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On bispindles $B(k_1, k_2; 1)$ in strongly connected digraphs with large chromatic number

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Abstract

A $(2+1)$-bispindle $B(k_1, k_2; k_3)$ is a digraph formed by the union of two (x, y)-dipaths of respective lengths k_1 and k_2, and one (y, x)-dipath of length k_3, all these dipaths being pairwise internally disjoint. Recently, Cohen et al. conjectured that for any positive integers k_1, k_2, k_3, there is an integer $g(k_1, k_2, k_3)$ such that every strongly connected digraph containing no subdivisions of $B(k_1, k_2; k_3)$ has a chromatic number at most $g(k_1, k_2, k_3)$, and they confirmed this only for the case where $k_2 = 1$. In this paper, we prove Cohen et al.'s conjecture for the case where k_1, k_2 are arbitrary and $k_3 = 1$, namely $g(k_1, k_2, 1) = O((k_1 + k_2)^2)$. Moreover, we show that if in addition D is Hamiltonian, then the chromatic number of D is at most $5k - 7$, with $k = \max\{k_1, k_2\}$.

1 Introduction

Throughout this paper, an *orientation* of a graph G is a digraph obtained by giving a direction to each edge of G, and the *underlying graph* of a digraph D, denoted by $G(D)$, is the graph obtained from D by ignoring the directions of its arcs. The *chromatic number* of a digraph D, denoted by $\chi(D)$, is the chromatic number of its underlying graph. The *chromatic number* of a class $\mathcal{D}$ of digraphs, denoted by $\chi(\mathcal{D})$, is the smallest integer k such that $\chi(D) \leq k$ for all $D \in \mathcal{D}$ or $+\infty$ if no such k exists. By convention, if $\mathcal{D} = \emptyset$, then $\chi(\mathcal{D}) = 0$. If $\chi(\mathcal{D}) \neq +\infty$, we say that $\mathcal{D}$ has a bounded chromatic number.

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A *directed path*, or simply a *dipath*, is an oriented path where all the arcs are oriented in the same direction from the initial vertex towards the terminal vertex. A classical result due to Gallai and Roy [11, 13] is the following:

Theorem 1.1 *(Gallai [11], Roy [13]) Let k be a non-negative integer. If $\chi(D) \geq k$, then D contains a directed path of order k.*

This raises the following question:

Problem 1 *Which are the digraph classes $\mathcal{D}$ such that every digraph with chromatic number at least k contains an element of $\mathcal{D}$ as a subdigraph?*

Denoting by $\mathrm{Forb}(\mathcal{D})$ the class of digraphs that do not contain an element of a class $\mathcal{D}$ of digraphs as a subdigraph, the above question can be restated in terms of $\mathrm{Forb}(\mathcal{D})$ as follows:

Which are the digraph classes $\mathcal{D}$ such that $\chi(\mathrm{Forb}(\mathcal{D})) < +\infty$?

Due to a famous theorem of Erdős [9] which guarantees the existence of graphs with arbitrarily high girth and arbitrarily high chromatic number, if H is a digraph containing an oriented cycle, then there exist digraphs with arbitrarily high chromatic number with no subdigraphs isomorphic to H. This means that the only possible candidates to generalize Theorem 1.1 are the oriented trees. In this context, Burr [6] proved that the chromatic number of $\mathrm{Forb}(T)$ for every oriented tree T of order k is at most $k^2 - 2k$ and he conjectured that this upper bound can be improved to $2k-3$. The best known upper bound, found by Addario-Berry et al. [1], is $k^2/2-k/2$. However, for oriented paths with two blocks (*blocks* are maximal directed subpaths), the best possible upper bound is known. Assuming that an oriented path P has two blocks of lengths k_1 and k_2, we say that P is a two-blocks path and we write $P = P(k_1, k_2)$.

Theorem 1.2 *(Addario-Berry et al. [2]) Let k_1 and k_2 be positive integers such that $k_1 + k_2 \geq 3$. Then $\mathrm{Forb}(P(k_1, k_2))$ has chromatic number equal to $k_1 + k_2$, for every two-blocks path $P(k_1, k_2)$.*

A *subdivision* of a digraph H is a digraph H' obtained from H by replacing each arc (x, y) by an (x, y)-dipath of length at least 1. A digraph D is said to be *H-subdivision-free* if it contains no subdivisions of H as a subdigraph. Inspired by the previous researches, Cohen et al. [8] asked about the existence of subdivisions of oriented cycles in highly chromatic digraphs. In other words, denoting by S-$\mathrm{Forb}(C)$ the class of digraphs that do not contain subdivisions of a given oriented cycle C as subdigraphs, Cohen et al. asked if the chromatic number of S-$\mathrm{Forb}(C)$ can be bounded. In the same article, Cohen et al. provided a negative answer to their question by proving a stronger theorem based on a construction built by Erdős and Lovász [10] that implies the existence of hypergraphs with large girth and large chromatic number:

Theorem 1.3 *(Cohen et al. [8]) For any positive integers b, c, there exists an acyclic digraph D with $\chi(D) \geq c$ in which all oriented cycles have more than b blocks.*

However, restricting Cohen et al.'s question to the class of strongly connected digraphs may lead to dramatically different results. A digraph D is said to be *strongly connected*, or simply *strong*, if for any two vertices x and y of D, there is a directed path from x to y. A *directed cycle*, or simply a *circuit*, is an oriented cycle whose all arcs have the same orientation. An example is provided by a famous result of Bondy [5]: *Every strong digraph D contains a directed cycle of length at least $\chi(D)$.* Denoting by $\mathcal{S}$ the class of strong digraphs, Bondy's theorem can be restated in the following way:

Theorem 1.4 *(Bondy [5]) For the circuit C_k^+ of length k, $\chi(S\text{-}Forb(C_k^+) \cap \mathcal{S}) = k-1$.*

Since any directed cycle of length at least k can be seen as a subdivision of C_k^+, Cohen et al. [8] conjectured that Bondy's theorem can be extended to all oriented cycles:

Conjecture 1 *(Cohen et al. [8]) For every oriented cycle C, there exists a constant $f(C)$ such that every strongly connected digraph with chromatic number at least $f(C)$ contains a subdivision of C.*

For two positive integers k_1 and k_2, a *cycle with two blocks* $C\,(k_1, k_2)$ is an oriented cycle formed of two internally disjoint directed paths of lengths k_1 and k_2 respectively. In their article, Cohen et al. [8] proved Conjecture 1 for the case of two-blocks cycles, where they showed that the chromatic number of strong digraphs with no subdivisions of a two-blocks cycle $C\,(k_1, k_2)$ is bounded from above by $O((k_1+k_2)^4)$. This upper bound was improved by Kim et al. [12] to $O((k_1+k_2)^2)$. In [2], Addario et al. asked if the upper bound of such digraphs can be improved to $O(k_1+k_2)$, which remains an open problem. However, this question is answered partially by Kim et al. [12] for the class $\mathcal{H}$ of Hamiltonian digraphs (a digraph D is said to be *Hamiltonian* if it contains a Hamiltonian directed cycle, that is, a directed cycle passing through all the vertices of D) and by Al-Mniny et al. [4] for the class of digraphs having a Hamiltonian directed path. Another contributions to Conjecture 1 were provided by Cohen et al. [8] for the case of four-blocks cycles $C(1,1,1,1)$ and by Al-Mniny [3] for the case of four-blocks cycles $C(k,1,1,1)$ for an arbitrary positive integer k.

A p*-spindle* is the union of p internally disjoint (x, y)-dipaths for some vertices x and y. In this case, x is the *tail* of the spindle and y is its *head*. A $(p+q)$*-bispindle* is the internally disjoint union of a p-spindle with tail x and head y and a q-spindle with tail y and head x. In other words, it is the union of p (x, y)-dipaths and q (y, x)-dipaths, all of these dipaths being pairwise internally disjoint. In this case, x and y are called the *left extremity* and the *right extremity* of the bispindle, respectively. Since directed cycles and two-blocks cycles can be seen as $(1+1)$-bispindles and 2-spindles respectively, Cohen et al. [7] asked about the existence of spindles and bispindles in strong digraphs with large chromatic number. First, they pointed out the existence of strong digraphs with large chromatic number that contain neither 3-spindle nor $(2+2)$-bispindle. Undoubtedly, this result guides them to focus in their study on the existence of $(2+1)$-bispindles in strong digraphs. Denoting by $B(k_1, k_2; k_3)$ the $(2+1)$-bispindle formed by the internally disjoint union of two (x, y)-dipaths, one of length k_1 and the other of length k_2, and one (y, x)-dipath of length k_3, Cohen et al. [7] conjectured the following:

Conjecture 2 *(Cohen et al. [7]) Let D be a digraph in S-*Forb $(B(k_1, k_2; k_3)) \cap \mathcal{S}$. *Then there exists an integer $g(k_1, k_2, k_3)$ such that $\chi(D) \leq g(k_1, k_2, k_3)$.*

In fact, Cohen et al. [7] confirmed their conjecture for $k_2 = 1$. The upper bound provided by Cohen et al. for the chromatic number of digraphs in S-Forb $(B(k_1,1; k_3))$ $\cap \mathcal{S}$ is very huge and certainly not the best possible. However, these authors attained a better bound for the case where k_1 (respectively, k_2) is arbitrary and $k_2 = k_3 = 1$ (respectively, $k_1 = k_3 = 1$). In this paper, we confirm Conjecture 2 for $k_3 = 1$, by following the lines of the proof of Kim et al. [12] dealing with the existence of subdivisions of two-blocks cycles $C(k_1, k_2)$ in strong digraphs, where going deep in their proofs leads us to generalize their work to the case of subdivisions of $(2+1)$-bispindle $B(k_1, k_2; 1)$. Indeed, we have noticed that the proof of Kim et al. [12] for the existence of subdivisions of two-blocks cycles $C(k_1, k_2)$ in strong digraphs proves the existence of subdivisions of $(2+1)$-bispindles $B(k_1, k_2; 1)$ in strong digraphs up to some modifications. In Section 4, we follow the overall proof of Kim et al. to show the existence of subdivisions of $B(k_1, k_2; 1)$ and we make some changes so that their proof is suitable for the digraphs we are interested in. Moreover, as a key step, we prove the existence of subdivisions of $B(k_1, k_2; 1)$ in Hamiltonian digraphs. This result will be presented in Section 3 and will be essential to generalize the work of Kim et al. on two-blocks cycles $C(k_1, k_2)$ to the case of $(2+1)$-bispindles $B(k_1, k_2; 1)$.

2 Preliminaries, Definitions and Notation

In what follows, we denote $[l] := \{1, 2, \ldots, l\}$ for every positive integer l. A graph G is said to be *d-degenerate* if any subgraph of G contains a vertex having at most d neighbors. Using an inductive argument, one may easily see the following statement:

Lemma 2.1 *If G is a d-degenerate graph, then G is $(d+1)$-colorable.*

The *union* of two digraphs D_1 and D_2, denoted by $D_1 \cup D_2$, is the digraph whose vertex set is $V(D_1) \cup V(D_2)$ and whose arc set is $A(D_1) \cup A(D_2)$. Note that $V(D_1)$ and $V(D_2)$ are not necessarily disjoint. The following statement is well-known:

Lemma 2.2 *For any two digraphs D_1 and D_2, $\chi(D_1 \cup D_2) \leq \chi(D_1) \times \chi(D_2)$.*

A consequence of the previous lemma is that, if we partition the arc set of a digraph D into $A_1, A_2, \ldots, A_l$, then bounding the chromatic number of all spanning subdigraphs D_i of D with arc set A_i gives an upper bound for the chromatic number of D.

Given a digraph D and a subset S of $V(D)$, the *contraction* of S (see Figure 1) into a new vertex v_S results in a new digraph D', denoted by D/S, whose vertex set is $V(D') = (V(D) \backslash S) \cup \{v_S\}$, and whose arc set $A(D')$ contains the arcs (x, y) of the following three kinds:

(i) $x, y \in V(D) \backslash S$, if $(x, y) \in A(D)$;

(ii) $x = v_S$ and $y \in V(D) \backslash S$, if there is $(z, y) \in A(D)$ for some $z \in S$;

(iii) $x \in V(D) \backslash S$ and $y = v_S$, if there is $(x, z) \in A(D)$ for some $z \in S$.

Going back from D/S to D through the *un-contraction* process, we define for every vertex $v \in V(D/S)$ the *preimage* $\phi(v)$ of v by $\phi(v) := S$ if $v = v_S$ and $\phi(v) := \{v\}$ otherwise. For $M \subset V(D/S)$, the *preimage* $\phi(M)$ of M is defined by $\phi(M) := \bigcup_{v \in M} \phi(v)$.

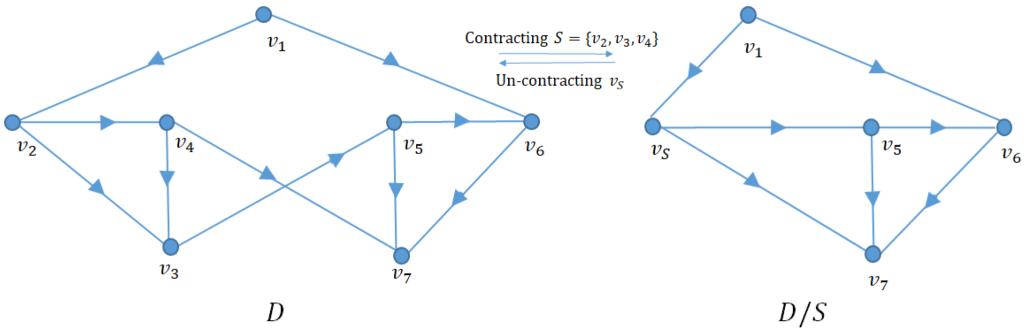

Figure 1: A tight example of the contraction and the un-contraction processes

The next lemma will be essential for the proof of our main result:

Lemma 2.3 *Let D be a digraph and let $S_1, S_2, \ldots, S_l$ be disjoint subsets of $V(D)$. If D' is the digraph obtained by contracting each S_i into one vertex v_{S_i}, then* $\chi(D) \leq \chi(D') \times \max\{\chi(D[S_i]); i \in [l]\}$.

Proof. Set $k_1 = \chi(D')$ and $k_2 = \max\{\chi(D[S_i]); i \in [l]\}$. Let α be a proper k_1-coloring of D' and let β_i be a proper k_2-coloring of $D[S_i]$ for all $i \in [l]$. Define ψ, the coloring of $V(D)$, as follows:

$$\psi(v) = \begin{cases} (\alpha(v), 1) & \text{if } v \notin S_i \forall i \in [l]; \\ (\alpha(v_{S_i}), \beta_i(v)) & \text{if } \exists i \in [l] \mid v \in S_i. \end{cases}$$

We verify now that ψ is a proper coloring of D with color-set $[k_1] \times [k_2]$. Let u and v be two adjacent vertices of D. If $u, v \in S_i$ for some $1 \leq i \leq l$, then $\beta_i(u) \neq \beta_i(v)$ and so $\psi(u) \neq \psi(v)$. Otherwise, the contraction definition implies that the vertices corresponding to u and v in D', say u' and v', are adjacent in D' as well. Thus, $\alpha(u') \neq \alpha(v')$ and so $\psi(u) \neq \psi(v)$. This completes the proof. □

Let D be a digraph. For a dipath or a directed cycle H of D and for any two vertices u, v of H, we denote by $H[u, v]$ the subdipath of H with initial vertex u

and terminal vertex v. Also, we denote by $H[u,v[$, $H]u,v]$ and $H]u,v[$ the dipaths $H[u,v]-v, H[u,v]-u$ and $H[u,v]-\{u,v\}$, respectively. For two dipaths P and Q of D, if the terminal vertex of P is the initial vertex of Q, we denote by $P \odot Q$ the dipath $P \cup Q$. Let G be a graph. For a vertex u of G, we denote by $N_G(u)$ the set of all neighbors of u in G and by $\delta(G) = \min_{u \in V(G)} |N_G(u)|$. In what follows, by a path or a cycle in a digraph we mean a directed one, unless otherwise specified.

In [12], Kim et al. introduced the notions of a *circuit-tree* and a *circuit-path* as follows. A strong digraph D is said to be a *circuit-tree*, denoted by $\mathcal{T}$, if D is covered by the union of l circuits for a positive integer l, for which there exists an ordering $\mathcal{E} := C_1, C_2, \ldots, C_l$ of all cycles of $\mathcal{T}$ such that $\left|V(C_i) \cap \left(\cup_{j \in [i-1]} V(C_j)\right)\right| = 1$ for all $2 \leq i \leq l$. Here, $\mathcal{E}$ is called a *circuit-tree ordering* of $\mathcal{T}$. A circuit-tree $\mathcal{T}$ is said to be a *circuit-path*, if there exists an ordering $C_1, C_2, \ldots, C_l$ of all cycles of $\mathcal{T}$ such that $|V(C_i) \cap V(C_j)| \leq 1$ for all $i, j \in [l]$ and $|V(C_i) \cap V(C_j)| = 1$ if and only if $|i-j| = 1$. Here, the circuits C_1 and C_l are called the *end-circuits* of $\mathcal{T}$. It is important to note that circuit-trees have some properties similar to those of trees. In fact, for any two distinct cycles C, C' of a circuit-tree $\mathcal{T}$, there exists a unique circuit-path in $\mathcal{T}$, denoted by $\mathcal{T}[C, C']$, with end-circuits C and C'. Consequently, it follows that for any two vertices $u, v \in V(\mathcal{T})$, there exists a unique path in $\mathcal{T}$ from u to v, denoted by $\mathcal{T}[u,v]$.

Given a spanning circuit-tree $\mathcal{T}$ of a digraph D with a circuit-tree ordering $\mathcal{E} := C_1, C_2, \ldots, C_l$, an arc (u,v) of D is called an *internal arc* with respect to $\mathcal{T}$ if there is an integer $1 \leq i \leq l$ such that $u, v \in V(C_i)$. Otherwise, (u,v) is called an *external arc* with respect to $\mathcal{T}$. In such case, u is called an *external in-neighbor* of v and v is called an *external out-neighbor* of u. Given a cycle C in $\mathcal{T}$ with $C \neq C_1$, the *ancestor cycle* of C, denoted by $a(C)$, is defined to be the second last cycle in $\mathcal{T}[C_1, C]$. The *ancestor vertex* of C is the unique vertex of $V(C) \cap V(a(C))$. For a vertex v of D, C_v denotes the circuit of $\mathcal{T}$ containing v and having the shortest circuit-path to the cycle C_1 and a_v denotes the ancestor vertex of C_v.

3 On bounding $\chi(S\text{-}\mathbf{Forb}(B(k_1,k_2;1)) \cap \mathcal{H})$

From now on, we consider k_1 and k_2 to be two positive integers and $k = \max\{k_1, k_2\}$. We assume that $k \geq 2$. The main aim of this section is to find an integer $f(k,k;1)$ such that every Hamiltonian digraph containing no subdivisions of $B(k,k;1)$ is $f(k,k;1)$-degenerate. By Lemma 2.1, this obviously implies an upper bound for $\chi(S\text{-Forb}(B(k,k;1)) \cap \mathcal{H}$.

Theorem 3.1 *Let D be a digraph in S-Forb$(B(k_1,k_2;1)) \cap \mathcal{H}$ and let $k = \max\{k_1,k_2\}$. Then $G(D)$ is $(5k-8)$-degenerate and thus $\chi(D) \leq 5k-7$.*

Proof. To prove that $G(D)$ is $(5k-8)$-degenerate, we have to show that every subgraph of $G(D)$ has a vertex of degree at most $5k-8$. To this end, we are going to show that $\delta(G) \leq 5k-8$ for every subgraph G of $G(D)$. Suppose not. Then there exists a subgraph G of $G(D)$ such that $\delta(G) \geq 5k-7$. Let H be the subdigraph of D whose underlying graph is G and let C be a Hamiltonian directed cycle of D. Since $\delta(G) \geq 5k-7$ and $k \geq 2$, it follows that $\delta(G) \geq 2k-1$. Thus this guarantees the existence of two vertices u, v of G such that $uv \in E(G) \backslash E(C)$ and $|V(C[u,v]) \cap V(G)| \geq 2k-1$. Assume that u, v are chosen such that $|V(C[u,v]) \cap V(G)|$ is minimal but at least $2k-1$. Consider now the vertex w of G with $|V(C[w,v]) \cap V(G)| = k$. Due to the fact that $|V(C[u,v]) \cap V(G)| \geq 2k-1$, we get that $|V(C[u,w]) \cap V(G)| \geq k$ (see Figure 2).

To reach the final contradiction, we argue on the neighbors of w in G. First, by the choice of the edge uv, note that w has at most $2k-3$ neighbors in $C[u,w] \cap V(G)$. Thus,

$$|N_G(w) \cap C[u,v]| = |N_G(w) \cap C[u,w]| + |N_G(w) \cap C[w,v]| \leq 3k-4. \tag{1}$$

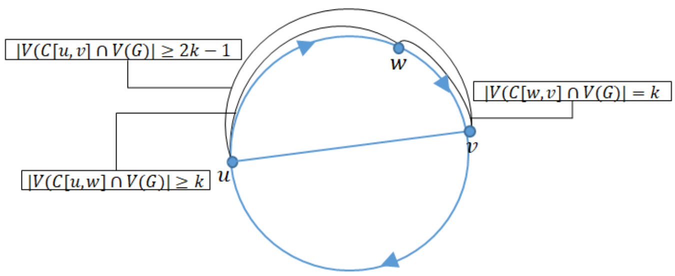

Figure 2: Figure for Theorem 3.1

Claim 3.2 $|N_H^+(w) \cap C]v, u[| \leq k-2$.

Subproof. Assume the contrary is true and consider the possible directions of the edge uv. For all $i \geq 1$, we denote by p_i the out-neighbor of w in $C]\,v, u\,[\cap H$ such that $|\, V(C]v, p_i]) \cap N_H^+(w)\, | = i$. If $(u,v) \in E(H)$, then the union of $(u,v) \odot C\,[v, p_{k-1}]\,, C[u,w] \odot (w, p_{k-1})$ and $C\,[p_{k-1}, u]$ is a subdivision of $B(k,k;1)$ in D, contradicting the fact that D is $B\,(k,k;1)$-subdivision-free (see Figure 3 (i)). Hence, $(v,u) \in E(H)$. But the union of $C[w,v] \odot (v,u), (w,p_1) \odot C\,[p_1, u]$ and $C[u,w]$ forms a subdivision of $B(k,k;1)$ in D, a contradiction (see Figure 3 (ii)). This proves our claim. $\Diamond$

Claim 3.3 $|N_H^-(w) \cap C]v, u[| \leq k-2$.

Subproof. Assume the contrary is true and consider the possible directions of the edge uv. For all $i \geq 1$, we denote by q_i the in-neighbor of w in $C]\,v, u\,[\cap H$ such that $|\, V(C]v, q_i]) \cap N_H^-(w)\, | = i$. If $(v,u) \in E(H)$, then the union of $(v,u) \odot C[u,w], C\,[v, q_{k-1}] \odot (q_{k-1}, w)$ and $C[w,v]$ forms a subdivision of $B(k,k;1)$ in D (see Figure 4 (i)), a contradiction. This means that $(u,v) \in E(H)$. But the union of $(q_1, w) \odot C[w,v], C\,[q_1, u] \odot (u,v)$ and $C\,[v, q_1]$ is a subdivision of $B(k,k;1)$ in D (see Figure 4 (ii)), a contradiction. This confirms our claim. $\Diamond$

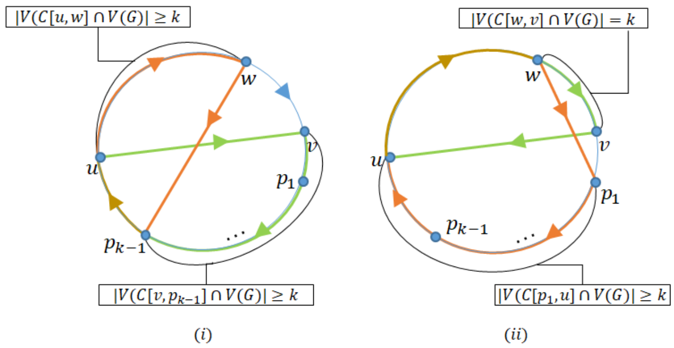

Figure 3: Figures for Claim 3.2

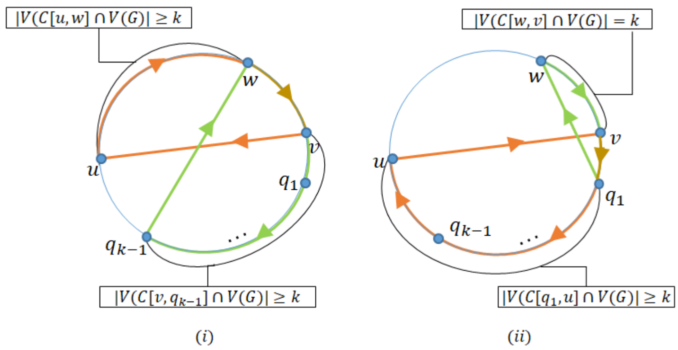

Figure 4: Figures for Claim 3.3

By the claims above, we conclude that

$$|N_G(w) \cap C]v,u[| \leq 2k-4. \tag{2}$$

Consequently, combining (1) with (2), we i obtain $d_G(w) \leq 5k-8$, which is a contradiction to the assumption that $\delta(G) \geq 5k-7$. This implies that $\delta(G) \leq 5k-8$ for every subgraph G of $G(D)$ and thus $G(D)$ is $(5k-8)$-degenerate. This ends the proof. $\square$

4 On bounding $\chi(S\text{-}\mathbf{Forb}(B(k_1,k_2;1)) \cap \mathcal{S})$

In this section, we study the chromatic number of strongly connected digraphs that are $B(k_1,k_2;1)$-subdivision-free, by following the lines of the result of Kim et al. in [12] and making some changes so that the technique they developed can fit with the digraphs of our interest. In what follows, for each subdivision of $B(k,k;1)$, we denote by P_1, P_2 and P_3 the three internally disjoint dipaths of lengths at least k, k and 1 respectively.

Let D be a strongly connected digraph containing no subdivisions of $B(k,k;1)$. We define a sequence of strong digraphs $D^0, D^1, \ldots, D^m$ and a sequence of cycles

$C^0, C^1, \ldots, C^{m-1}$ as follows: First, set $D^0 = D$. If $\chi(D^0) \leq 2k-3$, there is nothing to do. Otherwise, due to Theorem 1.4, D^0 contains a directed cycle of length at least $2k-2$. Let C^0 be a longest cycle of D^0 and let D^1 be the digraph obtained from D^0 by contracting $V(C^0)$. Clearly, D^1 is a strong digraph. If $\chi(D^1) \leq 2k-3$, we stop here. Otherwise, we keep proceeding in the same manner as before till reaching a digraph D^m such that $\chi(D^m) \leq 2k-3$.

Going backward from D^m to D^{m-1}, we define for each $v \in V(D^m)$ the *first preimage* $\phi^1(v)$ by $\phi^1(v) := \phi(v)$. Recursively, we define for every $v \in V(D^m)$ and $i \in \{2, \ldots, m\}$ the i^{th} *preimage* $\phi^i(v)$ by $\phi^i(v) := \phi(\phi^{i-1}(v))$. Clearly, $\phi^i(v)$ for each $v \in V(D^m)$ is a subset of the vertex-set of D^{m-i}, and for any two distinct vertices u, v of D^m we have $\phi^i(u) \cap \phi^i(v) = \emptyset$, implying that $\phi^i(v)$ for all $v \in V(D^m)$ form a partition of $V(D^{m-i})$. From now on, we denote by H^i the subdigraph of D^{m-i} induced by $\phi^i(v)$ for an arbitrary vertex $v \in V(D^m)$.

4.1 Properties on D^i and C^i

In this subsection, we introduce some properties on D^i and C^i that will be essential for the coming proofs. Keep in mind that C^i is a longest cycle of D^i whose length is at least $2k-2$ and that $D^{i+1} := D^i / V(C^i)$ for all $0 \leq i \leq m-1$. In what follows, we denote by v_{C^i} the new vertex created in D^{i+1} by contracting $V(C^i)$.

The following statements will be used frequently in the coming subsections:

Proposition 4.1 *For all $i \in \{0, 1, \ldots, m-2\}, l(C^i) \geq l(C^{i+1}) \geq 2k-2$.*

Proposition 4.2 *For all $i \in \{0, 1, \ldots, m\}, D^i$ is $B(k,k;1)$-subdivision-free.*

Proof. We proceed by induction on i. The case $i = 0$ follows by our initial assumption. We suppose now that D^i contains no subdivisions of $B(k,k;1)$ and we assume to the contrary that D^{i+1} contains a subdivision of $B(k,k;1)$, say $F := P_1 \cup P_2 \cup P_3$. If $v_{C^i} \notin F$, then F is a subdigraph of D^i, which yields a contradiction. Thus $v_{C^i} \in F$. Denoting by x and y the left and the right extremities of F respectively, we consider the following three possibilities: If $v_{C^i} \notin \{x, y\}$, it is straightforward to see that the subdigraph of D^i obtained from F by un-contracting v_{C^i} into the cycle C^i contains a subdivision of $B(k,k;1)$, a contradiction. Else if $v_{C^i} = x$, then there exists $x_p \in V(P_p)$ for $p \in [3]$ such that $(x, x_1), (x, x_2)$ and (x_3, x) are the three arcs of F incident to x. Note that probably $x_3 = y$. Un-contracting back to D^i, we see that there exist three arcs $(z_1, x_1), (z_2, x_2)$ and (x_3, z_3) of D^i for some $z_1, z_2, z_3 \in V(C^i)$. Possibly $z_p = z_q$ for any $1 \leq p \neq q \leq 3$. If $z_3 \in C^i[z_1, z_2]$, then the union of $C^i[z_2, z_1] \odot (z_1, x_1) \odot P_1[x_1, y], (z_2, x_2) \odot P_2[x_2, y]$ and $P_3[y, x_3] \odot (x_3, z_3) \odot C^i[z_3, z_2]$ is a subdivision of $B(k,k;1)$ in D^i, a contradiction. Else if $z_3 \in C^i[z_2, z_1]$, then the union of $(z_1, x_1) \odot P_1[x_1, y]$, $C^i[z_1, z_2] \odot (z_2, x_2) \odot P_2[x_2, y]$ and $P_3[y, x_3] \odot (x_3, z_3) \odot C^i[z_3, z_1]$ is a subdivision of $B(k,k;1)$ in D^i, a contradiction. Else if $v_{C^i} = y$, we proceed as the case where $v_{C^i} = x$. In this case, there exists $y_p \in V(P_p)$ for $p \in [3]$ such that $(y_1, y), (y_2, y)$ and (y, y_3) are the three arcs of F incident to y. Note that probably $y_3 = x$. Un-contracting back to D^i, we see that there exist three arcs $(y_1, z_1), (y_2, z_2)$

and (z_3, y_3) of D^i for some $z_1, z_2, z_3 \in V(C^i)$. Possibly $z_p = z_q$ for any $1 \leq p \neq q \leq 3$. If $z_3 \in C^i[z_1, z_2]$, then the union of $P_1[x, y_1] \odot (y_1, z_1)$, $P_2[x, y_2] \odot (y_2, z_2) \odot C^i[z_2, z_1]$ and $C^i[z_1, z_3] \odot (z_3, y_3) \odot P_3[y_3, x]$ is a subdivision of $B(k, k; 1)$ in D^i, a contradiction. Else if $z_3 \in C^i[z_2, z_1]$, then the union of $P_1[x, y_1] \odot (y_1, z_1) \odot C^i[z_1, z_2]$, $P_2[x, y_2] \odot (y_2, z_2)$ and $C^i[z_2, z_3] \odot (z_3, y_3) \odot P_3[y_3, x]$ is a subdivision of $B(k, k; 1)$ in D^i, a contradiction. This completes the proof. □

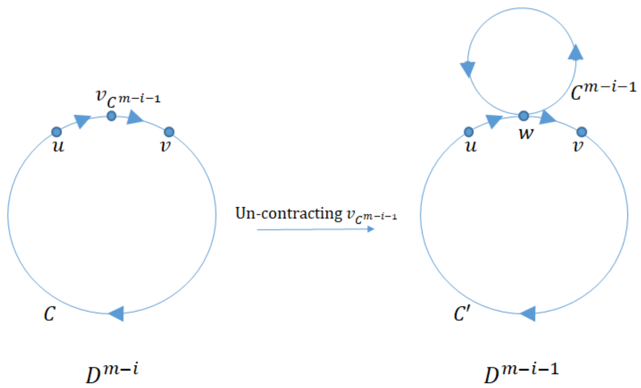

Figure 5: Figure for Lemma 4.3

4.2 Properties on H^i

The aim of this subsection is to study the structural properties of H^i for each $i \in [m]$. In the following, we denote $\mathcal{C} := \{C^0, C^1, \ldots, C^{m-1}\}$.

Lemma 4.3 *If C is a cycle in D^{m-i} of length at least $2k-2$ such that $v_{C^{m-i-1}} \in V(C)$, then the cycle C' of the digraph D^{m-i-1} obtained from C by un-contracting $v_{C^{m-i-1}}$ has the vertex set $V(C') = (V(C) \setminus \{v_{C^{m-i-1}}\}) \cup \{w\}$ with w is a vertex of C^{m-i-1}, and the arc set $A(C') = (A(C) \setminus \{(u, v_{C^{m-i-1}}), (v_{C^{m-i-1}}, v)\}) \cup \{(u, w), (w, v)\}$ with u, v are the unique in-neighbor and out-neighbor of $v_{C^{m-i-1}}$ in C respectively. Consequently, $l(C') = l(C)$.*

Proof. Since $v_{C^{m-i-1}} \in V(C)$, there exist $u, v \in V(C) \setminus \{v_{C^{m-i-1}}\}$ such that $C[u, v] = (u, v_{C^{m-i-1}}) \odot (v_{C^{m-i-1}}, v)$. Un-contracting $v_{C^{m-i-1}}$ back to D^{m-i-1}, we guarantee the existence of two arcs (u, u') and (v', v) in D^{m-i-1} for some vertices $u', v' \in V(C^{m-i-1})$. Clearly, the cycle $C' := Q \odot C^{m-i-1}[u', v']$ with $Q := (v', v) \odot C[v, u] \odot (u, u')$ is the cycle of D^{m-i-1} obtained from C by un-contracting $v_{C^{m-i-1}}$.

To reach our goal, we need to prove that $u' = v' = w$. Suppose not; then Q is a (v', u')-path of length at least k, because $l(Q) = l(C[v, u]) + 2 = l(C) - l(C[u, v]) + 2 = l(C) \geq 2k - 2 \geq k$, where the last inequality follows from the fact that $k \geq 2$. If $l(C^{m-i-1}[v', u']) \geq k$, then the union of $Q, C^{m-i-1}[v', u']$ and $C^{m-i-1}[u', v']$ is a subdivision of $B(k, k; 1)$ in D^{m-i-1} (see Figure 6 (*i*)), a contradiction to Proposition 4.2. Thus $l(C^{m-i-1}[v', u']) \leq k-1$ and so $l(Q) - l(C^{m-i-1}[v', u']) \geq k - 1 > 0$. This gives $l(C') = l(Q) + l(C^{m-i-1}[u', v']) = l(Q) + l(C^{m-i-1}) - l(C^{m-i-1}[v', u']) > l(C^{m-i-1})$, a contradiction to the fact that C^{m-i-1} is a longest cycle in D^{m-i-1} (see Figure 6 (ii)). This proves that $u' = v' = w$ (see Figure 5), which yields the desired result. □

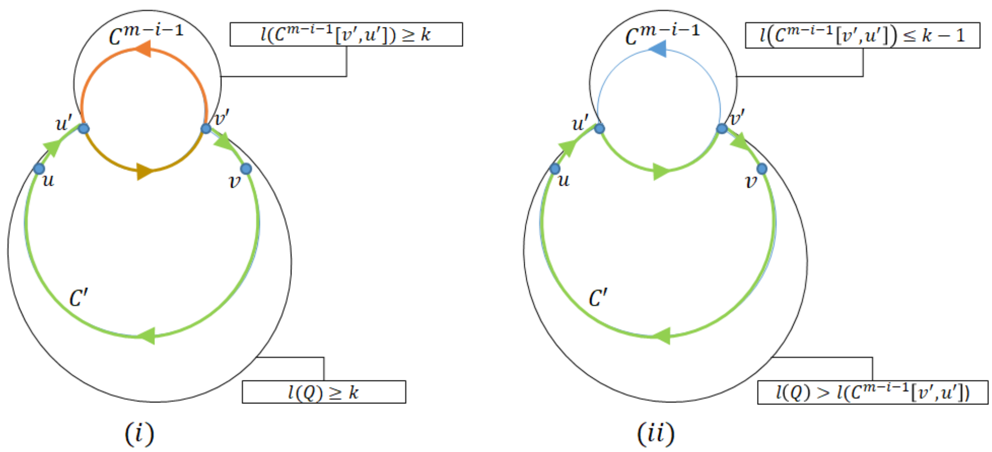

Figure 6: Figures for the proof of Lemma 4.3

Given a cycle C in D^{m-i} for $i \in \{2, \ldots, m\}$, we say that C is a C^{m-j}-*like cycle* for a cycle C^{m-j} of $\mathcal{C}$ with $1 \leq j < i \leq m$ if C is obtained from C^{m-j} by the successive un-contraction back to D^{m-l} of the vertices $v_{C^{m-l}}$ that belong to $V(C^{m-j})$ for each $j < l \leq i$. The next lemma shows that a cycle C of $\mathcal{C}$ and a C-like cycle have the same length:

Lemma 4.4 *Let C be a C^{m-j}-like cycle in D^{m-i} with $1 \leq j < i \leq m$. Then $V(C) = \left(V(C^{m-j}) \setminus \left\{\bigcup_{j<l\leq i} v_{C^{m-l}}\right\}\right) \cup \left\{\bigcup_{j<l\leq i} w_{m-l}\right\}$ with $v_{C^{m-l}}$, w_{m-l} being vertices of C^{m-j} and C^{m-l} respectively, and*

$$A(C) = \left(A(C^{m-j}) \setminus \bigcup_{j<l\leq i} \{(u_{m-l}, v_{C^{m-l}}), (v_{C^{m-l}}, v_{m-l})\}\right) \cup \bigcup_{j<l\leq i} \{(u_{m-l}, w_{m-l}), (w_{m-l}, v_{m-l})\},$$

where u_{m-l}, v_{m-l} are the unique in-neighbor and out-neighbors of $v_{C^{m-l}}$ in C^{m-j} respectively. Consequently, $l(C) = l(C^{m-j})$.

Proof. Without loss of generality, we may assume that C is not a subdigraph of D^{m-p} for all $p < i$. Due to this assumption, $v_{C^{m-i}} \in V(C^{m-j})$. The proof is by induction on the number of the un-contraction of the contracted vertices that belong to $V(C^{m-j})$ back to D^{m-i}, say t. For the base case $t = 1$, if $i = j + 1$, then the result follows directly by applying Lemma 4.3, knowing that C^{m-j} is a cycle in D^{m-j} of length at least $2k - 2$. Else if $i > j + 1$, we may easily see that C^{m-j} is a subdigraph of D^{m-i+1}, and so the result follows due to Lemma 4.3. Now we assume that the statement of Lemma 4.4 is true for t and we consider the case $t + 1$. Let C' be the C^{m-j}-like cycle in D^{m-q} where q is the nearest integer to i for which $v_{C^{m-q}} \in V(C^{m-j})$. Clearly, C' contains exactly one contracted vertex which is the vertex $v_{C^{m-i}}$. Hence, C' satisfies the induction hypothesis. Thus $l(C') = l(C^{m-j})$

and so $l\left(C'\right) \geq 2k-2$. Arguing on C' as in the base case, we prove it for $t+1$. This ends the proof. $\square$

Based on the two lemmas above, we are able to characterize the structure of H^i as follows.

Proposition 4.5 *For all $i \in [m]$, either $H^i = \{v\}$ or H^i contains a spanning circuit-tree $\mathcal{T}_i$ such that each cycle of $\mathcal{T}_i$ is either a cycle of $\mathcal{C}$ or a C-like cycle for a cycle C of $\mathcal{C}$.*

Proof. We proceed by induction on i. For the sake of simplicity, for each H^i satisfying the statement above, we say that H^i satisfies $\circledast$. The base case $i=1$ follows directly from the fact that $\phi^1(v)$ is either $\{v\}$ if $v \neq v_{C^{m-1}}$ or the cycle C^{m-1} if $v = v_{C^{m-1}}$. Now assume that H^i satisfies $\circledast$ and let us consider H^{i+1}. If $H^i = \{v\}$, it is obvious that H^{i+1} is either $\{v\}$ or the cycle C^{m-i-1}. This means that H^{i+1} satisfies $\circledast$ as well. Otherwise, H^i contains a spanning circuit-tree $\mathcal{T}_i$ such that each cycle of $\mathcal{T}_i$ is either a cycle of $\mathcal{C}$ or a C-like cycle for a cycle C of $\mathcal{C}$. If $H^i = H^{i+1}$, then H^{i+1} satisfies $\circledast$ with $\mathcal{T}_i = \mathcal{T}_{i+1}$. Else if $H^i \neq H^{i+1}$, then $v_{C^{m-i-1}}$ must belong to $V\left(H^i\right)$ and thus $v_{C^{m-i-1}}$ belongs to a non-empty set $\mathcal{R}$ of cycles in $\mathcal{T}_i$. Let C' be a cycle of $\mathcal{R}$ and let C'' be the cycle of H^{i+1} obtained from C' by un-contracting the vertex $v_{C^{m-i-1}}$. First, by the definition of a C-like cycle, it is clear that the cycle C'' is either a C'-like cycle in case that C' is a cycle of $\mathcal{C}$, or a C-like cycle for a cycle C of $\mathcal{C}$ in case that C' is a C-like cycle. Now observe that $l\left(C'\right) \geq 2k-2$. In fact, if C' is a cycle of $\mathcal{C}$, then Proposition 4.1 implies that $l\left(C'\right) \geq 2k-2$. Else if C' is a C-like cycle for a cycle C of $\mathcal{C}$, then Lemma 4.4 implies that $l\left(C'\right) = l(C) \geq 2k-2$, where the inequality follows from Proposition 4.1. But C' is a cycle in D^{m-i} such that $v_{C^{m-i-1}} \in V\left(C'\right)$, then Lemma 4.3 implies that $V\left(C''\right) = \left(V\left(C'\right) \setminus \{v_{C^{m-i-1}}\}\right) \cup \{w\}$ with w is a vertex of C^{m-i-1}, and $A\left(C''\right) = \left(A\left(C'\right) \setminus \{\left(u, v_{C^{m-i-1}}\right), \left(v_{C^{m-i-1}}, v\right)\}\right) \cup \{(u,w),(w,v)\}$ with u, v are the unique in-neighbor and out-neighbor of $v_{C^{m-i-1}}$ in C' respectively. In view of what precedes, it can be easily seen that the subdigraph $\mathcal{T}_{i+1}$ of H^{i+1} obtained from $\mathcal{T}_i$ by adding the cycle C^{m-i-1} and by replacing each cycle C' of $\mathcal{R}$ by its corresponding cycle C'' is a spanning circuit-tree of H^{i+1} with the described property. This ends the proof. $\square$

From now on, we assume that $H^m \neq \{v\}$ and we fix a spanning circuit-tree $\mathcal{T}_m$ of H^m with a circuit-tree ordering $\mathcal{E}_m := C_1, C_2, \ldots, C_l$. Note that the existence of $\mathcal{T}_m$ is guaranteed by Proposition 4.5. The next property will be fundamental for the next subsection:

Proposition 4.6 *Let (u,v) be an external arc of H^m and let C^u, C^v be the cycles of $\mathcal{T}_m$ containing u and v respectively such that $l\left(\mathcal{T}_m\left[C^u, C^v\right]\right)$ is minimal. If x and y are the common vertices of the first two and the last two cycles of the circuit-path $\mathcal{T}_m\left[C^u, C^v\right]$ respectively, then $l\left(\mathcal{T}_m[y,u]\right) \leq k-2$ and $l\left(\mathcal{T}_m[v,x]\right) \leq k-2$.*

Proof. Set $\mathcal{T}_m\left[C^u, C^v\right] = C_1, C_2, \ldots, C_t$ for some positive integer t. Since (u,v) is an external arc of H^m, then $t \neq 1$. Moreover, by the choices of the cycles C^u and C^v, note

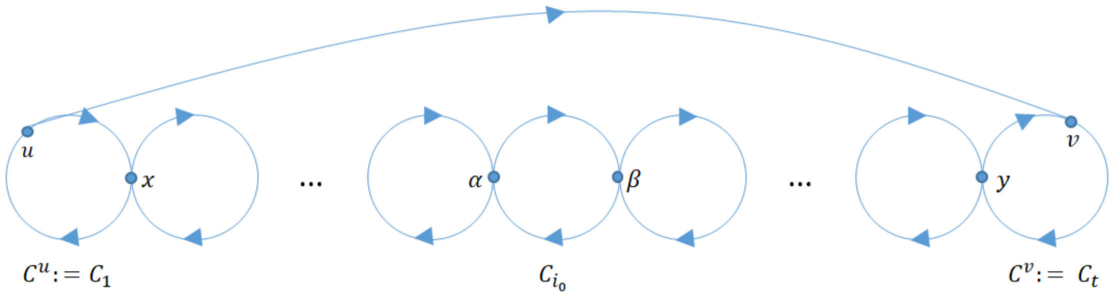

Figure 7: $\mathcal{T}_m\left[C^u, C^v\right] := C_1, C_2, \ldots, C_t$

that $u \notin \{x, y\}$ and $v \notin \{x, y\}$. Keep in mind that Proposition 4.5 gives that C_i, for all $1 \leq i \leq t$, is either a cycle of $\mathcal{C}$ or a C-like cycle for a cycle C of $\mathcal{C}$. Consequently, according to Lemma 4.4 and Proposition 4.1, the length of each cycle of $\mathcal{T}_m\left[C^u, C^v\right]$ is from $\mathcal{L}$ and thus at least $2k-2$, where $\mathcal{L} := \{l\left(C^0\right), l\left(C^1\right), \ldots, l\left(C^{m-1}\right)\}$. This guarantees the existence of $0 \leq l \leq m-1$ such that $l(C^l) = \gamma$, with $\gamma = \max\{l(C_i) \mid 1 \leq i \leq t\}$. Set $j = \min\{l \mid 0 \leq l \leq m-1\}$ for which $l(C^j) = \gamma$ and let C_{i_0} be a cycle of $\mathcal{T}_m\left[C^u, C^v\right]$ whose length is equal to γ.

In the rest of the proof, we will treat $H^m\left[\mathcal{T}_m\left[C^u, C^v\right]\right]$ as an induced subdigraph of D^j. This is because of the following observations: A cycle in D^0 exists in D^j if it is not contracted while passing from D^0 to D^j. But, by the minimality of j and due to the way we paved to contract the cycles, a cycle is contracted before D^j only if it has a length strictly greater than $l(C^j)$. This implies that every cycle whose length is less than or equal to $l(C^j)$ is not contracted before D^j. As $l(C_i) \leq l(C^j)$ for all $1 \leq i \leq t$, it follows that all the cycles of $\mathcal{T}_m\left[C^u, C^v\right]$ have not been contracted before D^j, and as a result, D^j contains $H^m\left[\mathcal{T}_m\left[C^u, C^v\right]\right]$ as an induced subdigraph.

Let us prove now that $l\left(\mathcal{T}_m[y, u]\right) \leq k-2$. Suppose not, then $l\left(\mathcal{T}_m[y, v]\right) \leq k-1$, since otherwise the union of $\mathcal{T}_m[y, v]$, $\mathcal{T}_m[y, u] \odot (u, v)$ and $\mathcal{T}_m[v, y]$ is a subdivision of $B(k, k; 1)$, a contradiction. This gives that $l\left(\mathcal{T}_m[v, y]\right) \geq k-1$ as $l\left(C^v\right) \geq 2k-2$. Similarly, we can prove that $l\left(\mathcal{T}_m[u, x]\right) \leq k-1$ and $l\left(\mathcal{T}_m[x, u]\right) \geq k-1$. Now observe that C_{i_0} is neither C_1 nor C_t. In fact, if $i_0 = 1$, then we consider the cycle $C := (u, v) \odot \mathcal{T}_m[v, y] \odot \mathcal{T}_m[y, u]$. As C is a subdigraph of $H^m\left[\mathcal{T}_m\left[C^u, C^v\right]\right]$ (which is a subdigraph of D^j), then C is a subdigraph of D^j as well, with

$$
\begin{aligned}
l(C) &= 1 + l(\mathcal{T}_m[v, y]) + l(\mathcal{T}_m[y, u]) \\
&\geq 1 + (k-1) + l\left(\mathcal{T}_m[y, x]\right) + l\left(\mathcal{T}_m[x, u]\right), \\
&\qquad \text{(because } l(\mathcal{T}_m[v, y]) \geq k-1 \text{ and } \mathcal{T}_m[y, u] = \mathcal{T}_m[y, x] \odot \mathcal{T}_m[x, u]) \\
&= k + l\left(\mathcal{T}_m[y, x]\right) + l(C_1) - l(\mathcal{T}_m[u, x]), \\
&\qquad \text{(because } C_1 = \mathcal{T}_m[x, u] \odot \mathcal{T}_m[u, x]) \\
&\geq k + l(C^j) - (k-1), (l(\mathcal{T}_m[u, x]) \leq k-1, \\
&\qquad \text{(because } l(C_1) = \gamma = l(C^j) \text{ and } l(\mathcal{T}_m[y, x]) \geq 0) \\
&> l\left(C^j\right),
\end{aligned}
$$

a contradiction to the fact that C^j is a longest cycle of D^j. In a similar way, we can prove that $i_0 \neq t$. Set α, β to be the vertices of C_{i_0} such that $V(C_{i_0}) \cap V(C_{i_0-1}) = \{\alpha\}$ and $V(C_{i_0}) \cap V(C_{i_0+1}) = \{\beta\}$. Note that probably $\alpha = x$ and $\beta = y$. To reach the final contradiction, we consider the possible lengths of $C_{i_0}[\alpha, \beta]$. If $l(C_{i_0}[\alpha, \beta]) \leq k-1$, then the cycle $C' = (u,v) \odot \mathcal{T}_m[v,y] \cup \mathcal{T}_m[y,\beta] \odot C_{i_0}[\beta,\alpha] \odot \mathcal{T}_m[\alpha,x] \odot \mathcal{T}_m[x,u]$ is a cycle in D^j whose length

$$\begin{aligned}
l(C') &= 1 + l(\mathcal{T}_m[v,y]) + l(\mathcal{T}_m[y,\beta]) + l(C_{i_0}[\beta,\alpha]) + l(\mathcal{T}_m[\alpha,x]) + l(\mathcal{T}_m[x,u]) \\
&\geq 1 + (k-1) + l(\mathcal{T}_m[y,\beta]) + l(C_{i_0}) - l(C_{i_0}[\alpha,\beta]) + l(\mathcal{T}_m[\alpha,x]) + l(\mathcal{T}_m[x,u]), \\
&\qquad \text{(because } l(\mathcal{T}_m[v,y]) \geq k-1 \text{ and } C_{i_0} = C_{i_0}[\beta,\alpha] \odot C_{i_0}[\alpha,\beta]) \\
&\geq k + l(\mathcal{T}_m[y,\beta]) + l(C^j) - (k-1) + l(\mathcal{T}_m[\alpha,x]) + (k-1), \\
&\qquad \text{(because } l(C_{i_0}) = l(C^j), l(C_{i_0}[\alpha,\beta]) \leq k-1 \text{ and } l(\mathcal{T}_m[x,u]) \geq k-1) \\
&> l(C^j),
\end{aligned}$$

contradicting the maximality of $l(C^j)$ in D^j. Else if $l(C_{i_0}[\alpha,\beta]) > k-1$, the union of $C_{i_0}[\alpha,\beta], \mathcal{T}_m[\alpha,u] \odot (u,v) \odot \mathcal{T}_m[v,\beta]$ and $C_{i_0}[\beta,\alpha]$ forms a subdivision of $B(k,k;1)$ in D^j, a contradiction to Proposition 4.2. This proves that $l(\mathcal{T}_m[y,u]) \leq k-2$.

Now we shall prove that $l(\mathcal{T}_m[v,x]) \leq k-2$. Suppose the contrary is true; then $P := (u,v) \odot \mathcal{T}_m[v,x]$ is a directed path of length at least k. Note that $\mathcal{T}_m[x,u]$ is a proper subpath of $\mathcal{T}_m[y,u]$ and thus of length at most $k-3$. But $l(C_1) \geq 2k-2$, so then $l(\mathcal{T}_m[u,x]) \geq k$. As a consequence, the union of $P, \mathcal{T}_m[u,x]$ and $\mathcal{T}_m[x,u]$ forms a subdivision of $B(k,k;1)$ in D^j, a contradiction. This proves that $l(\mathcal{T}_m[v,x]) \leq k-2$. □

4.3 Coloring H^m

This subsection is devoted finding a proper coloring of H^m. To this end, we partition the vertex-set of H^m into two subsets V_1 and V_2, where $V_1 = \{v \in H^m; C_v = C_1$ or $l(C_v[v,a_v]) \geq k-1\}$ and $V_2 = V(\mathcal{T}_m) \backslash V_1$. Note that all the notation used in this subsection are already introduced in Section 2. Now we partition the arc-set of H^m into $A_1 = \{(u,v) \mid (u,v)$ is an external arc of H^m and u, v are not vertices of the same subset $V_i\}$ and $A_2 = A(H^m) \backslash A_1$. For $i = 1, 2$, let H_i^m be the spanning subdigraph of H^m whose arc set is A_i. According to Lemma 2.2, it is enough to color H_i^m properly for each $i \in [2]$ to get a proper coloring of H^m.

By assigning the vertices of V_1 the color 1 and those of V_2 the color 2, the next lemma directly follows.

Lemma 4.7 *$\chi(H_1^m) \leq 2$.*

Now the rest of this subsection is dedicated to show that $\chi(H_2^m) \leq 6k-8$. In fact, Kim et al. [12] showed that for any digraph H with a spanning circuit-tree that satisfies Proposition 4.6 (which is H^m in our case), then the spanning subdigraph of H whose arc-set is A_2 (which is H_2^m in our case) satisfies the following properties. An external arc (u,v) of H^m is called *comparable* if either C_u is a cycle in $\mathcal{T}_m[C_v, C_1]$ or C_v

is a cycle in $\mathcal{T}_m[C_u, C_1]$. Actually, Kim et al. proved that all the external arcs of H_2^m are comparable. Recall that we have fixed a circuit-tree ordering $\mathcal{E}_m := C_1, C_2, \ldots, C_l$ of $\mathcal{T}_m$. For every $i \in \{2, \ldots, l\}, a_i$ denotes the ancestor vertex of C_i. And for every $i \in [l], (H_2^m)_i$ denotes the subdigraph of H_2^m induced by $V(C_1) \cup V(C_2) \cup \ldots \cup V(C_i)$. Note that $(H_2^m)_1 = H_2^m[V(C_1)]$ and $(H_2^m)_l = H_2^m$. Denoting by $E(v)$ the set of the exterior neighbors of a vertex $v \in V(C_i) \setminus \{a_i\}$ in $(H_2^m)_i$, Kim et al. proved the following:

Lemma 4.8 *(Kim et al. [12]) For all $i \in \{2, \ldots, l\}$ and for any vertex $v \in V(C_i) \backslash \{a_i\}$, we have $|E(v)| \leq k-2$.*

Based on what precedes, we are able to color H_2^m properly by $6k-8$ colors:

Proposition 4.9 *H_2^m is $(6k-9)$-degenerate and thus $\chi(H_2^m) \leq 6k-8$.*

Proof. The proof is exactly the same as that introduced by Kim et al. [12] to prove that the spanning subdigraph with arc set A_2 of a digraph having a circuit-tree and containing no subdivisions of two-blocks cycles $C(k, k)$ is $(3k-2)$-degenerate.

If H^m has no external arcs, consider the reverse of the circuit-tree ordering $C_l, \ldots, C_1$ one by one. Due to Theorem 3.1, we may see easily that H_2^m is $(5k-8)$-degenerate and thus $(6k-9)$-degenerate. Otherwise, we prove by induction on $i \in [l]$ that $(H_2^m)_i$ is $(6k-9)$-degenerate. Note that this is sufficient as $(H_2^m)_l = H_2^m$. The base case $i = 1$ follows directly from Theorem 3.1, since $(H_2^m)_1 = H_2^m[V(C_1)]$ is a Hamiltonian digraph with no subdivisions of $B(k, k; 1)$. Now suppose that $(H_2^m)_{i-1}$ is $(6k-9)$-degenerate and let us consider $(H_2^m)_i$. Note that $(H_2^m)_i$ is the union of $(H_2^m)_{i-1}, H_2^m[V(C_i)]$ and the external arcs between $V(C_i)$ and $(H_2^m)_{i-1}$. Let F be a subgraph of $G((H_2^m)_i)$. We shall prove that F contains a vertex whose degree is at most $6k-9$. If F is a subgraph of $G\left((H_2^m)_{i-1}\right)$, then the result follows directly by the hypothesis induction. If F is a subgraph of $G(H_2^m[V(C_i)])$, then we are done due to Theorem 3.1. Otherwise, $V(F) = N \cup M$ where $N \subset V\left((H_2^m)_{i-1}\right)$ and $M \subset V(C_i)$. Since $F[M]$ is a subgraph of $G(H_2^m[V(C_i)])$, then $F[M]$ contains a vertex $u \neq a_i$ whose degree in $F[M]$ is at most $5k-7$. According to Lemma 4.8, u has at most $k-2$ neighbors in $F[N]$ and thus u has at most $6k-9$ neighbors in F. This proves that H_2^m is $(6k-9)$-degenerate. Thus, due to Lemma 2.1, we get that $\chi(H_2^m) \leq 6k-8$. □

Now we are ready to state our main theorem on the existence of subdivisions of $B(k_1, k_2; 1)$ in strong digraphs:

Theorem 4.10 *Let D be a digraph in S-Forb$(B(k_1, k_2; 1)) \cap \mathcal{S}$ and let $k = \max\{k_1, k_2\}$. Then $\chi(D) \leq 2(2k-3)(6k-8)$.*

Proof. First, note that $\phi^m(v)$ for every $v \in V(D^m)$ are disjoint subsets of $V(D)$. Since D^m is the digraph obtained by contracting each $\phi^m(v)$ into v, due to Lemma 2.3 we find that $\chi(D) \leq (2k-3) \times \max\{\chi(D[\phi^m(v)]); v \in V(D^m)\}$. Choose $v \in V(D^m)$ to maximize $\chi(H^m)$, where $H^m = D[\phi^m(v)]$. Partition H^m into H_1^m and H_2^m as above.

According to Lemmas 2.2, 4.7, and Proposition 4.9, we get that $\chi(H^m) \leq 2(6k-8)$. This completes the proof. $\square$

Since any subdivision of the (2+1)-bispindle $B(k_1, k_2; 1)$ contains a subdivision of the two-blocks cycle $C(k_1, k_2)$, we get the following:

Corollary 4.11 *Let D be a digraph in S-*Forb$(C(k_1,k_2)) \cap \mathcal{S}$ *and let* $k = \max\{k_1, k_2\}$. *Then* $\chi(D) \leq 2(2k-3)(6k-8)$.

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