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The metamorphosis of λ-fold $(K_2 \times K_3)$-designs into λ-fold 6-cycle systems

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Abstract

Let $(X, \mathcal{B})$ be a λ-fold $(K_2 \times K_3)$-design of order n. From each B in $\mathcal{B}$, we retain one 6-cycle subgraph, and delete the remaining edges. If the deleted $3|\mathcal{B}|$ edges can be rearranged into further 6-cycles, then we get a λ-fold 6-cycle system. This is called a metamorphosis of a λ-fold $(K_2 \times K_3)$-design of order n into a λ-fold 6-cycle system. In this paper we determine the necessary and sufficient conditions for the existence of such a metamorphosis.

1 Introduction

Let λ be a positive integer, G and H be simple graphs, and let λH denote the graph H with its edges replicated λ times. A λ-fold G-*design* of λH is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is a collection of isomorphic copies of G, called *blocks*, which partitions the edge set of λH. If H is a complete graph K_n, we refer to such a λ-fold G-design as one of order n. Also we use the more common term 6-cycle system for a G-design where G is a 6-cycle.

Let $(X, \mathcal{B})$ be a λ-fold G_1-design of λH, and G_2 be a subgraph of G_1. From each B in $\mathcal{B}$, we retain one copy of G_2, which is put in $\mathcal{C}$, and delete the remaining edges. If the deleted edges of $\mathcal{B}$ can be rearranged into further copies of G_2, which are put in $\mathcal{D}$, then we get a λ-fold G_2-design of λH, which is $(X, \mathcal{C} \cup \mathcal{D})$. This is called a *metamorphosis* of a λ-fold G_1-design of λH into a λ-fold G_2-design of λH and is denoted by $(G_1 > G_2)$-$M_\lambda(H)$. If $\lambda = 1$, we simply write $(G_1 > G_2)$-$M(H)$. If H is a complete graph K_n, we use the notation $(G_1 > G_2)$-$M_\lambda(n)$ for $(G_1 > G_2)$-$M_\lambda(H)$.

The first paper on metamorphosis for graph designs was by Lindner and Street [13] in 2000. Afterwards, considerable work has been done on the problem and its variations. See [1, 3, 5, 7, 9, 10, 11, 12, 14, 16] for instance. In particular, results were given for the case $(G_1, G_2) = (K_{3,3}, C_6)$ in [4], and for the case $(G_1, G_2) =$

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$(\Theta(1,3,3), C_6)$ in [2]. Howerver, for the case $(G_1, G_2) = (K_2 \times K_3, C_6)$, it is still an open problem, which we will study in this paper.

Let K_r and K_c be complete graphs with vertex sets $V(K_r)$ and $V(K_c)$, respectively. The Cartesian product $K_r \times K_c$ is a graph whose vertex set is $V(K_r) \times V(K_c)$, where two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if $x_1 = x_2$ or $y_1 = y_2$. So the graph $K_2 \times K_3$ is a 3-regular simple graph with 6 vertices as shown in Figure 1. We use the notation

a	b	c
d	e	f

for this graph. Clearly the graph $K_2 \times K_3$ contains a 6-cycle as a subgraph.

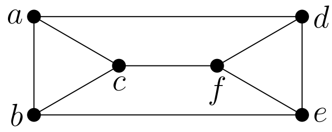

Figure 1: $K_2 \times K_3$.

In this paper we will investigate the existence of a λ-fold $(K_2 \times K_3)$-design of order n with a metamorphosis into a λ-fold 6-cycle system of the same order. We will prove the following theorem.

Theorem 1.1. *There exists a* $(K_2 \times K_3 > C_6)$-$M_\lambda(n)$ *if and only if* $\lambda(n-1) \equiv 0 \pmod 6$ *and* $\lambda n(n-1) \equiv 0 \pmod{36}$.

2 Preliminaries

We start with the necessary conditions. It is easily calculated that the necessary conditions for the existence of a λ-fold $(K_2 \times K_3)$-design of order n are $\lambda(n-1) \equiv 0 \pmod 3$ and $\lambda n(n-1) \equiv 0 \pmod{18}$, and those of a λ-fold 6-cycle system of order n are $\lambda(n-1) \equiv 0 \pmod 2$ and $\lambda n(n-1) \equiv 0 \pmod{12}$. For the possible existence of a λ-fold $(K_2 \times K_3)$-design of order n with a metamorphosis into a λ-fold 6-cycle system of the same order, we require the intersection of these conditions. So the necessary conditions for the existence of a $(K_2 \times K_3 > C_6)$-$M_\lambda(n)$ are $\lambda(n-1) \equiv 0 \pmod 6$ and $\lambda n(n-1) \equiv 0 \pmod{36}$ which we divide into six cases as follows:

$\lambda \pmod{18}$	order n
$1, 5, 7, 11, 13, 17$	$1 \pmod{36}$
$2, 4, 8, 10, 14, 16$	$1 \pmod 9$
$3, 15$	$1, 9 \pmod{12}$
$6, 12$	$0, 1 \pmod 3$
9	$1 \pmod 4$
0	any $n \geq 6$

In the recursive constructions we need the concepts of group divisible design and holey design.

Let K be a set of positive integers. A *group divisible design* (GDD) of index λ is a triple $(X, \mathcal{G}, \mathcal{A})$ where X is a set of v points, $\mathcal{G}$ is a partition of X into subsets (called *groups*), and $\mathcal{A}$ is a collection of subsets (called *blocks*) of X, such that any pair of distinct points from X occurs either in some group or in exactly λ blocks, but not both. A (K, λ)-GDD is a GDD with index λ and block sizes from K. If $K = \{k\}$, we simply write k for K. Also if $\lambda = 1$, we simply write k-GDD for $(k, 1)$-GDD. The *group type* (or *type*) of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We also use "exponential" notation for group types: the group type of $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ means there are u_i groups of size g_i for $1 \le i \le s$. We quote the following results for later use.

Lemma 2.1. *[8] The necessary and sufficient conditions for the existence of a* $(3, \lambda)$*-GDD of type* g^u *are (1)* $u \ge 3$*, (2)* $\lambda g(u-1) \equiv 0 \pmod 2$*, (3)* $\lambda g^2 u(u-1) \equiv 0 \pmod 6$.

Lemma 2.2. *[6, 15] The necessary and sufficient conditions for the existence of a* $(3, \lambda)$*-GDD of type* $g^u w^1$ *are (1) if* $g > 0$*, then* $u \ge 3$*, or* $u = 2$ *and* $w = g$*, or* $u = 1$ *and* $w = 0$*, or* $u = 0$*, (2)* $w \le g(u-1)$ *or* $gu = 0$*, (3)* $\lambda(g(u-1) + w) \equiv 0 \pmod 2$ *or* $gu = 0$*, (4)* $\lambda gu \equiv 0 \pmod 2$ *or* $w = 0$*, (5)* $\lambda(\frac{1}{2}g^2 u(u-1) + guw) \equiv 0 \pmod 3$.

Let $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a partition of a finite set X into subsets (called *holes*), where $|H_i| = h_i$ for $1 \le i \le t$. Let $K_{h_1, h_2, \ldots, h_t}$ be the complete t-partite graph on X with the i-th part on H_i. A λ-fold *holey G-design* (G-HD) is a triple $(X, \mathcal{H}, \mathcal{B})$ such that $(X, \mathcal{B})$ is a λ-fold G-design of $\lambda K_{h_1, h_2, \ldots, h_t}$. The *hole type* (or *type*) of the λ-fold G-HD is the multiset $\{h_1, h_2, \ldots, h_t\}$. We also use "exponential" notation to describe hole types: the hole type $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ denotes u_i occurrences of g_i for $1 \le i \le s$. A $(G_1 > G_2)$-$M_\lambda(K_{h_1, h_2, \ldots, h_t})$ is also denoted by $(G_1 > G_2)$-$M_\lambda(\{h_1, h_2, \ldots, h_t\})$ or by $(G_1 > G_2)$-$M_\lambda(g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s})$.

Construction 2.3. *(Weighting) Let* $(X, \mathcal{G}, \mathcal{A})$ *be a* (K, λ_1)*-GDD, and* $w : X \mapsto Z^+ \cup \{0\}$ *be a weight function. For each block* $A \in \mathcal{A}$*, suppose that there is a* $(G_1 > G_2)$*-*$M_{\lambda_2}(\{w(x) : x \in A\})$*. Then there exists a* $(G_1 > G_2)$*-*$M_{\lambda_1 \lambda_2}(\{\Sigma_{x \in G} w(x) : G \in \mathcal{G}\})$.

Proof. For every $x \in X$, let $S(x)$ be a set of $w(x)$ "copies" of x. For any $Y \subseteq X$, let $S(Y) = \cup_{x \in Y} S(x)$. For each block $A \in \mathcal{A}$, construct a λ_2-fold G_1-HD of type $\{|S(x)| : x \in A\}$, which is $(S(A), \{S(x) : x \in A\}, \mathcal{B}_A)$, with a metamorphosis into a λ_2-fold G_2-HD, which is $(S(A), \{S(x) : x \in A\}, \mathcal{C}_A \cup \mathcal{D}_A)$, where $\mathcal{C}_A$ is a collection of the retained copies of G_2, and $\mathcal{D}_A$ is a collection of the rearranged copies of G_2. Then it is readily checked that $(S(X), \{S(G) : G \in \mathcal{G}\}, \cup_{A \in \mathcal{A}} \mathcal{B}_A)$ is a $\lambda_1\lambda_2$-fold G_1-HD of type $\{\Sigma_{x \in G} w(x) : G \in \mathcal{G}\}$, with a metamorphosis into a $\lambda_1\lambda_2$-fold G_2-HD, which is $(S(X), \{S(G) : G \in \mathcal{G}\}, (\cup_{A \in \mathcal{A}} \mathcal{C}_A) \cup (\cup_{A \in \mathcal{A}} \mathcal{D}_A))$. □

Construction 2.4. *(Filling in Holes) Suppose there exists a* $(G_1 > G_2)$*-*$M_\lambda(\{h_1, h_2, \ldots, h_t\})$*, and let* $\varepsilon = 0, 1$*. For* $1 \le i \le t$*, suppose there exists a* $(G_1 > G_2)$*-*$M_\lambda(h_i + \varepsilon)$*. Then there exists a* $(G_1 > G_2)$*-*$M_\lambda(n)$*, where* $n = \sum_{i=1}^t h_i + \varepsilon$.

Proof. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_t\}$ be a partition of X, where $|H_i| = h_i$ for $1 \le i \le t$. Let Y be a set of cardinality ε such that $X \cap Y = \emptyset$.

Suppose $(X, \mathcal{H}, \mathcal{B})$ is a λ-fold G_1-HD of type $\{h_1, h_2, \ldots, h_t\}$, with a metamorphosis into a λ-fold G_2-HD, which is $(X, \mathcal{H}, \mathcal{C} \cup \mathcal{D})$, where $\mathcal{C}$ is a collection of the retained copies of G_2, and $\mathcal{D}$ is a collection of the rearranged copies of G_2. For $1 \le i \le t$, construct a λ-fold G_1-design of order $h_i + \varepsilon$, which is $(H_i \cup Y, \mathcal{B}_i)$, with a metamorphosis into a λ-fold G_2-design, which is $(H_i \cup Y, \mathcal{C}_i \cup \mathcal{D}_i)$, where $\mathcal{C}_i$ is a collection of the retained copies of G_2, and $\mathcal{D}_i$ is a collection of the rearranged copies of G_2. Then it is easily checked that $(X \cup Y, \mathcal{B} \cup (\cup_{i=1}^{t} \mathcal{B}_i))$ is a λ-fold G_1-design of order n, with a metamorphosis into a λ-fold G_2-design, which is $(X \cup Y, \mathcal{C} \cup (\cup_{i=1}^{t} \mathcal{C}_i) \cup \mathcal{D} \cup (\cup_{i=1}^{t} \mathcal{D}_i))$. □

The following construction shows how to increase λ without altering any of the other parameters; its proof is straightforward.

Construction 2.5. *Suppose there exists a $(G_1 > G_2)$-$M_{\lambda_1}(H)$ and a $(G_1 > G_2)$-$M_{\lambda_2}(H)$. Then there exists a $(G_1 > G_2)$-$M_{\lambda_1+\lambda_2}(H)$.*

3 Direct Constructions

In this section, we will construct some designs of small orders for future use. These designs are obtained by computer searches.

Usually it is difficult to find all the blocks of a design directly. A technique of "$+a \pmod{n}$" is used, meaning that we try to find a subset $\mathcal{S} \subseteq \mathcal{B}$ and an element $a \in Z_n$ such that $\{B + ka : B \in \mathcal{S}, 0 \le k \le n/a - 1\} = \mathcal{B}$. The blocks of $\mathcal{S}$ are called *base blocks*.

In some cases, we use multipliers or partial multipliers so that the required base blocks can be found in a shorter time. We say that $m \in Z_n^*$ is a *multiplier* of the design, if for each base block B, there exists some $g \in Z_n$ such that $m \cdot B + g$ is also a base block. We say that $m \in Z_n^*$ is a *partial multiplier* of the design, if for each base block $B \in \mathcal{T}$, where $\mathcal{T}$ is a subset of all the base blocks, there exists some $g \in Z_n$ such that $m \cdot B + g$ is also a base block.

Since in this paper we only consider the metamorphosis of λ-fold $(K_2 \times K_3)$-designs into 6-cycle systems, we abbreviate the notation $(K_2 \times K_3 > C_6)$-$M_\lambda(H)$ as $M_\lambda(H)$.

Let $(X, \mathcal{B})$ be a λ-fold $(K_2 \times K_3)$-design with a metamorphosis into a 6-cycle system $(X, \mathcal{C} \cup \mathcal{D})$. For each block $B = \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & f \\ \hline \end{array}$ in $\mathcal{B}$, in the direct constructions we will always retain the 6-cycle (a, d, f, e, b, c), which is put in $\mathcal{C}$, and delete the edges ab, de, cf. So it is not necessary to list $\mathcal{C}$ explicitly. We put the rearranged 6-cycles in $\mathcal{D}$.

Lemma 3.1. *For $n = 37, 73$, there exists an $M(n)$.*

Proof. Let $X = Z_n$. The required blocks $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycles $+1 \pmod{n}$, respectively.

$$n = 37: \quad \begin{array}{|c|c|c|}\hline 0 & 3 & 1 \\ \hline 4 & 18 & 9 \\ \hline \end{array} \cdot 2^i, \ i = 0, 9;$$
$$(0, 3, 11, 22, 32, 18).$$

$$n = 73: \quad \begin{array}{|c|c|c|}\hline 0 & 3 & 1 \\ \hline 4 & 65 & 10 \\ \hline \end{array} \cdot 5^i, \ i = 0, 9, 18, 27;$$
$$(0, 3, 20, 29, 59, 47) \cdot 5^i, \ i = 0, 18.$$

□

Lemma 3.2. *For $n = 10, 19$, there exists an $M_2(n)$.*

Proof. Let $X = Z_n$. For $n = 10$, $\mathcal{B}$ is obtained by developing the following base $K_2 \times K_3$ block $+1 \pmod n$, and $\mathcal{D}$ is obtained by developing the following base 6-cycle $+2 \pmod n$. For $n = 19$, $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycle $+1 \pmod n$, respectively.

$$n = 10: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 5 & 8 \\ \hline \end{array};$$
$$(0, 1, 3, 7, 8, 2).$$

$$n = 19: \quad \begin{array}{|c|c|c|}\hline 0 & 2 & 1 \\ \hline 3 & 8 & 5 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 10 & 4 \\ \hline 8 & 1 & 15 \\ \hline \end{array};$$
$$(0, 2, 6, 11, 1, 8).$$

□

Lemma 3.3. *For $n = 9, 13, 21$, there exists an $M_3(n)$.*

Proof. Let $X = Z_n$. For $n = 9, 21$, $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycles $+3 \pmod n$, respectively. For $n = 13$, $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycle $+1 \pmod n$, respectively.

$$n = 9: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 1 & 3 \\ \hline 2 & 4 & 6 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 4 & 8 \\ \hline 5 & 6 & 1 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 5 & 8 \\ \hline 4 & 1 & 2 \\ \hline \end{array};$$
$$(0, 1, 4, 2, 5, 8), \ (0, 1, 6, 2, 4, 3).$$

$$n = 13: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 4 & 6 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 2 & 7 \\ \hline 5 & 9 & 1 \\ \hline \end{array};$$
$$(0, 1, 3, 2, 8, 4).$$

$$n = 21: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 6 & 10 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 4 & 6 \\ \hline 5 & 1 & 10 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 5 & 6 \\ \hline 7 & 1 & 11 \\ \hline \end{array},$$
$$\begin{array}{|c|c|c|}\hline 0 & 6 & 13 \\ \hline 8 & 1 & 10 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 8 & 9 \\ \hline 10 & 1 & 17 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 9 & 19 \\ \hline 10 & 2 & 4 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 9 & 20 \\ \hline 11 & 1 & 14 \\ \hline \end{array},$$
$$\begin{array}{|c|c|c|}\hline 0 & 11 & 17 \\ \hline 13 & 15 & 8 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 14 & 16 \\ \hline 17 & 2 & 5 \\ \hline \end{array};$$
$$(0, 1, 3, 4, 7, 6), \ (0, 3, 7, 1, 5, 8), \ (0, 4, 10, 1, 8, 14),$$
$$(0, 5, 11, 1, 14, 9), \ (0, 8, 17, 7, 20, 9).$$

□

Lemma 3.4. *For $n = 6, 7, 12, 15, 16, 22$, there exists an $M_6(n)$.*

Proof. For $n = 6, 12, 15$, let $X = Z_{n-1} \cup \{\infty\}$. If $n = 6, 12$, $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycles $+1 \pmod{(n-1)}$, respectively. If $n = 15$, $\mathcal{B}$ is obtained by developing the following base $K_2 \times K_3$ blocks $+1 \pmod{(n-1)}$, and $\mathcal{D}$ is obtained by developing the following base 6-cycles $+2 \pmod{(n-1)}$.

$n = 6$: $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&4&\infty\\\hline\end{array}$ (repeated twice);
$(0, 1, 2, 3, 4, \infty)$.

$n = 12$: $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&4&\infty\\\hline\end{array}$ (repeated twice), $\begin{array}{|c|c|c|}\hline 0&2&4\\\hline 3&6&10\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&2&6\\\hline 5&7&1\\\hline\end{array}$;
$(0, 1, 2, 3, 4, \infty)$, $(0, 2, 4, 1, 7, 5)$.

$n = 15$: $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&4&\infty\\\hline\end{array}$ (repeated twice),
$\begin{array}{|c|c|c|}\hline 0&2&4\\\hline 3&5&9\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&2&7\\\hline 6&10&1\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&4&10\\\hline 7&11&2\\\hline\end{array}$;
$(0, 1, 2, 3, 5, \infty)$ (repeated twice),
$(0, 2, 1, 3, 7, 8)$, $(0, 2, 6, 8, 4, 10)$, $(0, 5, 1, 7, 3, 9)$.

For $n = 7, 16, 22$, let $X = Z_n$. If $n = 7$, $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycle $+1 \pmod{n}$, respectively. If $n = 16, 22$, $\mathcal{B}$ is obtained by developing the following base $K_2 \times K_3$ blocks $+1 \pmod{n}$, and $\mathcal{D}$ is obtained by developing the following base 6-cycles $+2 \pmod{n}$.

$n = 7$: $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&4&5\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&1&3\\\hline 2&4&5\\\hline\end{array}$;
$(0, 1, 2, 4, 5, 3)$.

$n = 16$: $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&4&5\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&5&7\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&2&5\\\hline 3&8&12\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&9&4\\\hline 7&1&11\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&6&12\\\hline 8&14&3\\\hline\end{array}$;
$(0, 1, 2, 3, 4, 5)$, $(0, 2, 1, 3, 5, 10)$, $(0, 2, 5, 8, 1, 6)$,
$(0, 5, 11, 4, 13, 6)$, $(0, 7, 1, 8, 15, 9)$.

$n = 22$: $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&4&5\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&1&2\\\hline 3&5&7\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&2&5\\\hline 3&7&11\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&4&8\\\hline 5&10&17\\\hline\end{array}$,
$\begin{array}{|c|c|c|}\hline 0&5&13\\\hline 9&15&1\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&6&13\\\hline 10&17&1\\\hline\end{array}$, $\begin{array}{|c|c|c|}\hline 0&7&16\\\hline 11&18&4\\\hline\end{array}$; □
$(0, 1, 2, 3, 4, 5)$, $(0, 2, 1, 3, 5, 9)$, $(0, 2, 5, 1, 4, 10)$, $(0, 4, 8, 1, 6, 12)$,
$(0, 5, 10, 1, 6, 12)$, $(0, 5, 11, 1, 13, 7)$, $(0, 7, 13, 1, 8, 15)$.

Lemma 3.5. *For $n = 17, 29$, there exists an $M_9(n)$.*

Proof. Let $X = Z_n$. The required blocks $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycles $+1 \pmod{n}$, respectively.

$$n=17: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 7 & 12 \\ \hline \end{array} \cdot 2^i, \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 3 \\ \hline 4 & 13 & 10 \\ \hline \end{array} \cdot 2^i, \ i=0,1,2,3;$$
$$(0,1,2,6,14,7)\cdot 2^i, \ i=0,1,2,3.$$

$$n=29: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 4 \\ \hline 5 & 7 & 26 \\ \hline \end{array} \cdot 4^i, \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 4 \\ \hline 2 & 20 & 25 \\ \hline \end{array} \cdot 4^i, \ i=0,1,2,3,4,5,6;$$
$$(0,1,2,4,15,7)\cdot 4^i, \ i=0,1,2,3,4,5,6.$$

□

Lemma 3.6. *For $n = 8, 11, 14, 20, 23$, there exists an $M_{18}(n)$.*

Proof. For $n = 8, 14, 20$, let $X = Z_{n-1} \cup \{\infty\}$. The required blocks $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycles $+1 \pmod{(n-1)}$, respectively.

$$n=8: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 5 & \infty \\ \hline \end{array} \text{ (repeated 6 times)}, \ \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 4 & 5 \\ \hline \end{array}, \ \begin{array}{|c|c|c|}\hline 0 & 1 & 3 \\ \hline 2 & 4 & 5 \\ \hline \end{array};$$
$$(0,1,2,4,6,\infty) \text{ (repeated 3 times)}, \ (0,1,2,4,5,3).$$

$$n=14: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 4 \\ \hline 2 & 7 & \infty \\ \hline \end{array} \text{ (repeated 6 times)},$$
$$\begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 5 & 11 \\ \hline \end{array} \cdot 5^i, \ i=0,1 \text{ (repeated 4 times)};$$
$$(0,1,2,7,12,\infty) \text{ (repeated 3 times)},$$
$$(0,1,3,2,6,4)\cdot 5^i, \ i=0,1 \text{ (repeated twice)}.$$

$$n=20: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 3 \\ \hline 11 & 5 & \infty \\ \hline \end{array} \cdot 2^i, \ i=0,3,6 \text{ (repeated twice)},$$
$$\begin{array}{|c|c|c|}\hline 0 & 1 & 3 \\ \hline 4 & 10 & 15 \\ \hline \end{array} \cdot 2^j, \ j=0,1 \text{ (repeated 7 times)};$$
$$(0,1,2,8,14,\infty)\cdot 2^i, \ i=0,3,6,$$
$$(0,1,3,8,14,7) \text{ (repeated 7 times)}.$$

For $n = 11, 23$, let $X = Z_n$. The required blocks $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycles $+1 \pmod{n}$, respectively.

$$n=11: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 2 \\ \hline 3 & 5 & 10 \\ \hline \end{array} \cdot 2^i, \ i=0,2,4,6,8 \text{ (repeated twice)};$$
$$(0,1,2,4,6,3)\cdot 2^i, \ i=0,2,4,6,8.$$

$$n=23: \quad \begin{array}{|c|c|c|}\hline 0 & 1 & 3 \\ \hline 4 & 19 & 10 \\ \hline \end{array} \cdot 5^i, \ i=0,2,4,\ldots,20 \text{ (repeated twice)};$$
$$(0,1,2,9,16,8)\cdot 5^i, \ i=0,2,4,\ldots,20.$$

□

Lemma 3.7. *For $(\lambda, h, u) \in \{(1,6,3), (2,3,3), (3,2,3)\}$, there exists an $M_\lambda(h^u)$.*

Proof. For $(\lambda, h, u) = (1, 6, 3)$, let $X = Z_{18}$, $\mathcal{H} = \{\{0, 3, 6, 9, 12, 15\} + i : i = 0, 1, 2\}$. The required blocks $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycles $+6 \pmod{18}$, respectively.

$$\begin{array}{|c|c|c|}\hline 0&2&1\\\hline 4&3&5\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&7&5\\\hline 14&3&10\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&13&8\\\hline 11&3&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&17&10\\\hline 16&3&8\\\hline\end{array};$$

$(0, 2, 4, 3, 11, 7)$, $(0, 13, 2, 9, 4, 17)$.

For $(\lambda, h, u) = (2, 3, 3)$, let $X = Z_9$, $\mathcal{H} = \{\{0, 3, 6\} + i : i = 0, 1, 2\}$. The required blocks $\mathcal{B}$ and $\mathcal{D}$ are obtained by developing the following base $K_2 \times K_3$ blocks and base 6-cycle $+3 \pmod 9$, respectively.

$$\begin{array}{|c|c|c|}\hline 0&2&1\\\hline 4&3&5\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&7&5\\\hline 4&2&6\\\hline\end{array};$$

$(0, 1, 5, 6, 4, 2)$.

For $(\lambda, h, u) = (3, 2, 3)$, let $X = Z_6$, $\mathcal{H} = \{\{0, 3\} + i : i = 0, 1, 2\}$. The required blocks $\mathcal{B}$ and $\mathcal{D}$ are listed as follows.

$$\mathcal{B}:\ \begin{array}{|c|c|c|}\hline 0&1&2\\\hline 4&5&3\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&1&5\\\hline 2&3&4\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&2&4\\\hline 5&1&3\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&4&5\\\hline 1&2&3\\\hline\end{array};$$

$\mathcal{D}:$ $(0, 1, 2, 3, 5, 4)$, $(0, 1, 5, 4, 3, 2)$.

□

4 Proof of Theorem 1.1

We now complete the proof of Theorem 1.1 by solving the cases $\lambda = 1, 2, 3, 6, 9, 18$.

Lemma 4.1. *For $n \equiv 1 \pmod{36}$, there exists an $M(n)$.*

Proof. For $n = 37, 73$, see Lemma 3.1.

For $n \geq 109$, a 3-GDD of type 6^u $(u \geq 3)$ exists by Lemma 2.1. Give every point of the GDD weight 6 and apply Construction 2.3 with an $M(6^3)$ from Lemma 3.7 to obtain an $M(36^u)$. Then apply Construction 2.4 with an $M(37)$ to get an $M(n)$, where $n = 36u + 1$, $u \geq 3$. □

Lemma 4.2. *For $n \equiv 1 \pmod 9$, there exists an $M_2(n)$.*

Proof. For $n = 10, 19$, see Lemma 3.2. For $n = 37$, apply Construction 2.5 with two copies of an $M(37)$.

For $n \equiv 10 \pmod{18}$ and $n \geq 28$, a 3-GDD of type 3^u exists by Lemma 2.1, where $u \equiv 1 \pmod 2$, $u \geq 3$. Give every point of the GDD weight 3 and apply Construction 2.3 with an $M_2(3^3)$ from Lemma 3.7 to obtain an $M_2(9^u)$. Then apply Construction 2.4 with an $M_2(10)$ to get the desired $M_2(n)$.

For $n \equiv 1 \pmod{18}$ and $n \geq 55$, give every point of a 3-GDD of type 6^u $(u \geq 3)$ weight 3 and apply Construction 2.3 with an $M_2(3^3)$ to obtain an $M_2(18^u)$. Then apply Construction 2.4 with an $M_2(19)$ to get the desired design. □

Lemma 4.3. *For $n \equiv 1, 9 \pmod{12}$, there exists an $M_3(n)$.*

Proof. For $n = 9, 13, 21$, see Lemma 3.3.

For $n = 25, 33$, a 3-GDD of type 4^u ($u = 3, 4$) exists by Lemma 2.1. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ from Lemma 3.7 to obtain an $M_3(8^u)$. Then apply Construction 2.4 with an $M_3(9)$ to get the desired designs.

For $n \equiv 1 \pmod{12}$ and $n \geq 37$, give every point of a 3-GDD of type 6^u ($u \geq 3$) weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to obtain an $M_3(12^u)$. Then apply Construction 2.4 with an $M_3(13)$ to get the desired $M_3(n)$.

For $n \equiv 9 \pmod{12}$ and $n \geq 45$, a 3-GDD of type 6^u4^1 ($u \geq 3$) exists by Lemma 2.2. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to get an $M_3(12^u8^1)$. Then apply Construction 2.4 with an $M_3(13)$ and an $M_3(9)$ to get the desired design. □

Lemma 4.4. *For $n \equiv 0, 1 \pmod{3}$, there exists an $M_6(n)$.*

Proof. For $n = 6, 7, 12, 15, 16, 22$, see Lemma 3.4. For $n = 9, 10, 13, 19, 21$, apply Construction 2.5 with copies of an $M_3(9)$, an $M_2(10)$, an $M_3(13)$, an $M_2(19)$ an $M_3(21)$, respectively.

For $n \equiv 0, 1 \pmod{6}$ and $n \geq 18$, a $(3, 2)$-GDD of type 3^u ($u \geq 3$) exists by Lemma 2.1. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to obtain an $M_6(6^u)$. Then apply Construction 2.4 with an $M_6(6)$ and an $M_6(7)$, respectively, to get an $M_6(n)$, where $n = 6u + \varepsilon$, $u \geq 3$, $\varepsilon = 0, 1$.

For $n \equiv 3 \pmod{6}$ and $n \geq 27$, a $(3, 2)$-GDD of type 3^u4^1 ($u \geq 3$) exists by Lemma 2.2. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to obtain an $M_6(6^u8^1)$. Then apply Construction 2.4 with an $M_6(7)$ and an $M_6(9)$ to get the desired $M_6(n)$.

For $n \equiv 4 \pmod{6}$ and $n \geq 28$, a $(3, 2)$-GDD of type 3^u5^1 ($u \geq 3$) exists by Lemma 2.2. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to obtain an $M_6(6^u10^1)$. Then apply Construction 2.4 with an $M_6(6)$ and an $M_6(10)$ to get the desired design. □

Lemma 4.5. *For $n \equiv 1 \pmod{4}$, there exists an $M_9(n)$.*

Proof. For $n = 17, 29$, see Lemma 3.5.

For $n \equiv 1, 9 \pmod{12}$, apply Construction 2.5 with three copies of an $M_3(n)$.

For $n = 41$, a $(3, 3)$-GDD of type 4^38^1 exists by Lemma 2.2. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to obtain an $M_9(8^316^1)$. Then apply Construction 2.4 with an $M_9(9)$ and an $M_9(17)$ to get the desired design.

For $n \equiv 5 \pmod{12}$ and $n \geq 53$, a $(3, 3)$-GDD of type 6^u8^1 ($u \geq 3$) exists by Lemma 2.2. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to obtain an $M_9(12^u16^1)$. Then apply Construction 2.4 with an $M_9(13)$ and an $M_9(17)$ to get the desired $M_9(n)$. □

Lemma 4.6. *For any $n \geq 6$, there exists an $M_{18}(n)$.*

Proof. For $n = 8, 11, 14, 20, 23$, see Lemma 3.6. For $n = 17$, apply Construction 2.5 with two copies of an $M_9(17)$.

For $n \equiv 0, 1 \pmod 3$, apply Construction 2.5 with three copies of an $M_6(n)$.

For $n \equiv 2 \pmod 6$ and $n \geq 26$, a $(3,6)$-GDD of type $3^u 4^1$ ($u \geq 3$) exists by Lemma 2.2. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to obtain an $M_{18}(6^u 8^1)$. Then apply Construction 2.4 with an $M_{18}(6)$ and an $M_{18}(8)$ to get the desired design.

For $n \equiv 5 \pmod 6$ and $n \geq 29$, a $(3,6)$-GDD of type $3^u 5^1$ ($u \geq 3$) exists by Lemma 2.2. Give every point of the GDD weight 2 and apply Construction 2.3 with an $M_3(2^3)$ to obtain an $M_{18}(6^u 10^1)$. Then apply Construction 2.4 with an $M_{18}(7)$ and an $M_{18}(11)$ to get the desired $M_{18}(n)$. □

Combining Construction 2.5 and Lemmas 4.1–4.6, we complete the proof of Theorem 1.1.

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