# Paths, Cycles and Wheels in Graphs without Antitriangles

*Stanisław P. Radziszowski and Xia Jin*

Department of Computer Science
Rochester Institute of Technology
Rochester, NY 14623

**Abstract.** We investigate paths, cycles and wheels in graphs with independence number of at most 2, in particular we prove theorems characterizing all such graphs which are hamiltonian. Ramsey numbers of the form $R(G,K_3)$, for $G$ being a path, a cycle or a wheel, are known to be $2n(G)-1$, except for some small cases. In this paper we derive and count all critical graphs for these Ramsey numbers.

## 1. Notation and Previous Work

For any graph $F$, $V(F)$ and $E(F)$ will denote the vertex and edge sets of the graph $F$, also let $n(F)=|V(F)|$ and $e(F)=|E(F)|$. The graph $\overline{F}$ denotes the complement of $F$. A graph $F$ will be called a $(G,H)$*–good* graph, if $F$ does not contain $G$ and $\overline{F}$ does not contain $H$. Any $(G,H)$-good graph on $n$ vertices will be called a $(G,H,n)$*–good* graph. The *Ramsey number* $R(G,H)$ is defined as the smallest integer $n$ such that no $(G,H,n)$-good graph exists. Any graph is called a *critical* graph for the Ramsey number $R(G,H)$ if it is $(G,H,R(G,H)-1)$-good. When the graph $F$ is fixed, then for any vertex $x \in V(F)$, $G_x$ and $H_x$ will denote the graphs induced by the neighbors of the vertex $x$ or by all the vertices disconnected from $x$, respectively. $P_i$ is a path on $i$ vertices, $C_i$ is a cycle of length $i$, and $W_i$ is a wheel with $i-1$ spokes, i.e. a graph formed by some vertex $x$, called a *hub* of the wheel, connected to all vertices of some cycle $C_{i-1}$, called a *rim*. $2K_i$ is the graph formed by two vertex disjoint copies of $K_i$. For notational convenience we define $C_i=K_i$ for $1 \leq i \leq 2$.

In this paper most of the graphs considered are $(T,K_3,n)$-good for $T$ being a path, a cycle, or a wheel. It is easy to see that if $F$ is any $(T,K_3,n)$-good graph and $x$ is a vertex in $V(F)$ of degree $deg(x)=d$, then:

(a) if $T=C_{i+1}$ then $G_x$ is a $(P_i,K_3,d)$-good graph,
(b) if $T=W_{i+1}$ then $G_x$ is a $(C_i,K_3,d)$-good graph,

and $H_x$ is a complete graph $K_{n-d-1}$. Observe that any graph without independent sets of size three and with more than one component is a vertex disjoint union of two cliques. We also note that the whole contents of this paper can be seen as a study of paths, cycles and wheels in the complements of triangle free graphs.

The value of the Ramsey number $R(P_i,K_3)=2i-1$ is a consequence of a well known theorem by Chvátal [3]. An interesting general related result in [2] says that $R(G,K_3)=2i-1$ for any connected graph $G$ of order $i\geq 4$ with at most $(17i+1)/15$ edges, which obviously applies to the cases of paths and cycles, but not wheels. Burr and Erdös [1] showed that $R(W_i,K_3)=2i-1$ for all $i\geq 6$, and the tables by Clancy [4] include the special value of $R(W_5,K_3)=11$. McKay and Faudree [5] generated and counted by computer all of the critical graphs for the Ramsey numbers $R(W_j,K_3)$ for all $j\leq 11$, and our proofs confirm their results. In two recent papers Sidorenko studied the general case: in [7] he showed that for any graph $G$ without isolated vertices we have $R(G,K_3)\leq 2e(G)+1$, which improved on his previous result in [6], where he also formulated an interesting conjecture that for any graph $G$ there is a general bound $R(G,K_3)\leq n(G)+e(G)$. Sidorenko's result in [7] proves Harary's conjecture formulated in 1980.

We derive a characterization of all hamiltonian graphs with independence number at most 2. For $T$ being any of $P_i$, $C_i$ or $W_i$ we will describe and count all of the critical graphs for the Ramsey numbers $R(T,K_3)$, in particular we will prove that almost all such critical graphs must contain $2K_{i-1}$. The latter will also give alternate proofs of previously known results that for the same possible $T$'s and for all $i\geq 1$ we have $R(T,K_3)=2i-1$, except some small cases listed in Theorems 3 and 5. We include these alternate proofs, so the results of this paper are self contained.

## 2. Paths

**Lemma 1:** *If the graph $\overline{F}$ has no triangles then all the components of $F$ have a hamiltonian path.*

*Proof:* Assume that $C$ is a component of $F$ without any hamiltonian path. Let $P$ on $r$ vertices be the longest path in $C$, and let $x$ and $y$ be the endpoints of $P$. Note that there must exist vertices $z$ and $t$ such that

$$z\in V(C)-V(P),\ \ t\in V(P)\ \ \text{and}\ \ \{z,t\}\in E(C).$$

Observe that $\{z,x\}$ and $\{z,y\}$ are not the edges in $C$, since otherwise $P$ would not be maximal, and that $\{x,y\}\in E(C)$, since $\overline{C}$ has no triangles. Then $C$ has a cycle $C_r$ with the vertex set $V(P)$, which with the edge $\{z,t\}$ produces a $P_{r+1}$, contradicting the maximality of $P$. □

Lemma 1 easily implies Corollary 1 below, which in turn gives us Corollary 2 including a characterization of critical graphs in the case of paths versus $K_3$.

**Corollary 1:** *For all $j\geq 1$, any $(P_j,K_3)$-good graph on at least $j$ vertices is a vertex disjoint union of two cliques of order at most $j-1$.*

**Corollary 2:** *For all $j \geq 1$, $R(P_j, K_3) = 2j-1$ and the unique up to isomorphism critical graph for this number is $2K_{j-1}$.*

## 3. Cycles

**Theorem 1:** *Any nonhamiltonian and nonempty graph $F$, without independent sets of size three, has a vertex $x$ such that $V(F)-\{x\}$ induces two vertex disjoint complete graphs. Furthermore, such $x$ is connected to all the vertices of at least one of these complete graphs.*

*Proof:* If the graph $F$ is disconnected then the theorem is obvious, hence we assume that $F$ is connected. For the first part of the theorem it is sufficient to show that for some vertex $x$, $V(F)-\{x\}$ induces a disconnected graph, since any not connected graph, without $K_3$ in the complement, must be a vertex disjoint union of two complete graphs.

Let $P = (a_1 a_2 \cdots a_n)$ be a hamiltonian path in $F$ guaranteed by Lemma 1, and note that $\{a_1, a_n\}$ is not an edge, since otherwise $F$ would be hamiltonian. Define

$$p = \max\{s : \{a_1, a_s\} \in E(F)\} \quad \text{and} \quad q = \min\{s : \{a_n, a_s\} \in E(F)\}, \tag{1}$$

so we have $2 \leq p, q \leq n-1$. First we claim that the sets

$$A = \{a_s : 1 \leq s < q\} \quad \text{and} \quad B = \{a_s : p < s \leq n\}$$

induce complete graphs in $F$, since any disconnected pair of vertices in $A$ or $B$ forms an independent set with $a_n$ or $a_1$, respectively. Hence by (1) we have $q \leq p+1$. If $q = p-1$ then $(a_1 a_2 \cdots a_q a_n \cdots a_p a_1)$ forms a hamiltonian cycle, which is a contradiction. For $q \leq p-2$, in order to avoid an independent set $\{a_{q-1}, a_{p-1}, a_n\}$ at least one of the pairs $\{a_{p-1}, a_n\}$ and $\{a_{q-1}, a_{p-1}\}$ is an edge. If the first one is an edge then $(a_1 \cdots a_{p-1} a_n \cdots a_p a_1)$ is a hamiltonian cycle, otherwise $\{a_{p-1}, a_{q-1}\}$ must be an edge and $(a_1 \cdots a_{q-1} a_{p-1} \cdots a_q a_n \cdots a_p a_1)$ is a hamiltonian cycle.

Thus we have $q = p+1$ or $q = p$, and we claim that $V(F)-\{a_r\}$, for $r = p$ or $r = q$, induces a disconnected graph. Recall that $A$ and $B$ induce complete graphs, and consider the two cases with respect to $p$ and $q$:

**Case of** $q = p$. Note that $V(F) = A \cup B \cup \{a_p\}$, and $r = p$. Assume the contrary to the claim, i.e. that for some $s$ and $t$, such that $1 < s < p < t < n$, $\{a_s, a_t\} \in E(F)$. Then $a_p HP_A(a_1, a_s) HP_B(a_t, a_n) a_p$ is a hamiltonian cycle in $F$, where $HP_X(x, y)$ denotes any hamiltonian path from $x$ to $y$ in a complete graph with a vertex set $X$. This is a contradiction.

**Case of** $q = p+1$. In this case $V(F) = A \cup B$, so since $F$ is not hamiltonian, the set of edges connecting $A$ to $B$ cannot contain two nonadjacent edges. However $\{a_p, a_q\}$ is an edge between $A$ and $B$, thus all the other such edges are either connected to $a_p$, in this case define $r = p$, or all of them are connected to $a_q$, in which case define $r = q$. Now clearly the graph induced by $V(F)-\{a_r\}$ is formed by two disjoint complete graphs with vertex sets $A$ and $B$. This completes the proof of the first part of the theorem.

For the second part, if for some vertices $a$ and $b$, in different components of the graph induced by $V(F)-\{x\}$, $\{a,x\}$ and $\{b,x\}$ are not the edges, then the set $\{x,a,b\}$ forms a triangle in $\overline{F}$, which is a contradiction. □

**Theorem 2:** *Let $F$ be any graph different from $C_4$ and $C_5$, and such that $\overline{F}$ has no triangles. Then, if $F$ is hamiltonian then it contains a cycle $C_j$ for all $1 \le j \le n(F)$.*

*Proof:* Let $F$ be any hamiltonian graph on $n$ vertices as in the theorem. The case $n \le 3$ is trivial. For $4 \le n \le 5$ adding one edge to $C_n$ creates cycles $C_j$ for all $j \le n$. If $n=6$ then $F$ contains $C_6$ with at least two additional edges also creating $C_j$ for all $j<6$, hence the theorem holds for $n \le 6$.

We will complete the proof by induction for $n \ge 7$. Let $(a_0 a_1 \cdots a_{n-1} a_0)$ be a hamiltonian cycle in $F$. If for some $i$, $\{a_i, a_{i+2}\} \in E(F)$, with arithmetic performed modulo $n$, then $V(F)-\{a_{i+1}\}$ easily induces a hamiltonian graph on $n-1$ vertices, hence by induction $F$ contains a cycle $C_j$ for all $j \le n-1$. Thus we may assume that for all $i$, $\{a_i, a_{i+2}\}$ is a nonedge, furthermore in order to avoid independent sets of the form $\{a_i, a_{i+2}, a_{i+4}\}$ we must have

$$\{\{a_i, a_{i+4}\} : 0 \le i \le n-1\} \subseteq E(F).$$

Now observe that

$$(a_0 a_4 a_5 a_1 a_2 a_6 \cdots a_{n-1} a_0)$$

is a hamiltonian cycle in the graph $G$ on $n-1$ vertices induced by $V(F)-\{a_3\}$. By induction $G$ has a cycle $C_j$ for all $j \le n-1$, therefore so does $F$ and the theorem follows. □

**Corollary 3:** *For all $j \ge 1$, any $(C_j, K_3)$-good graph $F$ on at least $j$ vertices, except for $F=C_4$ and $F=C_5$, has a vertex $x$ such that $V(F)-\{x\}$ induces two vertex disjoint complete graphs, and $x$ is connected to all the vertices of at least one of these complete graphs.*

*Proof:* By Theorem 2 any such graph must be nonhamiltonian, hence by Theorem 1 it has a required structure. □

**Theorem 3:** *$R(C_j, K_3)=2j-1$ for all $j \ge 1$ (but $j \ne 3$), and $R(C_3, K_3)=6$. Furthermore there are exactly two critical graphs for all $j \ge 4$, namely $2K_{j-1}$ with 0 or 1 edge joining two cliques, and unique critical graphs $\emptyset, 2K_1$, and $C_5$ for $j=1,2$ and 3, respectively.*

*Proof:* Since $C_j = K_j$ for $j \le 3$, $R(K_1, K_3)=1$, $R(K_2, K_3)=3$, $R(K_3, K_3)=6$, and $\emptyset, 2K_1$, and $C_5$ are the corresponding unique critical graphs, the theorem holds for $j \le 3$. For $j \ge 4$ and $n \ge \max(j, 6)$ consider any $(C_j, K_3, n)$-good graph $F$. By Corollary

3 we may assume that

$$V(F) = A \cup B \cup \{x\}, \quad \text{and} \quad A \cap B = \emptyset,$$

where $A$ and $B$ induce complete graphs and $x$ is connected to the whole $A$. Let $p = |A|$, $q = |B|$ and $e$ be the number of edges connecting $x$ to $B$. Then we clearly have

$$p \le j-2, \quad q \le j-1 \quad \text{and} \quad n = p+q+1, \tag{2}$$

furthermore if $e > 1$ then $q \le j-2$. Conditions (2) imply that $n \le 2j-2$, which gives a bound $R(C_j, K_3) \le 2j-1$. In addition we have an equality $n = 2j-2$ if and only if $p = j-2$, $q = j-1$ and $e = 0$ or 1, which obviously corresponds to the two critical graphs specified in the theorem. □

## 4. Wheels

### 4.1. A Characterization

In this section we will characterize all $(W_{j+1}, K_3, 2j)$-good graphs for all $j \ge 6$, in particular we will show that any such graph must contain $2K_j$. This in turn will permit us to conclude that $R(W_{j+1}, K_3) = 2j+1$ for all $j \ge 6$. We note that the proof of the latter could be simplified (as in [1]) if we do not derive a full characterization of critical graphs.

**Lemma 2:** *For all $j \ge 5$, if a $(W_{j+1}, K_3, 2j)$-good graph $F$ contains a $K_j$, then $F$ contains $2K_j$.*

*Proof:* Let $F$ be as in the lemma, so we may assume that $V(F) = A \cup B$, $|A| = |B| = j$, and $A$ induces a $K_j$. We will show that $B$ also induces a $K_j$. Assume the contrary, and let $y_1$ and $y_2$ be disconnected vertices in $B$. Denote by $p_i$, $i = 1, 2$, the number of vertices in $A$ connected to $y_i$. If $p_i \ge 3$ then the graph induced by $A \cup \{y_i\}$ contains a wheel $W_{j+1}$, hence $p_i \le 2$. Now $j \ge 5$ implies that there is a vertex $z \in A$ disconnected from both $y_1$ and $y_2$, and the set $\{z, y_1, y_2\}$ forms a triangle in $\overline{F}$, which is a contradiction. □

**Lemma 3:** *For all $j \ge 1$, every vertex in any $(W_{j+1}, K_3, 2j)$-good graph $F$ has the degree at least $j-1$. Furthermore for all $j \ge 5$, if $F$ has the minimum degree $j-1$ then it contains $2K_j$.*

*Proof:* For any vertex $x$ of any $(W_{j+1}, K_3, 2j)$-good graph $F$, the graph $H_x$ is complete, and so it can have at most $j$ vertices. Hence $|V(H_x)| = 2j - deg(x) - 1 \le j$ implies $deg(x) \ge j-1$. If $deg(x) = j-1$ then $H_x$ is a $K_j$, and thus by Lemma 2 for all $j \ge 5$ the graph $F$ contains also $2K_j$. □

**Theorem 4:** *For all* $j \geq 6$, $R(W_{j+1}, K_3) = 2j+1$ *and any* $(W_{j+1}, K_3, 2j)$*-good graph contains* $2K_j$.

*Proof:* First, in order to show that for all $j \geq 6$ any $(W_{j+1}, K_3, 2j)$-good graph contains $2K_j$, we assume the contrary and let $F$ be any such graph without $2K_j$. By Lemmas 2 and 3 it is sufficient to consider graphs $F$ with the minimum degree at least $j$ which do not contain $K_j$. For any vertex $x \in V(F)$, the structure of the $(C_j, K_3, deg(x))$-good graph $G_x$ implied by Corollary 3 is as follows:

$$V(G_x) = A \cup B \cup \{y\}, \quad \text{and} \quad A \cap B = \emptyset,$$

where $A$ and $B$ induce complete graphs and $y$ is connected to the whole set $A$. Denoting $p = |A|$, $q = |B|$, and using the assumption that $F$ does not contain $K_j$ we obtain:

$$1 \leq p, q, \quad j \leq 1 + p + q = deg(x), \quad p \leq j-3 \quad \text{and} \quad q \leq j-2, \tag{3}$$

so the maximum degree in $F$ is at most $2j-4$, and consequently every vertex in $V(H_x)$ is disconnected from at least one vertex in $A$ or $B$. Hence every vertex of $H_x$ is fully connected to either $A$ or $B$, and let $V(H_x) = HA \cup HB$, $h_A = |HA|$, $h_B = |HB|$ denote the corresponding subset of $V(H_x)$ fully connected to either $A$ or $B$. Using the latter, and the fact the we have no $K_j$, one can easily see that

$$h_A + p \leq j-1, \quad h_B + q \leq j-1, \quad \text{and} \quad h_A + h_B + deg(x) + 1 = 2j,$$

which in turn with (3) implies

$$h_A + p = j-1, \quad h_B + q = j-1 \quad \text{and} \quad 1 \leq h_A, h_B.$$

Observe that if $p \geq 3$ then any vertex $a \in A$ is a hub of a wheel $W_{j+1}$ with rim on $\{x, y\} \cup (A - \{a\}) \cup HA$, so

$$p \leq 2, \quad \text{and} \quad h_A \geq j-3 \geq 3, \tag{4}$$

and similarly, if $h_B \geq 2$ then any vertex $u \in HA$ is a hub of $W_{j+1}$ with rim on $A \cup (HA - \{u\}) \cup \{z_1, z_2\}$, for and any two vertices $z_1, z_2 \in HB$. Therefore

$$h_B = 1, \quad \{z\} = HB, \quad q = j-2. \tag{5}$$

and by (3) the degree of $x$, and thus of any vertex of $F$, satisfies

$$deg(x) \leq j+1,$$

which when applied to vertex $z \in HB$, by considering $q + h_A \leq deg(z)$, (4) and (5), gives $j = 6$. In this situation we further obtain $q = 4$, $h_A = 3$, $p = 2$, and hence the only possible counterexample is a 7-regular $(W_7, K_3, 12)$-good graph $F$. However, $z$ is disconnected from both vertices in $A$, so for $a \in A$ we have $deg(a) = 6$, which is a contradiction.

The graph $2K_j$ is $(W_{j+1}, K_3, 2j)$-good, so it remains to show that there does not exist any $(W_{j+1}, K_3, 2j+1)$-good graph $F$ for any $j \geq 6$. Assume that $F$ is such a graph, and let $x \in V(F)$. We know that the graph induced by $V(F) - \{x\}$ contains $2K_j$, hence we may assume that

$$V(F) = \{x\} \cup C \cup D, \quad |C| = |D| = j,$$

and both $C$, $D$ induce a $K_j$. Let $s$ and $t$ be the number of vertices in $C$ and $D$, respectively, connected to $x$. By Lemma 3 we have $j-1 \le deg(x) = s+t$. On the other hand $s \le 2$ and $t \le 2$, since $s \ge 3$ or $t \ge 3$ implies that $C \cup \{x\}$ or $D \cup \{x\}$ induces a graph containing $W_{j+1}$ respectively. This implies that $j \le 5$, which is a contradiction. □

## 4.2. Counting

We will count the number of nonisomorphic critical graphs for the Ramsey numbers $R(W_{j+1}, K_3)$ for all $j \ge 6$. We note that our counts agree with all the values obtained by McKay and Faudree [5] by computer enumeration.

**Lemma 4:** *For all $j \ge 4$ the number of nonisomorphic $(W_{j+1}, K_3, 2j)$-good graphs containing $2K_j$ is equal to*

$$s(j) = \sum_{i=0}^{j} h(i) f(j-i), \tag{6}$$

*where for all $i \ge 0$*

$$h(i) = \sum_{d=0}^{i/2} \left( \left\lfloor \frac{i+d}{3} \right\rfloor - d + 1 \right), \quad \text{and} \tag{7}$$

$$f(2i) = f(2i+1) = (i+1)(i+2)/2. \tag{8}$$

*Proof:* Any graph $F$ on $2j$ vertices containing $2K_j$ can be written as $F = 2K_j \cup G$, where $G$ is a subgraph of $K_{j,j}$. Observe that $\bar{F}$ has no triangles, and furthermore

$F$ does not contain $W_{j+1}$ *iff*
$G$ has a maximum degree at most 2 and $G$ has no $P_5$ *iff*
Any component of $G$ is isomorphic to $K_1$, $K_2$, $P_3$, $P_4$ or $C_4$.

Let us split the components of any $G$ as above into those on odd number of vertices, $K_1$ and $P_3$ forming $G_1$, and those on even number of vertices, $K_2$, $P_4$ and $C_4$ forming $G_2$, so

$$G = G_1 \cup G_2.$$

We will show that $h(i)$ defined in (7) and $f(i)$ defined in (8) count the number of nonisomorphic graphs on $2i$ vertices of the form of $G_1$ and $G_2$, respectively. Then (6) will certainly count all possible nonisomorphic graphs $G$, and the lemma will follow.

**Calculating h(i).** Let $V(G_1) = A \cup B$, so that $A$ and $B$ are independent sets of size $i$ in $G_1$. Let also $a$ and $b$ be the number of $P_3$'s with two vertices in $A$ or $B$, respectively. Furthermore we may assume that $a \le b$, and denote $d = b - a$. We can easily see that

$$a+2b \le i \quad \text{and} \quad 0 \le d \le i/2,$$

so

$$d \le b \le (i+d)/3,$$

and that different solutions to the above define all nonisomorphic $G_1$'s. Observe finally, that the number of such solutions is given by (7).

**Calculating f(i).** Let $a, b$ and $c$ denote the number of $K_2$, $P_4$ and $C_4$ components, respectively, in $G_2$. Similarly as before, the number of nonisomorphic $G_2$'s on $2i$ vertices is equal to the number of solutions to:

$$a+2b+2c = i \quad \text{and} \quad 0 \le d = b+c \le i/2,$$

which is equal to

$$\sum_{d=0}^{i/2} (d+1). \tag{9}$$

The proof is completed by noting that (9) reduces to (8). □

The following technical lemma is just a simplification of the formulas (7) and (6) for functions $h$ and $s$, which shows clearly their growth.

**Lemma 5:**

(a) *For all* $i \ge 0$, $h(6i) = 1+3i(i+1)$ *and* $h(6i+j) = (3i+j)(i+1)$ *for* $1 \le j \le 5$.

(b) *For all* $j \ge 0$

$$s(j) = 1 + \left\lfloor \frac{6j^5 + 165j^4 + 1700j^3 + 6t(j)}{17280} \right\rfloor,$$

*where* $t(j) = 1370j^2 + 3144j$ *for j even, and* $t(j) = 1325j^2 + 2649j$ *for j odd.*

*Proof:* (a) First use (7) and induction on $i$ to show that for all $i \ge 0$, $h(i+6) = h(i)+i+6$. Then by (7) compute $h(0)=1$ and $h(j)=j$ for $1 \le j \le 5$, and prove the first part of the lemma by induction on $i$ applied to $h(6i+j)$, for all $0 \le j \le 5$. (b) Using (a), (6) and (8), observe that $p_i(j) = s(6j+i)$ is a polynomial in $j$ of degree 5 for each fixed $i$, $0 \le i \le 5$. Then after computing $s(k)$ from (6) for $0 \le k < 36$ one can find all the coefficients of these polynomials, and some further technical work leads to the formula for $s(j)$ as in (b). □

### 4.3. All Critical Graphs

Now we can complete a description and count of all critical graphs for the Ramsey numbers $R(W_{j+1}, K_3)$, for all $j \ge 1$. The graph in Figure 1 from Lemma 6 was known to Clancy [4].

**Lemma 6:** *$R(W_5, K_3) = 11$ and there exists a unique $(W_5, K_3, 10)$-good graph as in Figure 1.*

*Proof:* By Theorem 3 $R(C_4, K_3) = 7$, so in any $(W_5, K_3, 10)$-good graph $F$ we certainly have $5 \leq deg(x) \leq 6$, for every $x$ in $V(F)$. Assume that $F$ has a vertex $x$ of degree 6, and consider a $(C_4, K_3, 6)$-good graph $G_x$. Also by Theorem 3, note that $G_x$ is critical so it contains two vertex disjoint triangles, say with vertex sets $A$ and $B$, and that there is at most one edge between $A$ and $B$. Observe that if some vertex $y \in V(H_x)$, $y$ is connected to the whole $A$ or $B$, then $\{x, y\} \cup A$ or $\{x, y\} \cup B$, respectively, induces a graph containing $W_5$. Hence let $a \in A$ and $b \in B$ be some vertices not connected to $y$. In order to avoid an independent set $\{a, b, y\}$, $\{a, b\}$ must be an edge in $F$, furthermore it has to be the only edge connecting $A$ to $B$, and any vertex $y \in V(H_x)$ is disjoint from $\{a, b\}$. Consequently, vertex $a$ is connected exactly to $\{x, b\} \cup (A - \{a\})$, i.e. $deg(a) = 4$, which is a contradiction. Thus the graph $F$ is regular of degree 5, and so it has 25 edges. For any $x \in V(F)$, $H_x$ is a $K_4$ and there are 8 edges between $G_x$ and $H_x$, hence $G_x$ must have 6 edges. Since $G_x$ is $(C_4, K_3, 5)$-good, using Corollary 3 we can easily conclude that it is isomorphic to two triangles sharing one vertex. Considering the latter property for all vertices in $F$, one can easily see that $F$ is isomorphic to the graph from Figure 1.

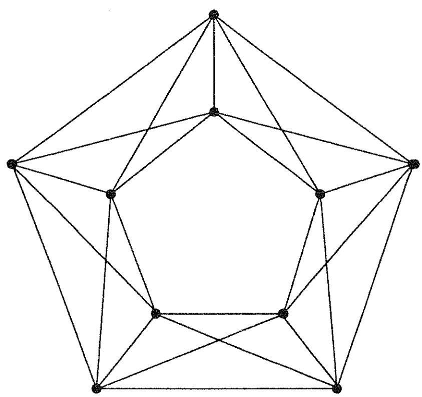

Figure 1. Unique $(W_5, K_3, 10)$-good graph.

It remains to be shown that there is no $(W_5, K_3, 11)$-good graph $F$. Observe that any such graph $F$ has to be regular of degree 6, but also $V(F) - \{x\}$ induces the unique $(W_5, K_3, 10)$-good graph, which is regular of degree 5. This is impossible, so the lemma follows. □

**Lemma 7:** *$R(W_6, K_3) = 11$ and there are exactly 37 nonisomorphic $(W_6, K_3, 10)$-good graphs. 36 of them contain $2K_5$, and the remaining one is as in Figure 1.*

*Proof:* By Lemma 4 there are $s(5)=36$ nonisomorphic $(W_6, K_3, 10)$-good graphs containing $2K_5$. Let $F$ be a $(W_6, K_3, 10)$-good graph without $2K_5$. It is sufficient to show that $F$, up to isomorphism, is as in Figure 1. By Lemma 2 $F$ has no $K_5$, and by Lemma 3 every vertex has the degree at least 5. If $deg(x) \geq 7$ then by Corollary 3 the graph $G_x$ has a $K_4$, and so $F$ has a $K_5$, which is impossible. Hence for every $x \in V(F)$ we have $5 \leq deg(x) \leq 6$. We may further assume that $F$ contains a $W_5$, since otherwise by Lemma 6 $F$ is as in Figure 1. Let $x$ be a hub of a wheel $W_5$ in $F$, and consider the graph $G_x$, which by the previous comments contains a $C_4$, but no $C_5$ neither $K_4$. Using Corollary 3, we can easily see that $G_x$ is isomorphic to $G_1$ if $deg(x)=5$ or to $G_2$ if $deg(x)=6$, as in Figure 2.

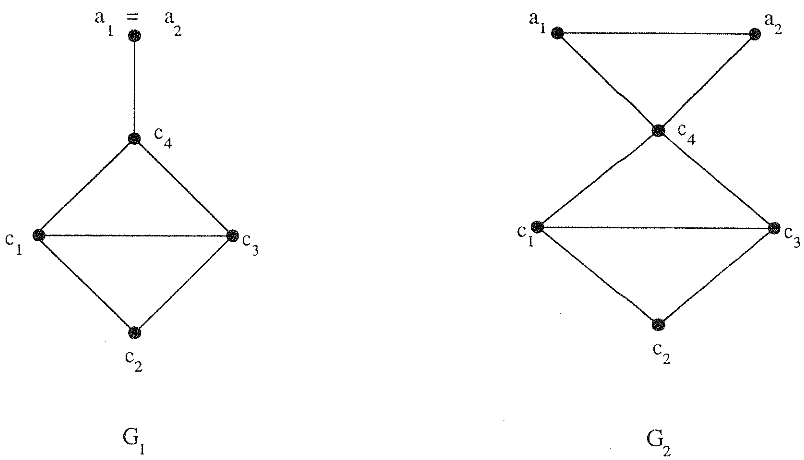


Figure 2.

In both cases a contradiction is derived by the same reasoning. In order to avoid $K_5$ at least one vertex $y \in V(H_x)$ is disconnected from $a_1$ or $a_2$. Then $y$ has to be connected to $c_i$, for $i=1$, 2 and 3, since otherwise $a_1$ or $a_2$, respectively, with $\{y, c_i\}$ forms an independent set. However now we have a wheel $W_6$ with a hub $c_1$ and a rim $yc_3c_4x\,c_2y$ in $F$.

It remains to be shown that there is no $(W_6, K_3, 11)$-good graph. Assume that $F$ is such a graph. As in Lemma 6, not all of 11 graphs induced by $V(F)-\{x\}$ can be isomorphic to the 5-regular graph in Figure 1, hence there exists $x \in V(F)$, such that the graph induced by $V(F)-\{x\}$ contains $2K_5$. Now, we easily have $deg(x) \geq 5$, which implies that $x$ is connected to at least three vertices in one of these $K_5$'s, inducing with it a graph containing $W_6$. This is a contradiction, so no $(W_6, K_3, 11)$-good graph exists. □

**Theorem 5:** *Table II summarizes the values of Ramsey numbers $R(W_j, K_3)$ and the number of corresponding critical graphs for all $j \geq 2$.*

*Proof:* The theorem holds for $j=5$ by Lemma 6, for $j=6$ by Lemma 7, and for all $j \geq 7$ by Theorem 4 and (6) in Lemma 4. For $2 \leq j \leq 4$, $W_j = K_j$, and it is well known that 3, 6, 9 are the values of the corresponding Ramsey numbers. For completeness we mention that $2K_1$ and $C_5$ are the unique critical graphs in the first two cases, and that the three $(K_4, K_3, 8)$-good graphs are the complements of $C_8$ with 2, 3 or 4 consecutive main diagonals. □

Observe finally that, by Theorem 4 and Lemmas 4 and 5, the number of nonisomorphic critical graphs for the Ramsey numbers $R(W_{j+1}, K_3)$ is of the form

$$s(j) = \frac{j^5}{2880} + \frac{11j^4}{1152} + \frac{85j^3}{864} + O(j^2).$$

| $j$ | $R(W_j, K_3)$ | order of critical graphs | number of critical graphs |
|---|---|---|---|
| 2 | 3 | 2 | 1 |
| 3 | 6 | 5 | 1 |
| 4 | 9 | 8 | 3 |
| 5 | 11 | 10 | 1 |
| 6 | 11 | 10 | 37 |
| 7 | 13 | 12 | 61 |
| 8 | 15 | 14 | 92 |
| 9 | 17 | 16 | 141 |
| 10 | 19 | 18 | 201 |
| 11 | 21 | 20 | 288 |
| 12 | 23 | 22 | 393 |
| 13 | 25 | 24 | 537 |
| ... | ... | ... | ... |
| $j$ | $2j-1$ | $2j-2$ | $s(j-1)$ |

Table II.

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