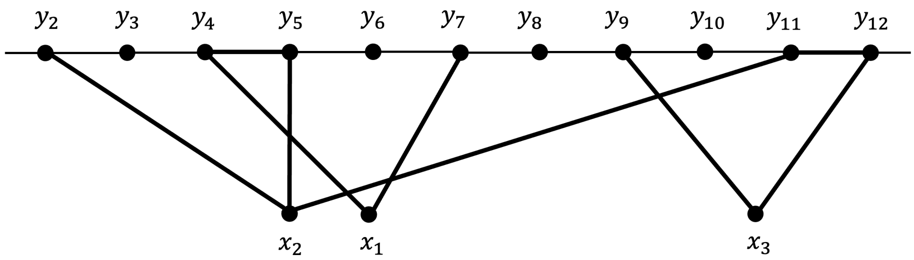

Figure 14: $T_{9,1}$ yielding the final contradiction.

is Claim 6.11. By Lemmas 3.2 and 4.2,

$$|M_4^w(u)| \le 15 \text{ and } |M_5^w(u)| \le |M_4^w(u) \cup M_5^w(u)| \le 525 \text{ for every } u \in N_{\ge 2}(w). \tag{6.1}$$

Claim 6.1 *Let $a_6a_5a_4a_3a_2a_1$ be a path in $G - w$ with $a_1 \in N(w)$ and $a_2, \dots, a_6 \in N_{\ge 2}(w)$. Then $\{a_6a_2, a_6a_1, a_5a_1\} \cap E(G) \ne \emptyset$.*

Proof. Set $X = \{a_2, \dots, a_6\}$. We have $X - N(X) = \emptyset$. Suppose that $\{a_6a_2, a_6a_1, a_5a_1\} \cap E(G) = \emptyset$. Then $E(G[X \cup \{a_1\}]) = \{a_6a_5, a_5a_4, a_4a_3, a_3a_2, a_2a_1\}$. Let Y_2 be as in Lemma 5.1. By Lemma 5.3 (i) and (6.1), $|Y_2| \le 5 \cdot 15 < |N(w)|$. Take $b_1 \in N(w) - Y_2$, and take b_2, $b_2' \in N(b_1) \cap N_{\ge 2}(w)$ with $b_2 \ne b_2'$ (see Figure 15). We have $E(G[\{a_1, w, b_1, b_2, b_2'\}]) = \{a_1w, wb_1, b_1b_2, b_1b_2'\}$. Since $E_G(X, \{w, b_1, b_2, b_2'\}) = \emptyset$ by Lemma 5.1(i), it follows that $X \cup \{a_1, w, b_1, b_2, b_2'\}$ induces a copy of $T_{9,2}$, a contradiction. □

Claim 6.2 *Let $a_5, a_5', a_4, a_3, a_2 \in N_{\ge 2}(w)$ be vertices such that $\{a_5a_4, a_5'a_4, a_4a_3, a_3a_2\} \subseteq E(G)$ and $a_2 \in N_2(w)$, and write $N(a_2) \cap N(w) = \{a_1\}$. Then $\{a_5a_1, a_5'a_1\} \cap E(G) \ne \emptyset$.*

Proof. Set $X = \{a_2, \dots, a_5, a_5'\}$. We have $X - N(X) = \emptyset$. Suppose that $\{a_5a_1, a_5'a_1\} \cap E(G) = \emptyset$. Then $E(G[X \cup \{a_1\}]) = \{a_5a_4, a_5'a_4, a_4a_3, a_3a_2, a_2a_1\}$. Let Z_1, Z_3 be as in Lemma 5.1. By Lemma 5.3 (ii), (iii) and (6.1), $|Z_1| \le 5 \cdot 525 = 2625$ and $|Z_3| \le 2630 \cdot 15 < |N(w)|$. Take $b_1 \in N(w) - Z_3$ and $b_2 \in N(b_1) \cap N_{\ge 2}(w)$. Since $|N(b_2) \cap N_{\ge 2}(w)| \ge 2$, we can take $b_3 \in N(b_2) \cap N_{\ge 2}(w)$ so that $a_1b_3 \notin E(G)$ (see Figure 16). Then $E(G[\{a_1, w, b_1, b_2, b_3\}]) = \{a_1w, wb_1, b_1b_2, b_2b_3\}$. Since $E_G(X, \{w, b_1, b_2, b_3\}) = \emptyset$ by Lemma 5.1 (ii), it follows that $X \cup \{a_1, w, b_1, b_2, b_3\}$ induces a copy of $T_{9,2}$, a contradiction. □

Claim 6.3 *We have $N_{\ge 4}(w) = \emptyset$ and $\Delta(G[N_3(w)]) \le 2$.*

Proof. Suppose that $N_{\ge 4}(w) \ne \emptyset$ or $\Delta(G[N_3(w)]) \ge 3$. If $N_{\ge 4}(w) \ne \emptyset$, then let $u \in N_4(w)$, if $\Delta(G[N_3(w)]) \ge 3$, then take $u \in N_3(w)$ so that $|N(u) \cap N_3(w)| \ge 3$. Then we can take $z, z', z'' \in N(u) \cap N_{\ge 3}(w)$ so that $z \in N_3(w)$. Take $y \in N(z) \cap N_2(w)$, and write $N(y) \cap N(w) = \{x\}$. Since $z', z'' \in N_{\ge 3}(w)$, we have $z'x$, $z''x \notin E(G)$.

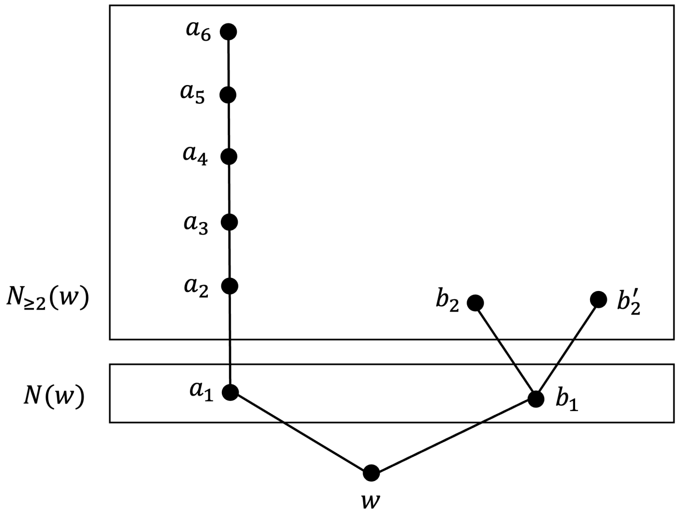

Figure 15: $T_{9,2}$ in Claim 6.1.

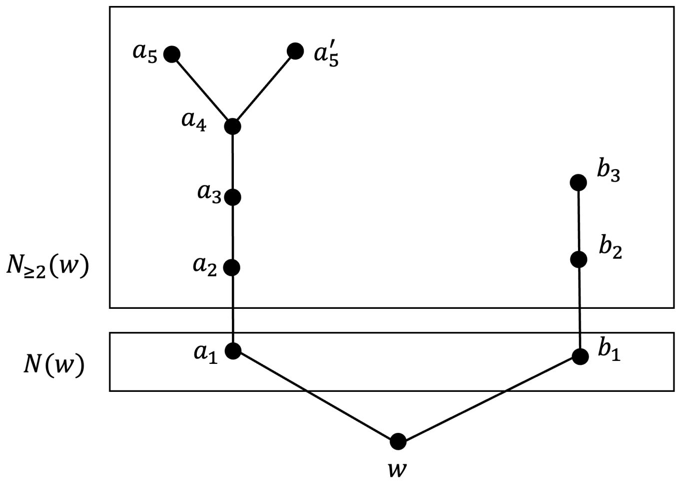

Figure 16: $T_{9,2}$ in Claim 6.2.

Hence we get a contradiction by applying Claim 6.2 to $\{z', z'', u, z, y\}$. $\square$

Set

$$R_1 = \{z \in N_3(w) \mid |N(z) \cap N_2(w)| = 1\},$$
$$R_2 = \{z \in N_3(w) \mid |N(z) \cap N_2(w)| \geq 2\}.$$

Claim 6.4 *Let $z \in R_1$ and $z' \in N_3(w)$ with $zz' \in E(G)$. Write $N(z) \cap N_2(w) = \{y\}$, let $y' \in N(z') \cap N_2(w)$, and write $N(y) \cap N(w) = \{x\}$ and $N(y') \cap N(w) = \{x'\}$. Then $x = x'$.*

Proof. Since G is C_3-free, $y \neq y'$. Suppose that $x \neq x'$. Then $yx' \notin E(G)$. Since $z \in R_1$, we can take $z'' \in N(z) \cap N_3(w)$ with $z'' \neq z'$. Since $x' \in N(w)$, $z''x' \notin E(G)$.

Consequently we get a contradiction by applying Claim 6.2 to $\{z'', z, y, z', y'\}$. □

As in Section 5, the following two claims follow from Claim 6.4.

Claim 6.5 *Let H be a component of $G[R_1]$. Then $N(V(H)) \cap N_2(w)$ is independent and $|N(N(V(H))) \cap N_2(w)) \cap N(w)| = 1$.* □

Claim 6.6 *We have $E_G(R_1,\ R_2) = \emptyset$.* □

It follows from Claims 6.3 and 6.6 that each component of $G[R_1]$ is a cycle, and is a component of $G[N_3(w)] = G[N_{\geq 3}(w)]$.

Claim 6.7 *Let H be a component of $G[R_1]$. Then $|V(H)| \equiv 0$ (mod 3), $N(V(H)) \cap N_2(w)$ is an independent set of cardinality 3, and we can write $H = z_1 z_2 \dots z_{3m} z_1$ and $N(V(H)) \cap N_2(w) = \{y_1, y_2, y_3\}$ so that for each $k \in \{1, 2, 3\}$, $N(z_j) \cap N_2(w) = \{y_k\}$ for every j with $j \equiv\ k$(mod 3).*

Proof. Take $u_1 \in V(H)$ and $u_2 \in N(u_1) \cap V(H)$, and write $(N(u_2) \cap V(H)) - \{u_1\} = \{u_3\}$ and $(N(u_3) \cap V(H)) - |u_2\} = \{u_4\}$. Write $N(u_k) \cap N_2(w) = \{y_k\}$ for each $k \in \{1, 2, 3\}$, and write $N(y_1) \cap N(w) = \{x\}$. Since G is $\{C_3, C_4\}$-free, $y_1 \neq y_2 \neq y_3 \neq y_1$. By Claim 6.5, $\{y_1, y_2, y_3\}$ is independent. Suppose that $u_4 y_1 \notin E(G)$. Since $u_3, u_4 \in N_3(w)$ and $x \in N(w)$, $u_3 x, u_4 x \notin E(G)$. Hence we get a contradiction by applying Claim 6.1 to $u_4 u_3 u_2 u_1 y_1 x$. Thus $u_4 y_1 \in E(G)$. Since $u_1 \in V(H)$ is arbitrary, this implies the desired conclusion. □

Claim 6.8 *For each $y \in N_2(w), y$ is adjacent to at most one component of $G[R_1]$.*

Proof. Suppose that there exists $y_1 \in N_2(w)$ such that y_1 is adjacent to two distinct components H, H' of $G[R_1]$. Having Claim 6.7 in mind, write $|V(H)| = 3m$ and $|V(H')| = 3m'$ $(m, m' \geq 2)$. Let $H = z_1 z_2 \dots z_{3m} z_1$ and $N(V(H)) \cap N_2(w) = \{y_1, y_2, y_3\}$ be as in Claim 6.7, and also let $H' = z'_1 z'_2 \dots z'_{3m'} z'_1$ and $N(V(H')) \cap N_2(w) = \{y_1, y'_2, y'_3\}$ be as in Claim 6.7. We have $y_2 \neq y'_2$ or $y_2 \neq y'_3$. By the symmetry of y'_2 and y'_3, we may assume that $y_2 \neq y'_2$. Write $N(y_1) \cap N(w) = \{x\}$. By Claim 6.5, $\{y_1, y_2, y_3, y'_2, y'_3\} \subseteq N(x)$. Hence $\{y_1, y_2, y_3, y'_2, y'_3\}$ is independent. Suppose that $N(y_2) - \{x\} - (V(H) \cup V(H'))) \neq \emptyset$ and take $z \in N(y_2) - \{x\} - (V(H) \cup V(H'))$. Then $z \in (N_2(w) - \{y_1, y_2, y_3, y'_2, y'_3\}) \cup (N_3(w) - (V(H) \cup V(H'))$. Since $N(V(H) \cup V(H')) \cap N_2(w) = \{y_1, y_2, y_3, y'_2, y'_3\}$ and H and H' are components of $G[N_3(w)]$, we obtain $N(z) \cap (V(H) \cup V(H')) = \emptyset$. Since $\{y_1, y_2, y'_2\} \subseteq N(x)$ and G is $\{C_3, C_4\}$-free, we also see that $N(z) \cap \{y_1, y_2, y'_2\} = \{y_2\}$. Consequently $\{z, y_2, z_5, z_2, z_1, y_1, z'_1, z'_2, y'_2, z'_5\}$ induces a copy of $T_{9,2}$, which is a contradiction (see Figure 17). Thus $N(y_2) - \{x\} \subseteq V(H) \cup V(H')$. By the symmetry of y_2 and y_3, we get $N(y_3) - \{x\} \subseteq V(H) \cup V(H')$. By the symmetry of H and H', we also obtain $(N(y'_2) \cup N(y'_3)) - \{x\} \subseteq V(H) \cup V(H')$. Therefore $\{x, y_1\}$ is a separator.

This contradicts the assumption that G is 3-connected, and completes the proof of Claim 6.8. □

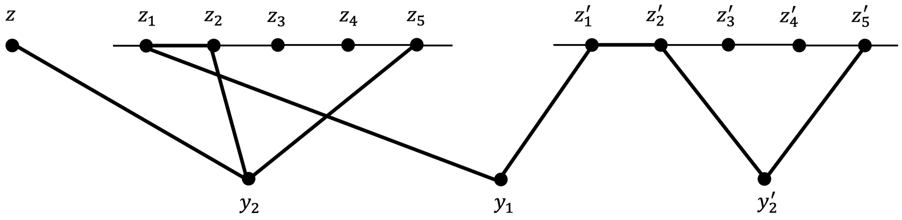

Figure 17: $T_{9.2}$ in Claim 6.8.

Claim 6.9 *For every $y \in N_2(w)$, $|N(y) \cap R_1| \leq 4$.*

Proof. Suppose that there exists $y_1 \in N_2(w)$ such that $|N(y_1) \cap R_1| \geq 5$. By Claim 6.8, there exists a component H of $G[R_1]$ such that $|N(y_1) \cap V(H)| \geq 5$. Let $H = z_1 z_2 \dots z_{3m} z_1$ and $N(V(H)) \cap N_2(w) = \{y_1, y_2, y_3\}$ be as in Claim 6.7. Since $|N(y_1) \cap V(H)| \geq 5$, we have $m \geq 5$. We now see that $\{z_8, y_2, z_{14}. z_{11}, z_{10}, y_1, z_4, z_3, y_3, z_6\}$ induces a copy of $T_{9,2}$, a contradiction (see Figure 18). □

Claim 6.10 *For every $y \in N_2(w)$, $|N(y) \cap (N_2(w) \cup R_2)| \leq 525$.*

Proof. Let $y \in N_2(w)$. By the definition of $L(\{y\})$ and R_2, $R_2 \subseteq L(\{y\})$. Hence $N(y) \cap N_2(w) \subseteq M_4^w(y)$ and $N(y) \cap R_2 \subseteq M_5^w(y)$ by Lemma 5.2. Consequently $|N(y) \cap (N_2(w) \cup R_2)| \leq |M_4^w(y) \cup M_5^w(y)| \leq 525$ by (6.1). □

Claim 6.11 *We have $\Delta(G[N_{\geq 2}(w)]) < 530$.*

Proof. Let $y \in N_{\geq 2}(w)$. If $y \in N_2(w)$, then it follows from Claims 6.9 and 6.10 that $\deg_{G[N_{\geq 2}(w)]}(y) = |N(y) \cap R_1| + |N(y) \cap (N_2(w) \cup R_2)| \leq 4 + 525 = 529$; if $y \in N_3(w)$, then $\deg_{G[N_{\geq 2}(w)]}(y) = |N(y) \cap N_2(w)| + |N(y) \cap N_3(w)| \leq |M_4^w(y)| + \Delta(G[N_3(w)]) \leq 15 + 2 \leq 529$ by Lemma 5.2, (6.1) and Claim 6.3. Since $y \in N_{\geq 2}(y)$ is arbitrary, we

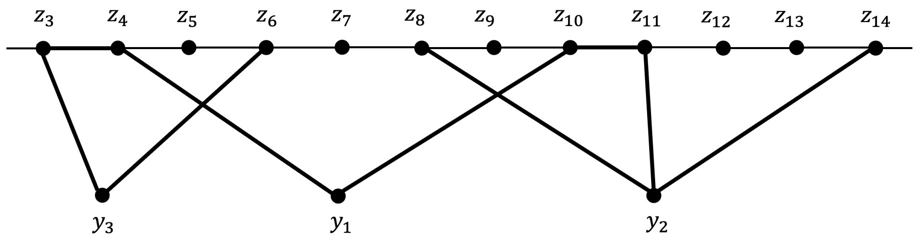

Figure 18: $T_{9,2}$ in Claim 6.9.

obtain $\Delta(G[N_{\geq 2}(w)]) < 530$. □

We divide the rest of the proof into two cases.

Case 1: There exists a component H_0 of $G[N_{\geq 2}(w)]$ such that $|V(H_0)| \geq 2.75 \cdot 10^{35}$.

In view of Lemma 1.3, it follows from Claim 6.11 that $\mathrm{diam}(H_0) \geq 14$. Take $y, y' \in V(H_0)$ with $\mathrm{dist}_{H_0}(y,\ y') = 14$, and let $P = y_1y_2 \ldots y_{15}$ be a shortest $y - y'$ path in H. We argue as in Claims 5.16 and 5.17.

Claim 6.12 *We have* $\{y_2, y_3, \ldots, y_{14}\} \subseteq N_2(w)$.

Proof. Suppose that $y_i \in N_3(w)$ for some $i \in \{2, \ldots, 14\}$. By symmetry, we may assume that $i \leq 8$. Take $z \in N(y_i) - \{y_{i-1}, y_i\}$. Then $z \in N_{\geq 2}(w)$. Since G is $\{C_3, C_4\}$-free and P is a shortest $y-y'$ path in $G[N_{\geq 2}(w)]$, it follows that $\{y_{i-1}, y_i, \ldots, y_{i+7},\ z\}$ induces a copy of $T_{9,2}$, a contradiction. □

For each $i \in \{2, \ldots, 14\}$, write $N(y_i) \cap N(w) = \{x_i\}$.

Claim 6.13 *(i) Let* $2 \leq i \leq 11$*. Then we have* $x_iy_{i+3} \in E(G)$ *or* $x_iy_{i+4} \in E(G)$*.*
(ii) Let $5 \leq i \leq 14$*. Then we have* $x_iy_{i-3} \in E(G)$ *or* $x_iy_{i-4} \in E(G)$*.*

Proof. Let $2 \leq i \leq 11$, and suppose that $x_iy_{i+3},\ x_iy_{i+4} \notin E(G)$. Since P is an induced path, $y_iy_{i+4} \notin E(G)$. Hence we get a contradiction by applying Claim 6.1 to $y_{i+4}y_{i+3} \ldots y_ix_i$. This proves (i), and (ii) can be verified in a similar way. □

Claim 6.14 *(i) Let* $2 \leq i \leq 10$*, and suppose that* $x_iy_{i+4} \in E(G)$*. Then* $x_{i+1}y_{i+5} \in E(G)$*.*
(ii) Let $6 \leq i \leq 14$*, and suppose that* $x_iy_{i-4} \in E(G)$*. Then* $x_{i-1}y_{i-5} \in E(G)$*.*

Proof. Let $2 \leq i \leq 10$ and $x_iy_{i+4} \in E(G)$, and suppose that $x_{i+1}y_{i+5} \notin E(G)$. Then by Claim 6.13 (i), $x_{i+1}y_{i+4} \in E(G)$. Since $N(y_{i+4}) \cap N(w) = \{x_{i+4}\}$, this implies that $x_{i+1} = x_{i+4} = x_i$, which contradicts the fact that G is C_3-free. Thus $x_{i+1}y_{i+5} \in E(G)$. This proves (i), and (ii) is verified in a similar way. □

Claim 6.15 *Let* $2 \leq i \leq 9$*, and suppose that* $x_iy_{i+3} \in E(G)$*. Then* $x_{i+1}y_{i+4} \in E(G)$*.*

Proof. Suppose that $x_{i+1}y_{i+4} \notin E(G)$. Then by Claim 6.13(i), $x_{i+1}y_{i+5} \in E(G)$, which implies $x_{i+1} = x_{i+5}$, and hence $x_{i+5}y_{i+1} \in E(G)$. Applying Claim 6.14 (ii) with i replaced by $i + 5$, we get $x_{i+4}y_i \in E(G)$, i.e., $x_iy_{i+4} \in E(G)$. Since $x_iy_{i+3} \in E(G)$ by assumption, this contradicts the fact that G is C_3-free. □

Note that we have $x_2y_5 \in E(G)$ or $x_2y_6 \in E(G)$ by Claim 6.13. Suppose that $x_2y_6 \in E(G)$. Then by Claim 6.14(i), for each $k \in \{2, 3, 4, 5\}$, $N(y_i) \cap N(w) = \{x_k\}$ for every $i \in \{2, \ldots, 14\}$ with $i \equiv k (\mathrm{mod}\ 4)$. Since G is $\{C_3, C_4\}$-free, this implies that x_2, x_3, x_4, x_5 are distinct. We now see that $\{y_{14}, x_2, y_6, y_2, y_3, x_3, y_{11}, y_{12}, x_4, y_8\}$ induces a copy of $T_{9,2}$, which contradicts the assumption that G is $T_{9,2}$-free (see

Figure 19). Consequently $x_2y_5 \in E(G)$. By Claim 6.15, for each $k \in \{2,3,4\}$, $N(y_i) \cap N(w) = \{x_k\}$ for every $i \in \{2,\dots 13\}$ with $i \equiv k (\mathrm{mod}\ 3)$. Therefore $\{y_2, x_2, y_8, y_5, y_6, x_3, y_{12}, y_{13}. x_1, y_{10}\}$ induces a copy of $T_{9,2}$, which again contradicts the assumption that G is $T_{9,2}$-free (see Figure 19). This concludes the discussion for Case 1. □ **Case 2:** $|V(H)| < 2.75 \cdot 10^{35}$ for every component H of $G[N_{\geq 2}(w)]$.

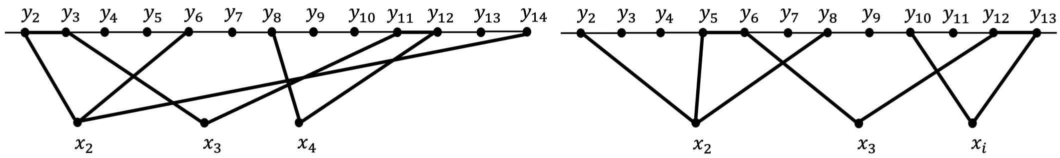

Figure 19: $T_{9,2}$ in Case 1.

Let $H_1, \dots, H_p$ be the components of $G[N_{\geq 2}(w)]$. Since $|N_{\geq 2}(w)| \geq 2|N(w)| \geq 1.1 \cdot 10^{36}$ by (5.1), we have $p \geq 5$. For each $x \in N(w)$, define a subset $g(x)$ of $\{1, \dots, p\}$ by $g(x) = \{\alpha \in \{1, \dots, p\} \mid N(x) \cap V(H_\alpha) \neq \emptyset\}$. Since $N(N(w)) \supseteq N_2(w)$, we have $\cup_{x \in N(w)} g(x) = \{1, \dots, p\}$.

Claim 6.16 *If $x,\ x' \in N(w)$, then $g(x) \subseteq g(x')$ or $g(x') \subseteq g(x)$ or $g(x) \cap g(x') = \emptyset$.*

Proof. Suppose that $g(x) - g(x') \neq \emptyset$, $g(x') - g(x) \neq \emptyset$ and $g(x) \cap g(x') \neq \emptyset$, and take $\alpha \in g(x) - g(x')$, $\beta \in g(x') - g(x)$ and $\gamma \in g(x) \cap g(x')$. Take $y \in N(x) \cap V(H_\gamma)$ and $y' \in N(x') \cap V(H_\gamma)$ so that $\mathrm{dist}_{H_\gamma}(y, y')$ is as small as possible, and let $y_1 \dots y_r$ ($y_1 = y, y_r = y'$) be a shortest $y - y'$ path in H_γ. We have $r \geq 2$ by (5.1). Take $u \in N(x) \cap V(H_\alpha)$. Since $\delta(H_\alpha) \geq 2$, we can take $u', u'' \in N(u) \cap V(H_\alpha)$ so that $u' \neq u''$. Take $v \in N(x') \cap V(H_\beta)$. Since $\delta(H_\beta) \geq 2$, we can take $v', v'' \in V(H_\beta) - \{v\}$ so that $vv', v'v'' \in E(G)$ (see Figure 20). By the minimality of r, we have $x, x' \notin N(\{y_2, \dots, y_{r-1}\})$. By (5.1), $x \notin N(\{y', v\})$ and $x' \notin N(\{y, u\})$. Since $\beta \notin g(x)$ and $\alpha \notin g(x')$, we also get $x \notin N(\{v', v''\})$ and $x' \notin N(\{u', u''\})$. Since G is $\{C_3, C_4\}$-free, it follows that $\{u', u, u'', x, y_1, \dots, y_r, x', v, v', v''\}$ induces a copy of $S_{r+7}(\{2\}, \emptyset)$ (see the second paragraph of Section 1 for the definition of $S_n(I, J)$). Since $T_{9,2} = S_9(\{2\}, \emptyset)$ is an induced subgraph of $S_{r+7}(\{2\}, \emptyset)$, this contradicts the assumption that G is $T_{9,2}$-free. □

Set $A = \{x \in N(w) \mid g(x) = \{1, \dots, p\}\}$. We distinguish three subcases according as $|A| \leq 1$, $|A| = 2$, or $|A| \geq 3$.

Subcase 2-1: $|A| \leq 1$.

Claim 6.17 *We have $\cup_{x \in N(w) - A} g(x) = \{1, \dots, p\}$.*

Proof. Suppose that $\cup_{x \in N(w) - A} g(x) \neq \{1, \dots, p\}$. Since $\cup_{x \in N(w)} g(x) = \{1, \dots, p\}$, this implies $|A| = 1$. Take $\alpha \in \{1, \dots, p\} - (\cup_{x \in N(w) - A} g(x))$. By the definition of $g(x)$, $N(V(H_\alpha)) \cap N(w) = A$. Consequently H_α is a component of $G - A$, which contradicts the assumption that G is 3-connected. □

Let $I_1, \dots, I_q (\subseteq \{1, \dots, p\})$ be the maximal sets among $g(x)$ ($x \in N(w) - A$).

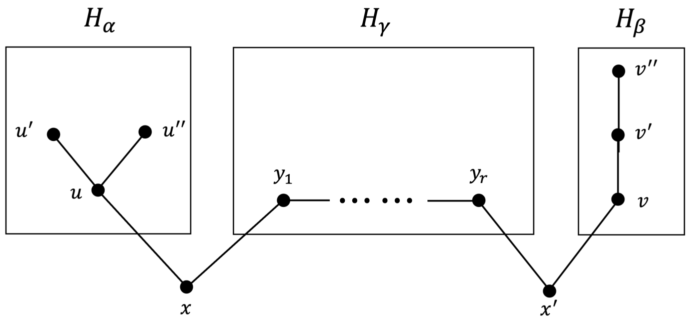

Figure 20: $S_{r+7}(\{2\},\emptyset)$ in Claim 6.16.

By Claim 6.17, $I_1 \cup \ldots \cup I_q = \{1,\ldots,p\}$. By the definition of A, $q \geq 2$. Set $B = \cup_{\alpha \in I_1} V(H_\alpha)$. Since $q \geq 2$, $B \neq N_{\geq 2}(w)$. By Claim 6.16 and the maximality of I_1, $g(x) \subseteq I_1$ for all $x \in (N(B) \cap N(w)) - A$ which, by the definition of $g(x)$, implies that $N((N(B)\cap N(w))-A)\cap N_2(w) \subseteq B$. Therefore $G[B \cup ((N(B)\cap N(w))-A)]$ is a component of $G-(A\cup\{w\})$, which contradicts the assumption that G is 3-connected.

Subcase 2-2: $|A| = 2$.

Write $A = \{x, x'\}$. Since G is 3-connected, H_1 is not a component of $G - \{x, x'\}$. Hence $N(N(w) - \{x, x'\}) \cap V(H_1) \neq \emptyset$. Take $u \in N(\{x, x'\}) \cap V(H_1)$ and $u' \in N(N(w) - \{x, x'\}) \cap V(H_1)$ so that $\mathrm{dist}_{H_1}(u, u')$ is as small as possible. By the symmetry of x and x', we may assume that $N(u) \cap N(w) = \{x\}$. Write $N(u') \cap N(w) = \{x''\}$. Then $x'' \notin A$. Let $u_1, \ldots, u_s$ ($u_1 = u, u_s = u'$) be a shortest u–u' path in H_1 ($s \geq 2$). From the minimality of s, it follows that $x, x', x'' \notin N(\{u_2, \ldots, u_{s-1}\})$. Take $u'' \in (N(x'')\cap N_2(w)) - \{u'\}$. Then $u'' \notin \{u_1, \ldots, u_s\}$. Since G is $\{C_3, C_4\}$-free, the minimality of s implies that $u'' \notin N(\{u_1, \ldots, u_s\})$. Let α be the index with $u'' \in V(H_\alpha)$ (it is possible that $\alpha = 1$). Since $x'' \notin A$, there exists β with $\beta \notin g(x'')$. Take $y' \in N(x')\cap V(H_\beta)$ and $y \in N(x)\cap V(H_\beta)$ so that $\mathrm{dist}_{H_\beta}(y', y)$ is as small as possible, and let $y_1 \ldots y_r (y_1 = y', y_r = y)$ be a shortest $y' - y$ path in $H_\beta (r \geq 2)$. Recall that $p \geq 5$. Take $\gamma, \gamma' \in \{1, \ldots, p\} - \{1, \alpha, \beta\}$ with $\gamma \neq \gamma'$, and take $v \in N(x') \cap V(H_\gamma)$ and $v' \in N(x')\cap V(H_{\gamma'})$ (see Figure 21). Since $\beta \notin g(x''), x'' \notin N(\{y_2, \ldots, y_{r-1}\})$. By the minimality of r, we have $x', x \notin N(\{y_2, \ldots, y_{r-1}\})$. Consequently it follows from (5.1) that $\{v, x', v', y_1, \ldots, y_r, x, u_1, \ldots, u_s, x'', u''\}$ induces a copy of $S_{r+s+5}(\{2\},\emptyset)$, which contradicts the assumption that G is $T_{9,2}$-free.

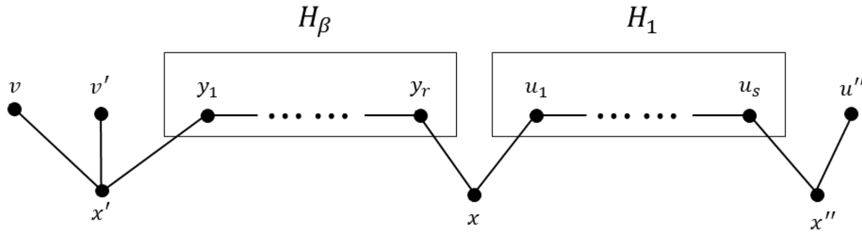

Figure 21: $S_{r+s+5}(\{2\},\emptyset)$ in Subcase 2-2.

Subcase 2-3: $|A| \geq 3$.

Take $x, x', x'' \in A$ with $x \neq x' \neq x'' \neq x$. Choose $y' \in N(x') \cap V(H_1)$ and $y \in N(x) \cap V(H_1)$ so that $\mathrm{dist}_{H_1}(y', y)$ is as small as possible, and let $y_1 \dots y_r$ ($y_1 = y', y_r = y$) be a shortest $y' - y$ path in H_1 ($r \geq 2$). Then $x', x \notin N(\{y_2, \dots, y_{r-1}\})$. Choose $z \in N(x) \cap V(H_2)$ and $z' \in N(x'') \cap V(H_2)$ so that $\mathrm{dist}_{H_2}(z, z')$ is as small as possible, and let $z_1 \dots z_s$ ($z_1 = z, z_s = z'$) be a shortest $z - z'$ path in H_2 ($s \geq 2$). Then $x, x'' \notin N(\{z_2, \dots, z_{s-1}\})$. We assume that we have labeled x, x' and x'' so that $r + s$ is as small as possible. If $x'' \in N(\{y_2, \dots, y_{r-1}\})$, say $x'' \in N(y_i)$, then replacing $x, x'', y_1 \dots y_r$ and $z_1 \dots z_s$ by $x'', x, y_1 \dots y_i$ and $z_s \dots z_1$, respectively, we get a contradiction to the minimality of $r + s$. Thus $x'' \notin N(\{y_2, \dots, y_{r-1}\})$. Similarly $x' \notin N(\{z_2, \dots, z_{s-1}\})$. Therefore $N(\{y_2, \dots, y_{r-1}, z_2, \dots, z_{s-1}\}) \cap \{x, x', x''\} = \emptyset$. Take $u \in N(x'') \cap V(H_3)$, $v \in N(x') \cap V(H_4)$ and $v' \in N(x') \cap V(H_5)$ (see Figure 22). It now follows from (5.1) that $\{v, x', v', y_1, \dots, y_r, x, z_1, \dots, z_s, x'', u\}$ induces a copy of $S_{r+s+5}(\{2\}, \emptyset)$. This contradicts the assumption that G is $T_{9,2}$-free.

This concludes the discussion for Case 2, and completes the proof of Proposition 1.6.

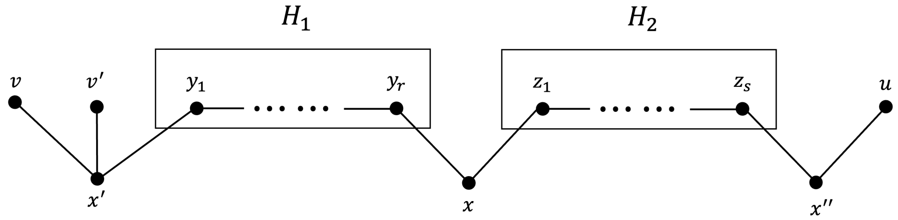

Figure 22: $S_{r+s+5}(\{2\}, \emptyset)$ in Subcase 2-3.

7 Proof of Proposition 1.7

In this section, we prove Proposition 1.7. Suppose that G is T_8-free and $\Delta(G) \geq 1221$. By Claims 3.3 and 4.3,

$$|M_4^w(u)| \leq 5 \text{ and } |M_5^w(u)| \leq |M_4^w(u) \cup M_5^w(u)| \leq 60 \text{ for every } u \in N_{\geq 2}(w). \quad (7.1)$$

Claim 7.1 *Let $a_5a_4a_3a_2a_3'$ be a path in $G[N_{\geq 2}(w)]$ with $a_2 \in N_2(w)$, and write $N(a_2) \cap N(w) = \{a_1\}$. Then $\{a_5a_3', a_5a_1\} \cap E(G) \neq \emptyset$.*

Proof. Set $X = \{a_2, \dots, a_5, a_3'\}$. We have $X - N(X) = \emptyset$. Suppose that $\{a_5a_3', a_5a_1\} \cap E(G) = \emptyset$. Then $E(G[X \cup \{a_1\}]) = \{a_5a_4, a_4a_3, a_3a_2, a_2a_1, a_2a_3'\}$. Let Y_2 be as in Lemma 5.1. By Lemma 5.3 (i) and (7.1), $|Y_2| \leq 5 \cdot 5 < |N(w)|$. Take $b_1 \in N(w) - Y_2$, and take $b_2, b_2' \in N(b_1) \cap N_{\geq 2}(w)$ with $b_2 \neq b_2'$ (see Figure 23). We have $E(G[\{a_1, w, b_1, b_2, b_2'\}]) = \{a_1w, wb_1, b_1b_2, b_1b_2'\}$. Since $E_G(X, \{w, b_1, b_2, b_2'\}) = \emptyset$ by Lemma 5.1 (i), it follows that $X \cup \{a_1, w, b_1, b_2, b_2'\}$ induces a copy of T_8, a contradiction. □

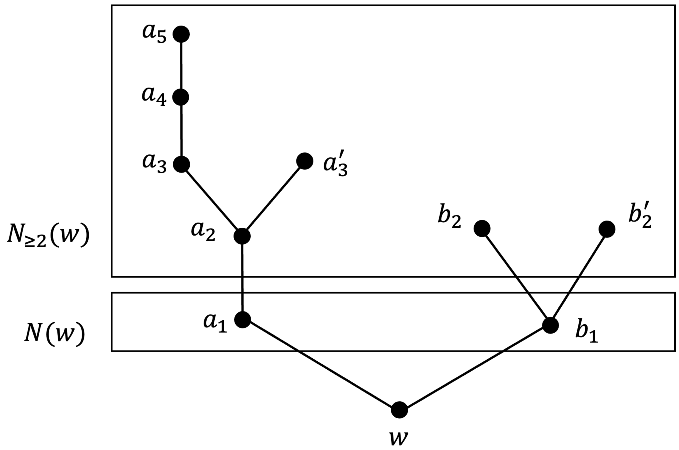

Figure 23: T_8 in Claim 7.1.

Claim 7.2 *We have $N_{\geq 3}(w) = \emptyset$, and $G[N_2(w)]$ is 2-regular.*

Proof. Recall that $\delta(G[N_{\geq 2}(w)]) \geq 2$. By way of contradiction, suppose that $N_{\geq 3}(w) \neq \emptyset$ or $\Delta(G[N_2(w)]) \geq 3$. If $N_{\geq 3}(w) \neq \emptyset$, then take $a_3 \in N_3(w)$; if $\Delta(G[N_2(w)] \geq 3$, then take $a_3 \in N_2(w)$ so that $|N(a_3) \cap N_2(w)| \geq 3$. Then we can take $a_2, a_4, a'_4 \in N(a_3) \cap N_{\geq 2}(w)$ so that $a_2 \in N_2(w)$. Write $N(a_2) \cap N(w) = \{a_1\}$. Set $X = \{a_2, a_3, a_4, a'_4\}$. We have $X - N(X) = \emptyset$ and $E(G[X \cup \{a_1\}]) = \{a_4a_3, a'_4a_3, a_3a_2, a_2a_1\}$. Let Z_1, Z_3 be as in Lemma 5.1. By Lemma 5.3 (ii), (iii) and (7.1), $|Z_1| \leq 4 \cdot 60 = 240$ and $|Z_3| \leq 244 \cdot 5 < |N(w)|$. Take $b_1 \in N(w) - Z_3$ and $b_2 \in N(b_1) \cap N_{\geq 2}(w)$. Since $|N(b_2) \cap N_{\geq 2}(w)| \geq 2$, we can take $b_3 \in N(b_2) \cap N_{\geq 2}(w)$ so that $a_1b_3 \notin E(G)$. Then $E(G[\{a_1, w, b_1, b_2, b_3\}]) = \{a_1w, wb_1, b_1b_2, b_2b_3\}$. By Lemma 5.1 (ii), $E_G(X, \{w, b_1, b_2, b_3\}) = \emptyset$. By (5.1), $|N(X \cup \{b_2, b_3\}) \cap N(w)| \leq 6 < |N(w)|$. Take $c \in N(w) - (N(X \cup \{b_2, b_3\}) \cap N(w))$ (see Figure 24). It now follows that $X \cup \{a_1, w, b_1, b_2, b_3, c\}$ induces a copy of T_8, a contradiction. □

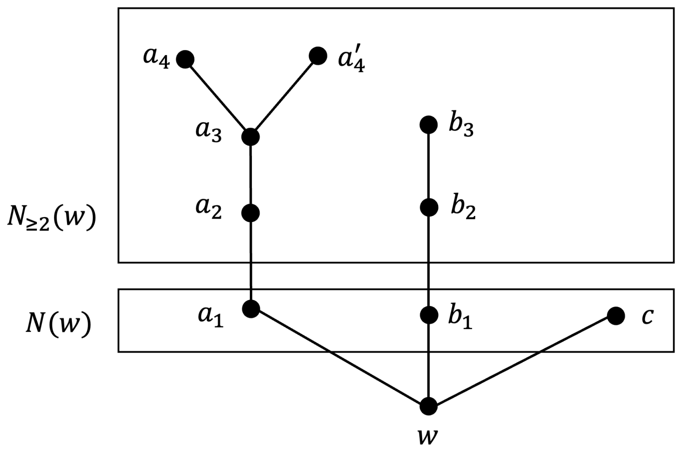

Figure 24: T_8 in Claim 7.2.

Let $H_1, \dots, H_p$ be the components of $G[N_2(w)]$. By Claim 7.2, each H_α is a cycle.

Claim 7.3 *Let $\alpha \in \{1, \dots, p\}$, and suppose that $|V(H_\alpha)| \neq 5$. Then $|V(H_\alpha)| \equiv 0 \pmod 3$, $|N(V(H_\alpha)) \cap N(w)| = 3$, and we can write $H_\alpha = y_1 y_2 \cdots y_{3m} y_1$ and $N(V(H_\alpha)) \cap N(w) = \{x_1, x_2, x_3\}$ so that for each $k \in \{1, 2, 3\}$, $N(y_j) \cap N_2(w) = \{x_k\}$ for every j with $j \equiv k \pmod 3$.*

Proof. Since G is $\{C_3, C_4\}$-free, $|V(H_\alpha)| \geq 6$. Take $u_1 \in V(H_\alpha)$, and write $N(u_1) \cap V(H_\alpha) = \{u_0, u_2\}$, $(N(u_2) \cap V(H_\alpha)) - \{u_1\} = \{u_3\}$ and $(N(u_3) \cap V(H_\alpha)) - \{u_2\} = \{u_4\}$. Also write $N(u_1) \cap N(w) = \{x\}$. Suppose that $u_4 x \notin E(G)$. Since H_α is a cycle with $|V(H_\alpha)| \geq 6$, $u_4 u_0 \notin E(G)$. Hence we get a contradiction by applying Claim 7.1 to $u_4 u_3 u_2 u_1 u_0$. Thus $u_4 x \in E(G)$. Since u_1 is arbitrary, this implies the desired conclusion. □

For each $\alpha \in \{1, \dots, p\}$, if $|V(H_\alpha)| = 5$, then $|N(V(H_\alpha)) \cap N(w)| = 5$ by (5.1) and, if $|V(H_\alpha)| \neq 5$, then $|N(V(H_\alpha)) \cap N(w)| = 3$ by Claim 7.3. Hence $p \geq |N(w)|/5 \geq 5$. For each $x \in N(w)$, set $g(x) = \{\alpha \in \{1, \dots, p\} \mid N(x) \cap V(H_\alpha) \neq \emptyset\}$. We have $\cup_{x \in N(w)} g(x) = \{1, \dots, p\}$.

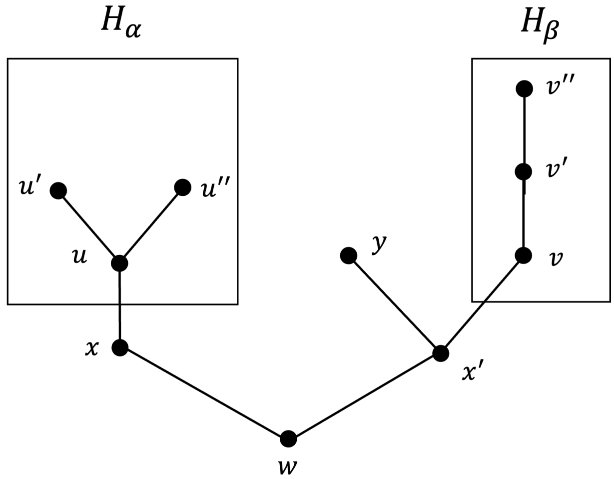

Figure 25: T_8 in Claim 7.4.

Claim 7.4 *If $x, x' \in N(w)$, then $g(x) \subseteq g(x')$ or $g(x') \subseteq g(x)$ or $g(x) \cap g(x') = \emptyset$.*

Proof. Suppose that $g(x) - g(x') \neq \emptyset$, $g(x') - g(x) \neq \emptyset$ and $g(x) \cap g(x') \neq \emptyset$, and take $\alpha \in g(x) - g(x')$, $\beta \in g(x') - g(x)$ and $\gamma \in g(x) \cap g(x')$. Take $y \in N(x') \cap V(H_\gamma)$. Take $u \in N(x) \cap V(H_\alpha)$, and write $N(u) \cap V(H_\alpha) = \{u', u''\}$. Take $v \in N(x') \cap V(H_\beta)$ and $v' \in N(v) \cap V(H_\beta)$, and write $(N(v') \cap V(H_\beta)) - \{v\} = \{v''\}$ (see Figure 25). By the choice of α and β, $x \notin N(\{v', v''\})$ and $x' \notin N(\{u', u''\})$. Since G is $\{C_3, C_4\}$-free, it now follows from (5.1) that $\{u', u, u'', x, w, x', y, v, v', v''\}$ induces a copy of T_8, a contradiction. □

Set $A = \{x \in N(w) \mid g(x) = \{1, \dots, p\}\}$. If $|A| \leq 1$, then, in view of Claim 7.4, we get a contradiction by arguing in Subcase 2-1 in Section 6. Thus $|A| \geq 2$. Take $x, x' \in A$ with $x \neq x'$. Suppose that there exists an index $\alpha_0 \in \{1, \dots, p\}$ for which we can take $y_0 \in N(x) \cap V(H_{\alpha_0})$ and $y_0' \in N(x') \cap V(H_{\alpha_0})$ so that $y_0 y_0' \in E(G)$. We may assume that $\alpha_0 = 1$. Take $y \in N(x') \cap V(H_2)$. Since G is $\{C_3, C_4\}$-free, one of the two neighbors of y in H_2, say y', satisfies $N(y') \cap \{x, x'\} = \emptyset$. Write $N(y') \cap N(w) = \{x''\}$. Take $v \in N(x') \cap V(H_3)$, $u \in N(x) \cap V(H_4)$ and $u' \in N(x) \cap V(H_5)$ (see Figure 26). By (5.1), $\{u, x'', u', y_0, y_0', x', v, y, y', x''\}$ induces a copy of T_8, a contradiction.

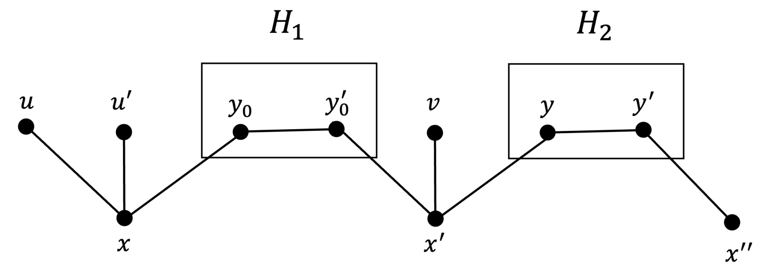

Figure 26: T_8 in the paragraph following the proof of Claim 7.4.

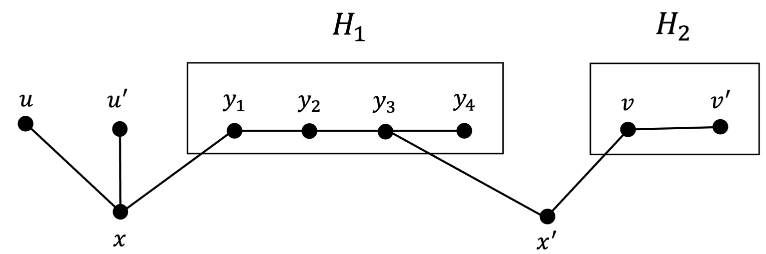

Figure 27: T_8 yielding the final contradiction.

Consequently for each $\alpha \in \{1, \dots, p\}$, $yy' \notin E(G)$ for every $y \in N(x) \cap V(H_\alpha)$ and every $y' \in N(x') \cap V(H_\alpha)$. In view of Claim 7.3, thus means that $|V(H_\alpha)| = 5$ for every α. Since G is $\{C_3,\ C_4\}$-free, it follows that $|N(x) \cap V(H_\alpha)| = |N(x') \cap V(H_\alpha)| = 1$ for every α. Write $H_1 = y_1 y_2 y_3 y_4 y_5 y_1$ so that $N(x) \cap V(H_1) = \{y_1\}$ and $N(x') \cap V(H_1) = \{y_3\}$. Take $v \in N(x') \cap V(H_2)$ and $v' \in N(v) \cap V(H_2)$. Since $yv \notin E(G)$ for every $y \in N(x) \cap V(H_2)$, we see that $xv' \notin E(G)$. Finally take $u \in N(x) \cap V(H_3)$ and $u' \in N(x) \cap V(H_4)$ (see Figure 27). Since $|N(x) \cap V(H_1)| = |N(x') \cap V(H_1)| = |N(x') \cap V(H_2)| = 1$, it follows from (5.1) that $\{u, x, u', y_1, y_2, y_3, y_4, x, v, v'\}$ induces a copy of T_8, which contradicts the assumption that G is T_8-free.

This completes the proof of Proposition 1.7.

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