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Counting symmetric and asymmetric peaks in Motzkin paths with air pockets

Jean-Luc Baril

LIB, Université de Bourgogne Franche-Comté
B.P. 47 870, 21078, Dijon Cedex, France
barjl@u-bourgogne.fr

Rigoberto Flórez

Department of Mathematical Sciences
The Citadel, Charleston, SC, U.S.A.
rigo.florez@citadel.edu

José L. Ramírez

Departamento de Matemáticas
Universidad Nacional de Colombia, Bogotá, Colombia
jlramirezr@unal.edu.co

Abstract

In this paper, we study four subfamilies of Motzkin paths. Two of these subfamilies are well-established in the literature and are known as Motzkin paths with air pockets of the first kind and Motzkin paths with air pockets of the second kind. The remaining two subfamilies extend the concept of non-decreasing paths to the first two families mentioned earlier. Within these four subfamilies, we define two distinct types of subpaths, namely symmetric peaks and asymmetric peaks. Our analysis focuses on understanding the distribution of these symmetric and asymmetric peaks across these subfamilies of paths.

To facilitate this analysis, we present trivariate generating functions. These functions take into consideration parameters such as path length and the numbers of symmetric and asymmetric peaks. These generating functions allow us to calculate, for instance, the total number of symmetric and asymmetric peaks for paths of specific lengths. Furthermore, we conduct an asymptotic analysis of the relationship between these two quantities.

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1 Introduction

A *Motzkin path* is a lattice path in $\mathbb{Z}^2_{\geq 0}$ that starts at the origin, ends on the x-axis, and consists of steps $U = (1, 1)$, $H = (1, 0)$, and $D_k = (1, -k)$ for any $k \geq 1$. A *Motzkin path with air pockets of the first kind* (MAP1) is a Motzkin path where two consecutive down-steps cannot be adjacent. Similarly, a *Motzkin path with air pockets of the second kind* (MAP2) is a Motzkin path where every step H or D_k is immediately followed by an up-step. We denote the sets of all MAP1 and MAP2 as $\mathcal{M}^1$ and $\mathcal{M}^2$, respectively. These two path families were originally introduced by Baril and Barry (see, for example, [6]).

A *valley* is a subpath of the form D_kU, HU, or D_kH. A subpath in the form of U^kD_k is called a *symmetric peak* if it cannot be extended to a subpath in the form of $U^{k+1}D_k$. An asymmetric peak is a subpath of the form $U^\ell D_k$ whether either $\ell > k$, or $\ell < k$ and the subpath cannot be extended to a subpath of the form $U^\ell D_k$. We use $\mathsf{sp}(P)$ (and $\mathsf{ap}(P)$) to denote the number of symmetric (and asymmetric) peaks in the path P.

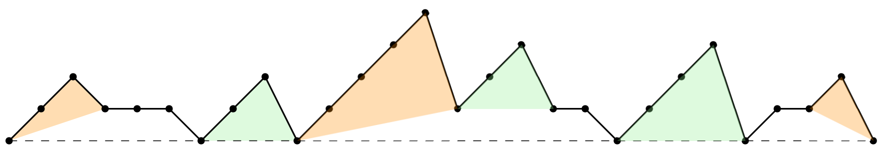

Figure 1: Symmetric and asymmetric peaks of a MAP1.

A path of the form of MAP1 or of the form of MAP2 is called *non-decreasing* if the sequence of ordinates of valleys D_kU, D_kH, HU (considered from left to right) is non-decreasing.

It is worth noting that MAPs without horizontal steps are referred to as *Dyck paths with air pockets*. This term was introduced in a recent paper by Baril et al. [4]. As mentioned in their work, these paths also correspond to a stack evolution with (partial) reset operations, where consecutive resets are not allowed (refer to [14] for more details). Recently, Prodinger [16] used the kernel method to study the partial Dyck paths with air pockets.

In a separate study by Baril et al. [3], the authors explored the prevalence of symmetric and asymmetric peaks in these paths and provided asymptotic approximations for their occurrences.

Furthermore, in a related work [5], these paths were further generalized by allowing them to extend below the x-axis, leading to the concept of *grand Dyck paths with air pockets*. The paper presents enumerative results for these paths, considering parameters such as path length and restrictions on the minimum and maximum ordinates reached.

It is worth mentioning that MAP1 is enumerated by the sequence A114465 in Sloane's On-line Encyclopedia of Integer Sequences [17], while MAP2 corresponds to the sequence of Motzkin numbers (as seen in A001006). These sequences were originally derived using generating functions. However, in our paper, we employ recurrence relations to achieve the same counting results.

In this paper, we center our focus on analyzing the distribution of symmetric and asymmetric peaks within various classes of Motzkin paths with air pockets, encompassing those of both the first and second kinds, including non-decreasing paths. To accomplish this, we present trivariate generating functions that account for the path's length, the count of symmetric peaks, and the number of asymmetric peaks.

Through the utilization of these generating functions, we are able to determine, for instance, the cumulative count of $\mathsf{sp}(P)$ (as well as $\mathsf{ap}(P)$) for paths of specific lengths. Additionally, we delve into an asymptotic analysis to explore the relationship between these two quantities.

For the sake of simplicity, we adopt the abbreviation 'g.f.' to represent 'generating function'.

The concept of symmetric and asymmetric peaks was first introduced by Asakly in 2018 in the context of words [1]. Since then, several related studies have been published on this subject. For instance, Flórez and Ramírez [12] explored the concept of symmetric and asymmetric peaks for Dyck paths. The concept was further extended to non-decreasing Dyck paths by Elizalde et al. [9] and Flórez et al. [11], to Motzkin paths by Flórez and Ramírez [10], and to partial Dyck paths by Sun et al. [18]. Mansour et al. introduced this concept for integer compositions [15]. Elizalde also contributed other significant findings on Dyck paths [8]. In some of these papers, the authors referred to these objects as 'symmetric pyramids' instead of 'symmetric peaks'.

2 Symmetric and asymmetric peaks in MAP1

In this section, our research is focused on Motzkin paths with air pockets of the first kind, denoted as $\mathcal{M}^1$. We introduce a trivariate generating function that depends on three key parameters: path length, the count of symmetric peaks, and the number of asymmetric peaks. As a corollary to this power series, we derive both a generating function and a closed-form expression for the total number of symmetric peaks. Additionally, we conduct an asymptotic analysis to explore the ratio of the number of symmetric peaks to the total number of peaks for paths of specific lengths. The same analysis is performed for asymmetric peaks. Towards the conclusion of this section, we present a recursive relation for counting the number of paths in $\mathcal{M}^1$ with a given length.

Consider the generating function with the parameters of length, symmetric peaks, and asymmetric peaks:

$$M_{sp,ap}(x,y,z) = \sum_{P\in\mathcal{M}^1} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)}.$$

We establish the following theorem:

Theorem 2.1 *The generating function $M_{sp,ap}(x, y, z)$ for the number of MAP1 with respect to length, number of symmetric peaks, and number of asymmetric peaks is given by:*

$$\frac{2(1-x)}{1-x-x^2y-x^3(y-2z)+\sqrt{(1-x-x^2y-x^3(y-2z))^2-\alpha}},$$

where $\alpha := \alpha(x, y, z) = 4x(1 - x - x^2(y - z))(1 - 2x + x^2(2 - y + z) - x^3)$.

Proof. We denote the generating function as $M := M_{sp,ap}(x, y, z)$. Now, let us consider various cases. Except for Case (1) where the paths start with H, we deal with paths UAD_kB where the first return to ground occurs just after D_k for some $k \geq 1$. We consider seven cases (2) – (8) grouped into three types of paths: Case (2) deal with paths where UAD_k is a symmetric peak; Cases (3), (4), and (5) deal with paths where A starts with a symmetric peak; and Cases (6), (7), and (8) deal with the other paths.

Case (1). If the path, denoted as P, is of the form HQ, where Q is another MAP1, then the generating function for this case is simply xM.

Case (2). If the path P is of the form U^aD_aQ (where $a \geq 1$), and Q is a MAP1 (see Figure 2, left-hand side), then the generating function for this case is $\frac{x^2}{1-x}yM$.

Case (3). When P is of the form $P = UU^aD_aQU^bD_{b+1}R$ (with $a, b \geq 1$) and both Q and R are MAP1 (see Figure 2, right-hand side), the generating function becomes

$$x\frac{x^2}{1-x}zM\frac{x^2}{1-x}zM = \frac{x^5}{(1-x)^2}z^2M^2.$$

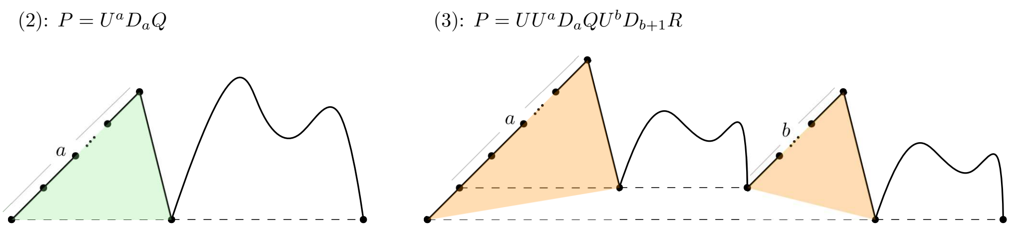

Figure 2: Decomposition of cases (2) and (3).

Case (4). If the path P takes the form of $UU^aD_a\bar{Q}R$ (with $a \geq 1$) and both Q and R are MAP1, with Q ending in a down-step and not concluding with a symmetric peak, and $\bar{Q}$ being an adjusted version of Q after increasing the size of its last down-step by one, the generating function is

$$x\frac{x^2}{1-x}zBM,$$

where $B := B(x, y, z)$ represents the generating function for nonempty MAP1 paths that do not end with a symmetric peak nor with a horizontal step. Considering the complement, we easily obtain $B = M - 1 - \frac{x^2}{1-x}yM - xM$.

Case (5). If the path P is of the form UU^aD_aQHDR (with $a \geq 1$) and both Q and R are MAP1 (see Figure 3 left-hand side), the generating function takes the shape of

$$x\frac{x^2}{1-x}zx^2M^2.$$

Case (6). When P has the form of $UQU^aD_{a+1}R$ (with $a \geq 1$) and both Q and R are MAP1, with Q not starting with a symmetric peak (see Figure 3, right-hand side), the generating function becomes

$$xMB'\frac{x^2}{1-x}z,$$

where $B' := B'(x, y, z)$ denotes the generating function for nonempty MAP1 paths that do not start with a symmetric peak. Considering the complement, we have $B' = M - 1 - \frac{x^2}{1-x}yM$.

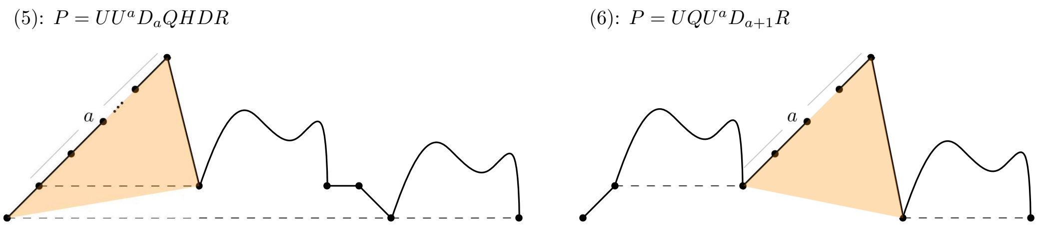

Figure 3: Decomposition of cases (5) and (6).

Case (7). If the path P takes the form of $U\bar{Q}R$, where both Q and R are MAP1, with Q being nonempty, ending with a down-step, not starting or ending with a symmetric peak, and $\bar{Q}$ being an adjusted version of Q, after increasing the size of its last down-step by one, see Figure 4. So, the generating function for this case given by xCM, where $C := C(x, y, z)$ represents the generating function for nonempty MAP1 that end with a down-step and do not start or end with a symmetric peak. Considering the complement, we deduce $C = B' - \frac{x^2y}{1-x}B' - x(B' + 1)$.

Case (8). In the final case, if the path P is of the form $UQHDR$ where both Q and R are MAP1, and Q does not start with a symmetric peak, see Figure 4, the generating function becomes $x^3(B' + 1)M$.

Figure 4: Decomposition of cases (7) and (8).

Summing up these cases, we derive the following functional equation:

$$M = 1 + xM + \frac{x^2}{1-x}yM + \frac{x^5}{(1-x)^2}z^2M^2 + \frac{x^3}{1-x}zM\left(M - 1 - \frac{x^2}{1-x}yM - xM\right) + \frac{x^5 z}{1-x}M^2 + \frac{x^3 z}{1-x}MB' + xM\left(B' - \frac{x^2 y}{1-x}B' - x(B'+1)\right) + x^3(B'+1)M,$$

which leads to the desired result. □

The first terms of the Taylor expansion of M are as follows:

$$1 + x + (1+y)x^2 + (3y+2)x^3 + \boldsymbol{(y^2 + 6y + z + 5)x^4} + (5y^2 + z^2 + 12y + 6z + 12)x^5 + O(x^6).$$

In Figure 5, the MAP1 of length 4 are displayed, with their corresponding weights highlighted in boldface in the previous expansion.

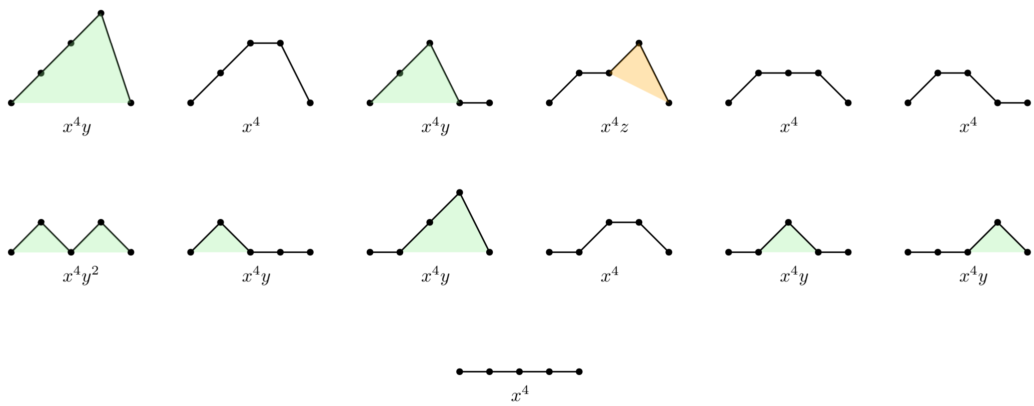

Figure 5: The MAP1 of length 4 and their contribution in $M_{sp,ap}(x, y, z)$.

Corollary 2.2 *The generating function for the number of MAP1 that avoid symmetric and asymmetric peaks is given by:*

$$M(x, 0, 0) = \frac{2}{1 + \sqrt{1 - 4x + 4x^2 - 4x^3}}.$$

The Taylor expansion of this generating function yields:

$$1 + x + x^2 + 2x^3 + 5x^4 + 12x^5 + 29x^6 + 73x^7 + 190x^8 + 505x^9 + O(x^{10}).$$

Remarkably, the coefficients in this sequence correspond to A152171 in [17], which counts Dyck paths of semi-length n without peaks at height 2 (mod 3) and valleys at height 1 (mod 3).

From the formulas provided in A152171 and A025265, we can determine the number of MAP1 of length n that avoid symmetric and asymmetric peaks. Hence, we obtain the expression:

$$\sum_{\ell=0}^{n}\sum_{k=0}^{\ell}\sum_{i=0}^{n-\ell}\sum_{j=0}^{k+1} C_k \binom{j}{\ell-k-j}\binom{k+1}{j}\binom{n-\ell-i}{i}(-1)^{-\ell+k+i},$$

where C_k represents the kth Catalan number.

Corollary 2.3 *The generating function for the number of MAP1 that avoid symmetric peaks is given by:*

$$M(x,0,1) = \frac{2(1-x)}{1-x+2x^3+\sqrt{(1-x)(1-5x+8x^2-12x^3+8x^4-8x^5)}}.$$

The Taylor expansion of this generating function is as follows:

$$1+x+x^2+2x^3+6x^4+19x^5+58x^6+173x^7+519x^8+1585x^9+O(x^{10}).$$

It is worth noting that this sequence of coefficients does not appear in [17].

Corollary 2.4 *The generating function for the number of MAP1 that avoid asymmetric peaks is given by:*

$$M(x,1,0) = \frac{2(1-x)}{1-x-x^2-x^3+\sqrt{(1-x)(1-5x+6x^2-2x^3+x^4+3x^5)}}.$$

The Taylor expansion of this generating function is as follows:

$$1+x+2x^2+5x^3+12x^4+29x^5+73x^6+190x^7+508x^8+1391x^9+O(x^{10}).$$

This sequence of coefficients does not appear in [17].

To find the total number of symmetric peaks and asymmetric peaks in all MAP1, we calculate $\partial_y(M(x,y,1))|_{y=1}$ and $\partial_z(M(x,1,z))|_{z=1}$. This yields two generating functions that we formally state in the following two corollaries.

Corollary 2.5 *The g.f. for the total number of symmetric peaks in all MAP1 is given by:*

$$\frac{2x^2(1-3x+x^2-3x^3+(1+x)\sqrt{\beta})}{(1-x)\sqrt{\beta}(1-x^2+\sqrt{\beta})^2},$$

where $\beta = 1-4x+2x^2-4x^3+x^4$. *An asymptotic expression for the n-th coefficient is given by:*

$$\frac{(2+\sqrt{3})^n}{3\sqrt{2(12+7\sqrt{3})\pi n}}.$$

The Taylor expansion of this generating function is

$$x^2 + 3x^3 + 8x^4 + 22x^5 + 64x^6 + 196x^7 + 625x^8 + 2053x^9 + O(x^{10}).$$

It is important to note that this sequence of coefficients is not found in [17].

Corollary 2.6 *The g.f. for the total number of asymmetric peaks in all MAP1 is*

$$\frac{4x^3(1-x+2x^2+\sqrt{\beta})}{(1-x)\sqrt{\beta}(1-x^2+\sqrt{\beta})^2},$$

where $\beta = 1-4x+2x^2-4x^3+x^4$. *An asymptotic expression for the n-th coefficient is given by:*

$$\frac{(2+\sqrt{3})^n(-17+10\sqrt{3})}{6\sqrt{2(-12+7\sqrt{3})\pi n}}.$$

The Taylor expansion of this generating function is

$$x^4 + 8x^5 + 38x^6 + 154x^7 + 590x^8 + 2204x^9 + O(x^{10}).$$

It is important to note that this sequence of coefficients is not found in [17].

Let $p_1(n)$ represent the total number of peaks in all MAP1 of length n. From Corollaries 2.5 and 2.6 we can approximate $p_1(n)$ as follows:

$$p_1(n) \sim \frac{(2+\sqrt{3})^n\left(2\sqrt{(-12+7\sqrt{3})\pi} - (17-10\sqrt{3})\sqrt{(12+7\sqrt{3})\pi}\right)}{6\sqrt{6n}\pi}.$$

Now, let $s_1(n)$ and $t_1(n)$ denote the numbers of symmetric and asymmetric peaks, respectively, in all MAP1 of length n. We can derive the following asymptotic ratios.

Corollary 2.7 *The asymptotic for the ratio between the number of symmetric peaks and the number of all peaks in all MAP1 is*

$$\lim_{n\to\infty}\frac{s_1(n)}{p_1(n)} = \frac{2\sqrt{(-12+7\sqrt{3})\pi}}{2\sqrt{(-12+7\sqrt{3})\pi} - (17-10\sqrt{3})\sqrt{(12+7\sqrt{3})\pi}} \sim 0.309401077.$$

The asymptotic for the ratio between the number of asymmetric peaks and the number of all peaks in all MAP1 is

$$\lim_{n\to\infty}\frac{t_1(n)}{p_1(n)} = \frac{(-17+10\sqrt{3})\sqrt{(12+7\sqrt{3})\pi}}{2\sqrt{(-12+7\sqrt{3})\pi} + (-17+10\sqrt{3})\sqrt{(12+7\sqrt{3})\pi}} \sim 0.690598923.$$

The asymptotic for the ratio between the numbers of asymmetric and symmetric peaks in all MAP1 is

$$\lim_{n\to\infty}\frac{t_1(n)}{s_1(n)} = \frac{1}{2} + \sqrt{3} \sim 2.232050808.$$

We denote the set of paths of length n in $\mathcal{M}^1$ as $\mathcal{M}^1_n$, and we use $m_1(n)$ to represent the cardinality of $\mathcal{M}^1_n$. Additionally, let $\mathcal{B}_n \subseteq \mathcal{M}^1_n$ denote the subset of paths that do not have valleys at ground level and do not contain sub-paths of the form H at ground level. Essentially, these are MAP1s without sub-paths of the form H and D_kU that touch the x-axis, where $k \geq 1$. It is worth to note that $\mathcal{B}_2 = \{UD_1\}$ and $\mathcal{B}_3 = \{U^2D_1, UHD_1\}$.

Lemma 2.8 *For $n \geq 4$, $|\mathcal{B}_n| = m_1(n-1) - m_1(n-2) + m_1(n-3)$.*

Proof. The set $\mathcal{M}^1_n$ can be partitioned into three disjoint sets: A_n, B_n, and C_n. In A_n we include all paths where the last step D_a satisfies $a \geq 1$. B_n comprises all paths of the form PHH, where $P \in \mathcal{M}^1_{n-2}$. Finally, $C_n = \mathcal{M}^1_n \setminus (A_n \cup B_n)$.

By adding an initial north-east step and replacing the last South-East step of length a (D_a-step) with a step D_{a+1} in all paths in A_n, we obtain a set $W_A \subset \mathcal{B}_{n+1}$. Similarly, by adding an initial north-east step and replacing the last horizontal step H with a step D_1 in all paths in B_n, we obtain a set $W_B \subset \mathcal{B}_{n+1}$.

Notably, $W_A \cup W_B = \mathcal{B}_{n+1}$. Consequently, no path in C_n gives rise to a path in $\mathcal{B}_{n+1}$. This construction establishes the bijection between $\mathcal{B}_{n+1}$ and $\mathcal{M}^1_n \setminus \mathcal{S}$ where $\mathcal{S}$ is the subset of paths from $\mathcal{M}^1_n$ that end with precisely one H step, which induces $|\mathcal{B}_n| = m_1(n-1) - m_1(n-2) + m_1(n-3)$. □

Theorem 2.9 *For $n > 3$, we have*

$$m_1(n) = 3m_1(n-3) + 2m_1(n-1) + \sum_{k=4}^{n-1} \big(m_1(k-1) - m_1(k-2) + m_1(k-3)\big) m_1(n-k),$$

anchored with the initial values $m_1(1) = 1$, $m_1(2) = 2$, and $m_1(3) = 5$.

Proof. Consider a path $P \in \mathcal{M}^1_n$. Such a path P can be decomposed into one of the following forms: HR_{n-1} or Q_kR_{n-k}, where $Q_k \in \mathcal{B}_k$ and $R_{n-k} \in \mathcal{M}^1_{n-k}$, for $2 \leq k < n$, and if $k = n$, we consider R_0 as the empty path.

It is evident that all paths of the form HR_{n-1} are enumerated by $m_1(n-1)$. Since $|\mathcal{B}_2| = 1$ and $|\mathcal{B}_3| = 2$, it follows that all paths of the form Q_2R_{n-2} and Q_3R_{n-3} are enumerated by $m_1(n-2)$ and $2m_1(n-3)$, respectively.

By Lemma 2.8, paths of the form Q_nR_0 are enumerated by $m_1(n-1) - m_1(n-2) + m_1(n-3)$. Furthermore, by Lemma 2.8 paths of the form Q_kR_{n-k}, for a fixed k, $4 \leq k \leq n-1$, are counted as

$$\Big(m_1(k-1) - m_1(k-2) + m_1(k-3)\Big) m_1(n-k).$$

By varying k within the set $\{4, \ldots, n-1\}$, we obtain the desired result. □

The first eleven values of the sequence $m_1(n)$ for $n = 1, \ldots, 11$ are as follows:

1, 2, 5, 13, 36, 105, 317, 982, 3105, 9981, 32520.

3 Symmetric and asymmetric peaks in MAP2

In this section, our research is focused on Motzkin paths with air pockets of the second kind, denoted as $\mathcal{M}^2$. It is worth recalling that in $P \in \mathcal{M}^2$, two consecutive down-steps cannot occur, and any horizontal-step and down-step (except the last of the path) are immediately followed by an up-step.

We introduce a trivariate generating function that depends on three key parameters: path length, the count of symmetric peaks, and the number of asymmetric peaks. As a corollary to this power series, we derive a generating function for the total number of paths avoiding symmetric peaks, with coefficients corresponding to the Fibonacci numbers. Additionally, we conduct an asymptotic analysis to explore the ratio of the number of symmetric peaks to the total number of peaks for paths of specific lengths. The same analysis is performed for asymmetric peaks. Towards the conclusion of this section, we present a recursive relation for counting the number of paths in $\mathcal{M}^2$ with a given length.

Consider the generating function with the parameters of length, symmetric peaks, and asymmetric peaks:

$$M'_{sp,ap}(x, y, z) = \sum_{P\in\mathcal{M}^2} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)}.$$

For brevity, we set $M' := M'_{sp,ap}(x, y, z)$.

Theorem 3.1 *The generating function $M'_{sp,ap}(x, y, z)$ for the number of MAP2s with respect to length, number of symmetric peaks, and number of asymmetric peaks is as follows:*

$$\frac{(1-x)^2\left(1-x^2y-x^3(1+y-2z)-\sqrt{(1-x)\gamma}\right)}{2x(1-x-x^2y+x^2z)(1-x^2-x^2y+x^2z)},$$

where $\gamma := \gamma(x, y, z)$ is the polynomial:

$$1-3x+x^2(1-2y)-x^4(1-y^2)-x^5(1+y-2z)^2+x^3(3+4y-4z).$$

Proof. We consider a MAP2s which can be categorized into three groups: H, HQ, or Q, where Q is non-empty and starts with U. Now, let $\mathcal{S}$ be the set of non-empty MAP2 that start with U, and we denote $S := S(x, y, z)$ as its trivariate generating function. Clearly, we have:

$$M' = 1 + x + (1 + x)S.$$

Let us determine the generating function S for P, where P is a path in $\mathcal{S}$. We distinguish several cases according to the first return decomposition UAD_kB of P. We consider five cases $(1)-(5)$ grouped into three types of paths: Case (1) deal with paths where UAD_k is a symmetric peak; Cases (2) and (3) deal with paths where A starts with a symmetric peak; and Cases (4) and (5) deal with the other paths.

Case (1). When $P = U^a D_a Q$ and $a \geq 1$, where Q can be either empty or belong to $\mathcal{S}$, the generating function for these paths is $\frac{x^2 y}{1-x}(S+1)$. (See Figure 2 (left-hand side).)

Case (2). When $P = UU^a D_a Q U^b D_{b+1} R$ and $a, b \geq 1$, where Q and R are possibly empty or belong to $\mathcal{S}$, see Figure 2 (right-hand side), the generating function for these paths is

$$x\frac{x^2}{1-x}z(S+1)\frac{x^2}{1-x}z(S+1) = \frac{x^5}{(1-x)^2}z^2(S+1)^2.$$

Case (3). When $P = UU^a D_a \bar{Q} R$ and $a \geq 1$, where Q and R are in $\mathcal{S}$, R can be empty, Q does not end with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing the size of the last down-step by one, the generating function for these paths is

$$x\frac{x^2}{1-x}zV(S+1),$$

where $V := V(x, y, z)$ is the generating function for the paths in $\mathcal{S}$ that do not end with a symmetric peak. Using the complement, it is clear that $V = S - (S+1)\frac{x^2 y}{1-x}$.

Case (4). When $P = UQU^a D_{a+1} R$ and $a \geq 1$, where Q, R are some MAP2, Q does not start with a symmetric peak, see Figure 3 (right-hand side). the contribution is

$$xV'\frac{x^2}{1-x}z(S+1).$$

Here, $V' := V'(x, y, z)$ is the generating function for paths in $\mathcal{S}$ that do not start with a symmetric peak (as in case (3)), plus the g.f. for MAP2 starting with H, that is $V' = V + x(S+1)$.

Case (5). When $P = U\bar{Q}R$, where Q and R are some MAP2, with R in $\mathcal{S}$ and Q does not start or end with symmetric peaks, Q different from H, $\bar{Q}$ is obtained from Q by increasing the size of the last down-step by one, see Figure 4 (left-hand side), the contribution is $xW(S+1)$, where $W := W(x, y, z)$ is the g.f. for paths in $\mathcal{S}$ that do not start or end with a symmetric peak, and different from H. Clearly, we have $W = V' - x - V'\frac{x^2 y}{1-x}$.

Summarizing all these cases, we obtain the following functional equation:

$$S = \frac{x^2 y}{1-x}(S+1) + \frac{x^5}{(1-x)^2}z^2(S+1)^2 + \frac{x^3}{1-x}zV(S+1)$$
$$+ V'\frac{x^3}{1-x}z(S+1) + xW(S+1).$$

This functional equation leads to the desired result. $\square$

The Taylor expansion of this generating function is

$$1 + x + x^2y + 2x^3y + (\boldsymbol{y^2 + 2y + z})x^4 + (3y^2 + z^2 + 2y + 3z)x^5 + (y^3 + 5y^2 + 3yz + 4z^2 + 2y + 6z)x^6 + O(x^7).$$

In Figure 6, the MAP2 of length 4 are displayed, with their corresponding weights highlighted in boldface in the previous expansion.

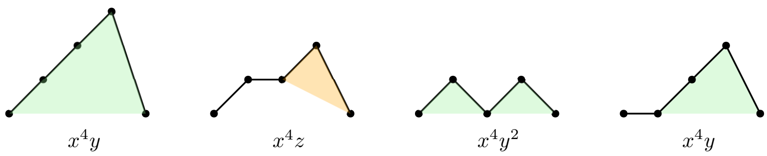

Figure 6: The MAP2 of length 4 and their contribution in $M'_{sp,ap}(x, y, z)$.

Corollary 3.2 *The generating function for the number MAP2 avoiding symmetric peaks is given by:*

$$M'(x, 0, 1) = \frac{(1 - x^2)(1 + x^3 - \sqrt{1 - 4x + 4x^2 - 2x^3 + x^6})}{2x(1 - x + x^2)}.$$

The Taylor expansion is $1 + x + x^4 + 4x^5 + 10x^6 + 23x^7 + 54x^8 + 131x^9 + O(x^{10})$, where the sequence of coefficients does not appear in [17].

Corollary 3.3 *The generating function for the number MAP2 avoiding asymmetric peaks is given by:*

$$M'(x, 1, 0) = \frac{1 - x^2}{1 - x - x^2}.$$

The Taylor expansion is $1+x+x^2+2x^3+3x^4+5x^5+8x^6+13x^7+21x^8+34x^9+O(x^{10})$, where the sequence of coefficients corresponds to the Fibonacci sequence A212804 in [17].

By calculating $\partial_y(M'(x, y, 1))|_{y=1}$ and $\partial_z(M'(x, 1, z))|_{z=1}$ we obtain the following corollaries.

Corollary 3.4 *The g.f. for the total number of symmetric peaks in all MAP2 is*

$$\frac{x(-1 + 3x + 2x^2 + \sqrt{1 - 2x - 3x^2})}{2(1 - x)\sqrt{1 - 2x - 3x^2}},$$

and an asymptotic for the n-th coefficient is

$$\frac{\sqrt{3}\, 3^n}{36\sqrt{\pi n}}.$$

The Taylor expansion is $x^2+2x^3+4x^4+8x^5+18x^6+43x^7+109x^8+286x^9+O(x^{10})$, where the sequence of coefficients does not appear in [17].

Corollary 3.5 *The generating function for the total number of symmetric peaks in all MAP2 is given by:*

$$\frac{x(2-3x-3x^2-(2-x)\sqrt{1-2x-3x^2}}{2(1-x)\sqrt{1-2x-3x^2})},$$

and an asymptotic for the n-th coefficient is

$$\frac{\sqrt{3}\,3^n}{12\sqrt{\pi n}}.$$

The Taylor expansion is $x^4+5x^5+17x^6+53x^7+158x^8+464x^9+O(x^{10})$, where the sequence of coefficients does not appear in [17].

Let $p_2(n)$ be the number of peaks in all MAP2 on length n. From Corollaries 3.4 and 3.5 we can deduce that:

$$p_2(n) \sim \frac{3^{n-1}}{\sqrt{3n\pi}}.$$

Now, let us define $s_2(n)$ and $t_2(n)$ as the number of symmetric peaks and asymmetric peaks, respectively, in all MAP2s of length n.

Corollary 3.6 *The asymptotic for the ratio between the number of symmetric peaks and the number of all peaks in all MAP2 is*

$$\lim_{n\to\infty}\frac{s_2(n)}{p_2(n)} = \frac{1}{4}.$$

The asymptotic for the ratio between the number of asymmetric peaks and the number of all peaks in all MAP2 is

$$\lim_{n\to\infty}\frac{t_2(n)}{p_2(n)} = \frac{3}{4}.$$

The asymptotic for the ratio between the numbers of asymmetric and symmetric peaks in all MAP2 is 3.

Corollary 3.7 *The generating function for the total number of peaks in all MAP2 is given by:*

$$\frac{z(1+z-\sqrt{1-2z-3z^2})}{2\sqrt{1-2z-3z^2}}$$

and an asymptotic for the n-th coefficient is

$$\frac{\sqrt{3}\,3^n}{9\sqrt{\pi n}}.$$

The Taylor expansion is $z^2 + 2z^3 + 5z^4 + 13z^6 + 96z^7 + 267z^8 + 750z^9 + O(x^{10})$, where the sequence of coefficients corresponds to A005773 in [17], which also counts the directed animals of size n (see [7] for instance).

We use $\mathcal{M}_n^2$ to denote the set of paths of length n in $\mathcal{M}^2$, and we use $m_2(n)$ to denote the cardinality of $\mathcal{M}_n^2$. Let $\mathcal{S}_n^2 \subseteq \mathcal{M}_n^2$ to represent all paths starting with a U step. Define $\mathcal{B}_n^2 := \mathcal{M}_n^2 \setminus \{HQ_{n-1} | Q_{n-1} \in \mathcal{S}_{n-1}\}$. We use $s(n)$ to denote the cardinality of $\mathcal{S}_n$.

Theorem 3.8 *For $n \geq 3$, we have*

$$m_2(n) = s(n-2) + s(n-1) + m_2(n-1) + \sum_{k=2}^{n-3} m_2(k) s(n-k-1),$$

where $m_2(2) = 1$, $s(2) = 1$, and $s(n) = m_2(n) - s(n-1)$.

Proof. Let us consider a path $P \in \mathcal{M}_n^2$. Such a path P can be decomposed into one of the following forms: HQ_{n-1}, UD_1Q_{n-2}, UM_{n-1}, or $UM_kQ_{n-(k+1)}$, where $Q_i \in \mathcal{S}_i$, and M_i is obtained from a path in $\mathcal{M}_i^2$ by increasing the size of the last down-step by one.

From definition of $\mathcal{B}_n$, we can deduce that $s(i) = m_2(i) - s(i-1)$. Therefore, from the decomposition, we can see that all paths of the form HQ_{n-1} and UD_1Q_{n-2} are counted by $s(n-1)$ and $s(n-2)$, respectively. Paths of the form UM_{n-1} are counted by $m_2(n-1)$.

For a fixed k, $2 \leq k \leq n-3$, once again from the decomposition, we have that all paths of the form $UM_kQ_{n-(k+1)}$ are counted by $m_2(k)s(n-k-1)$. So, by varying k in the set $\{2, \ldots, n-3\}$, and adding $s(n-1)$, $s(n-2)$, and $m_2(n-1)$, we obtain the desired result. □

The first eleven values of the sequence $m_2(n)$ for $n = 2, \ldots, 12$ are as follows:

1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, 5798.

This sequence is related to the Motzkin numbers, see A001006.

4 The non-decreasing MAPS

In this section we introduce the concept of non-decreasing Motzkin paths with air pockets of both kinds. The concept of non-decreasing path was first introduced by Barcucci et al. [2] in the context of the classical Dyck paths. Recently, Flórez and Ramírez [13] studied this concept for Motzkin paths.

We present a trivariate generating function that depends on three parameters: path length, the number of symmetric peaks, and the number of asymmetric peaks. We then count the number of non-decreasing paths in the form of MAP1 and MAP2, and provide a table with a similar structure to those counted in the previous sections.

4.1 Symmetric and asymmetric peaks in non-decreasing MAP1

In this section, we focus our attention to the set $\mathcal{NM}^1$ of non-decreasing MAP1. These are MAP1 in which the sequence of valley ordinates, specifically D_kU, D_kH for $k \geq 1$, and HU, forms a non-decreasing pattern when read from left to right.

Theorem 4.3 provides the generating function

$$N_{sp,ap}(x, y, z) = \sum_{P \in \mathcal{NM}^1} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)}.$$

For brevity, we will refer to this generating function as N. To facilitate our analysis, we will begin by introducing some essential lemmas.

This lemma explores non-decreasing MAP1, with all its valleys positioned at ground level.

Lemma 4.1 *Let $V := V_{sp,ap}(x, y, z)$ denote the generating function for the number of non-decreasing MAP1 having all its valleys at ordinate 0, with respect to the path's length and the number of symmetric and asymmetric peaks. The generating function is given by:*

$$V = \frac{(1-x)^2}{1 - 3x + (3-y)x^2 - (2-y)x^3}.$$

It is important to note that non-decreasing MAP1 with valleys at ground level do not contain any asymmetric peaks. Consequently, the variable z does not appear in the expression for V.

Proof. To establish the generating function for non-decreasing MAP1 with all its valleys at ground level (or equivalently on the x-axis), we consider non-empty non-decreasing paths of the form $H^{a_0} R_1 H^{a_1} \cdots R_k H^{a_k}$. Here, $k \geq 0$, $a_i \geq 0$ for $0 \leq i \leq k$, and R_i can either be a symmetric peak $U^a D_a$ or a truncated symmetric peak $U^a H^b D_a$, where $U^a H^b D_a$, $a, b \geq 1$, for $1 \leq i \leq k$. Therefore, the generating function V comes from the decomposition of paths into a run of initial H steps, followed by a sequence of subpaths of the form $U^k D_k H^\ell$ or $U^k H^m D_k H^\ell$:

$$V = \frac{1}{1-x} \cdot \frac{1}{1 - \frac{1}{1-x} \cdot \left(\frac{x^2 y}{1-x} + \frac{x^3}{(1-x)^2}\right)}.$$

This completes the proof. □

Lemma 4.2 *The g.f. $B := B_{sp,ap}(x, y, z)$ for the number of non-decreasing MAP1 ending with a down-step, that do not end end with a symmetric peak, with respect to the length and the numbers of symmetric and asymmetric peaks satisfies*

$$B = N - 1 - xV - \frac{x^2 y V}{1-x},$$

where V is given in the previous lemma.

Proof. By reasoning with the complement, a non-decreasing MAP1 ending with a down-step, that does not end with a symmetric peak, is a non-decreasing MAP1 different from (i) the empty path, (ii) QH, and (iii) QU^aD_a, where Q has all of its valleys at ordinate 0. Summarizing all these cases, we deduce functional equation. □

Theorem 4.3 *The g.f. $N_{sp,ap}(x,y,z)$ for the number of non-decreasing MAP1 with respect to length, number of symmetric peaks, and number of asymmetric peaks is $\frac{P(x,y,z)}{Q(x,y,z)}$, where $P(x,y,z)$ is*

$$1-6x-x^2(-14+y)-x^3(17-4y+z)-x^4(-11+6y-4z)+x^5(-2+4y-5z+z^2)\\ -x^6(1-3z+2z^2)-x^7(-1+2y-z^2)-x^8(-y+z),$$

and $Q(x,y,z)$ is

$$(1-3x-x^2(-3+y)+x^3(-2+y))(1-4x-x^2(-5+y)-x^4(y-z)-x^3(3-2y+z)).$$

Proof. Let P be a non-decreasing MAP1 (NMAP1 for short). Except for the first case where P starts with H, we distinguish several cases according to the first return decomposition UAD_kB of P. We consider eight cases (2)–(9) grouped into four types of paths: Case (2) deals with paths where UAD_k is a symmetric peak; Case (3) deals with paths where A starts with H; Cases (4), (5) and (6) deal with paths where A starts with a symmetric peak; and Cases (7), (8) and (9) deal with the other paths.

Case (1). If $P=HQ$ where Q also is a NMAP1, then the g.f. for this case is xN.

Case (2). If $P=U^aD_aQ$ for $a\geq 1$ and Q a NMAP1, then the g.f. is $\frac{x^2y}{1-x}N$.

Case (3). If $P=U^aH^bD_aQ$, for $a,b\geq 1$ and Q a non-empty NMAP1, then the g.f. is $\frac{x^3}{(1-x)^2}N$.

Case (4). If $P=UU^aD_aQU^bD_{b+1}$, for $a,b\geq 1$ and Q a NMAP1 having all its valleys on the x-axis, then the g.f. is

$$\frac{x^3z}{1-x}V\frac{x^2z}{1-x},$$

where $V:=V(x,y,z)$ is given in Lemma 4.1.

Case (5). If $P=UU^aD_aQHD$ for $a\geq 1$, Q having all its valleys at ordinate 0, then the g.f. is $\frac{x^5z}{1-x}V$.

Case (6). If $P=UU^aD_a\bar{Q}$ for $a\geq 1$, Q a non-empty NMAP1 ending with a down-step and that does not end with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing by one the size of the last down step, then the g.f. is

$$\frac{x^3z}{1-x}B,$$

where $B:=B(x,y,z)$ is the g.f. of Lemma 4.2.

Case (7). If $P = UQU^aD_{a+1}$ for $a \geq 1$, Q a non-empty NMAP1 that does not start with a symmetric peak, and having all its valleys on the x-axis, then the g.f. is

$$\left(V - 1 - \frac{x^2y}{1-x}V\right)\frac{x^3z}{1-x}.$$

Case (8). If $P = U\bar{Q}$, where Q does not start with a symmetric peak, ends with a down step but does not end with a symmetric peak, and $\bar{Q}$ is obtained from Q by increasing the last down-step, then the g.f. is

$$x\left(B - \frac{x^2yB}{1-x} - \frac{x^3}{(1-x)^2}\right),$$

where B satisfies Lemma 4.2. Indeed, $B - \frac{x^2yB}{1-x}$ corresponds to the paths Q that do not start with a symmetric peak, ending with a down step, but not end with a symmetric peak, and we must substract $\frac{x^3}{(1-x)^2}$ in order to eliminate paths of the form $U^aH^bD_a$, $a, b \geq 1$.

Case (9). If $P = UQHD$ where Q does not start with a symmetric peak and having all its valleys at ordinate 0, and different from H^a, $a \geq 1$, then the g.f. is

$$x^3\left(V - 1 - \frac{x^2yV}{1-x} - \frac{x}{1-x}\right).$$

Summarizing all these cases and using the the previous lemmas, we obtain the result. □

The Taylor expansion of this generating function is

$$1+x+(y+1)x^2+(2+3y)x^3+(y^2+6y+z+5)x^4+(5y^2+z^2+12y+5z+12)x^5+O(x^6)$$

Corollary 4.4 *The g.f. for the number of non-decreasing MAP1 is*

$$N(x,1,1) = \frac{1-6x+13x^2-14x^3+9x^4-2x^5}{(1-3x+2x^2-x^3)(1-4x+4x^2-2x^3)}.$$

The Taylor expansion of this generating function is

$$1+x+2x^2+5x^3+13x^4+35x^5+96x^6+265x^7+734x^8+2040x^9+O(x^{10}),$$

where the sequence does not appear in [17].

We can adapt the same proofs used in the previous sections to non-decreasing MAPs of both kinds. Therefore, we will only provide in Table 1 a summary of the results without including the proofs.

Notice that the sequences associated to the statistics given in Table 1 do not appear in [17].

Corollary 4.5 *An asymptotic for the ratio between the numbers of asymmetric and symmetric peaks in non-decreasing MAP1 is*

$$\frac{\left(\sqrt{33}+7\right)\left(26+6\sqrt{33}\right)^{\frac{1}{3}} - 2\sqrt{33} - \left(26+6\sqrt{33}\right)^{\frac{2}{3}} + 2}{3\left(26+6\sqrt{33}\right)^{\frac{2}{3}}} \sim 0.5436890133.$$

Statistic population	Generating function
Non-decreasing MAP1 avoiding symmetric peaks	$N(x,0,1) = \frac{1-5x+8x^2-5x^3+2x^4+x^5-x^6}{(1-2x)(1-3x+x^2)(1-x+x^2)}$
Non-decreasing MAP1 avoiding asymmetric peaks	$N(x,1,0) = \frac{1-6x+13x^2-13x^3+5x^4+2x^5-x^6-x^7+x^8}{(1-3x+2x^2-x^3)(1-4x+4x^2-x^3-x^4)}$
Non-decreasing MAP1 avoiding symmetric and asymmetric peaks	$N(x,0,0) = \frac{(1-3x+2x^2-x^3)(1-3x+3x^2-x^3-x^4)}{(1-2x)(1-x+x^2)(1-4x+5x^2-3x^3)}$
Symmetric peaks in all non-decreasing MAP1	$\frac{(1-x)x^2(1-x+x^2)(1-9x+31x^2-52x^3+48x^4-32x^5+22x^6-16x^7+8x^8-2x^9)}{(1-3x+2x^2-x^3)^2(1-4x+4x^2-2x^3)^2}$
Asymmetric peaks in all non-decreasing MAP1	$\frac{(1-x)x^4(1-3x-x^2+8x^3-10x^4+6x^5-2x^6)}{(1-3x+2x^2-x^3)(1-4x+4x^2-2x^3)^2}$

Table 1: Some statistics for non-decreasing MAP1.

4.2 Symmetric and asymmetric peaks in non-decreasing MAP2

Finally, we focus on the set $\mathcal{NM}^2$ of non-decreasing Dyck paths with air pockets of second kind. The following theorem provides the generating function

$$N'_{sp,ap}(x,y,z) = \sum_{P\in\mathcal{NM}^2} x^{|P|} y^{\mathsf{sp}(P)} z^{\mathsf{ap}(P)}.$$

For short, we set $N' := N'_{sp,ap}(x,y,z)$.

Theorem 4.6 *The g.f. $N'_{sp,ap}(x,y,z)$ for the number of non-decreasing MAP2 with respect to the numbers of symmetric and asymmetric peaks is*

$$\frac{(1+x)(1-3x+2x^2+x^3-x^4-x^2y+2x^3y-x^4y-x^3z+2x^4z-x^5z+x^5z^2)}{(1-x-x^2y)(1-2x+x^3-x^2y+x^3y-x^3z)}.$$

Proof. We set $N' = N'_{sp,ap}(x,y,z)$. Any nonempty non-decreasing MAP2 (NMAP2 for short) is either H, HQ, or Q, where Q is non-empty and Q starts with U.

Now, let $\mathcal{S}$ be the set of non-empty NMAP2 that starts with U, and $S := S(x,y,z)$ its associated generating function. Obviously, we have $N' = 1 + x + (1+x)S$. We distinguish several cases according to the beginning and the end of A in the first return decomposition UAD_kB of P.

Case (1). If $P = U^aD_aQ$, $a \geq 1$, where Q is either empty or $Q \in \mathcal{S}$, the g.f. for these paths is $\frac{x^2y}{1-x}(S+1)$.

Case (2). If $P = UU^aD_aQU^bD_{b+1}$, $a, b \geq 1$, where Q is either empty or $Q \in \mathcal{S}$ with all its valleys on the x-axis, then the g.f. is

$$x\frac{x^2}{1-x}zW\frac{x^2}{1-x}z = \frac{x^5}{(1-x)^2}z^2W,$$

where $W := W(x, y, z)$ is the g.f. for NMAP2 in $\mathcal{S} \cup \{\epsilon\}$ having valleys on the x-axis, i.e., W is the solution of $W = 1 + \frac{x^2y}{1-x}W$.

Case (3). If $P = UHQU^aD_{a+1}$, $a \geq 1$, where Q is either empty or $Q \in \mathcal{S}$ having all its valleys on the x-axis, then the g.f. is $\frac{x^4z}{1-x}W$.

Case (4). If $P = UU^aD_a\bar{Q}$, $a \geq 1$, where $Q \in \mathcal{S}$, Q ends with an asymmetric peak, and $\bar{Q}$ is obtained from Q after increasing by one the size of the last down-step, then the g.f. is

$$x\frac{x^2z}{1-x}z(S-(W-1)).$$

Case (5). If $P = UH\bar{Q}$, where $Q \in \mathcal{S}$, Q ends with an asymmetric peak, and $\bar{Q}$ is obtained from Q after increasing by one the size of the last down-step, then the g.f. is

$$x^2(S-W+1).$$

Case (6). If $P = U\bar{Q}$, where Q is a NMAP2 in $\mathcal{S}$, Q starts and ends with an asymmetric peak, $\bar{Q}$ is obtained from Q by increasing by one the last down-step, then the contribution is

$$x\left(S-W+1-\frac{x^2y}{1-x}(S-W+1)\right).$$

Summarizing all these cases, we obtain the following functional equation.

$$S = \frac{x^2y}{1-x}(S+1) + \frac{x^5z^2W}{(1-x)^2} + \frac{x^4zW}{1-x} + \frac{x^3z}{1-x}(S-W+1)$$
$$+ x^2(S-W+1) + x(S-W+1)\left(1-\frac{x^2}{1-x}\right),$$

which induces the result for S, and thus for N'. □

The first terms of the Taylor expansion of N' are

$$1 + x + yx^2 + 2yx^3 + (2y + y^2 + z)x^4 + (2y + 3y^2 + 3z + z^2)x^5$$
$$+ (2y + 5y^2 + y^3 + 6z + 2yz + 4z^2)x^6$$
$$+ (2y + 7y^2 + 4y^3 + 11z + 8yz + 11z^2 + 2yz^2)x^7 + O(x^8).$$

Corollary 4.7 *The g.f. for the number of non-decreasing MAP2 is*

$$N'(x,1,1) = \frac{(1+x)(1-2x)}{1-2x-x^2+x^3}.$$

The first terms of the Taylor expansion are

$$1 + x + x^2 + 2x^3 + 4x^4 + 9x^5 + 20x^6 + 45x^7 + 101x^8 + 227x^9 + O(x^{10}),$$

where the sequence of coefficients corresponds to the sequence A052534 in [17].

In Table 2 we summarized the results for the MAP2.

Statistic population	Generating function
Non-decreasing MAP2 avoiding symmetric peaks	$N'(x,0,1) = \frac{(1+x)(1-3x+2x^2+x^4)}{(1-x)(1-2x)}$
Non-decreasing MAP2 avoiding asymmetric peaks	$N'(x,1,0) = \frac{1-x^2}{1-x-x^2}$, Fibonacci sequence
Symmetric peaks in all non-decreasing MAP2	$\frac{x^2(1+x)(1-4x+3x^2+3x^3-x^4-x^5)}{(1-x-x^2)(1-2x-x^2+x^3)^2}$
Asymmetric peaks in all non-decreasing MAP2	$\frac{x^4(1+x)(1-x-3x^2+x^3+x^4)}{(1-x-x^2)(1-2x-x^2+x^3)^2}$

Table 2: Some statistics for non-decreasing MAP2.

Corollary 4.8 *An asymptotic for the ratio between the numbers of asymmetric and symmetric peaks in NMAP2 is*

$$\frac{a^2\left(a^4+a^3-3a^2-a+1\right)}{-a^5-a^4+3a^3+3a^2-4a+1} \sim 0.8019374457,$$

where

$$a = \frac{1}{3} - \frac{\sqrt{7}\sin\left(\frac{\arctan\left(3\sqrt{3}\right)}{3}+\frac{\pi}{6}\right)}{3} + \frac{\sqrt{3}\,\sqrt{7}\cos\left(\frac{\arctan\left(3\sqrt{3}\right)}{3}+\frac{\pi}{6}\right)}{3} \sim 0.4450418680.$$

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