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Labeled packing of non-star trees into their k^{th} power, $k \geq 5$

Maidoun Mortada* Sara Nasser†

Faculty of Sciences (I), Mathematics Department
KALMA Laboratory, Lebanese University
Beirut, Lebanon
maydoun.mortada@ul.edu.lb sara.nasser@liu.edu.lb

Hamamache Kheddouci

Lyon 1 University, LIRIS Lab. UMR CNRS 5205
Bêtiment Nautibus, Lyon
France
hamamache.kheddouci@univ-lyon1.fr

Abstract

Duchêne et al. [*Australas. J. Combin.* 57 (2013), 109–126] introduced a new variant of the graph packing problem called the labeled packing of a graph, which aims to study the packing of a vertex labeled graph. In this paper, we show that there exists a labeled packing of a non-star tree T into its k^{th} power T^k, $k \geq 5$, with $m_T + 1$ labels, where m_T denotes the maximum number of leaves which can be removed from T in such a way that the tree so obtained is a non-star one.

1 Introduction

All graphs considered in this paper are finite. For a graph G, $V(G)$ and $E(G)$ will denote its vertex set and edge set respectively. For any two vertices x and y in $V(G)$, the edge between x and y is denoted by xy. We denote the cardinality of $V(G)$ and

* Also at:
Faculty of Sciences (IV), Mathematics Department, KALMA Laboratory, Lebanese University, Baalbek;
(*)Graph Theory and Operation Research, Department of Mathematics and Physics, Lebanese International University (LIU), Beirut, Lebanon;
(#)Graph Theory and Operation Research, Department of Mathematics and Physics, The International University of Beirut (BIU), Beirut, Lebanon.
† Also at (*) and (#).

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$E(G)$ by $v(G)$ and $e(G)$ respectively. We denote by $N_G(x)$ the set of neighbors of a vertex x in G. The degree $d_G(x)$ of a vertex x in G is the cardinality of the set $N_G(x)$. For short, we use $d(x)$ instead of $d_G(x)$ and $N(x)$ instead of $N_G(x)$. The distance between two vertices of G, say x and y, is denoted by $\text{dist}_G(x, y)$, and for simplicity we usually use $\text{dist}(x, y)$. We say that a graph H is a subgraph of G if and only if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let H be a subgraph of G; we say that H is an induced subgraph of G if every edge in $E(G)$ whose vertices are in $V(H)$ is also an edge in $E(H)$. For $U \subseteq V(G)$, we denote by $G - U$ the graph obtained from G after deleting all the vertices in U and their incident edges. For $F \subseteq E(G)$, we define $G - F$ as the graph obtained from G after removing all the edges in F. A graph G is said to be connected if any two vertices of G are joined by a path. A connected component of a graph G is a maximal induced connected subgraph (with respect to $\subseteq$). We denote by $C_G(x)$ the connected component of G containing x.

Let k be an integer with $k \geq 2$. We denote by G^k the k^{th} power of G, and it is the graph obtained from G after adding the edges xy whenever $xy \notin E(G)$ and $\text{dist}(x, y) \leq k$.

A vertex of degree one in a tree is called a leaf. One can easily notice that the removal of a set of leaves from a non-star tree may result in either a non-star tree or a star one. For a non-star tree T, we denote by m_T the maximum number of leaves that can be removed from T in such a way that the obtained graph is a non-star tree. The number of edges of a path P is its length $l(P)$. We denote by P_n a path of order n, and we say that a path is an xy-path if x and y are its ends. Let $P = v_1v_2 \dots v_n$ be a path, if $P_1, P_2, \dots, P_k$, $k \geq 2$, are pairwise vertex disjoint subpaths of P such that $V(P) = \bigcup_{i=1}^{k} V(P_i)$, P_i is an x_iy_i-path with $x_1 = v_1$, $y_k = v_n$ and $y_ix_{i+1} \in E(P)$ for $i = 1, 2, \dots, k - 1$, then we write $P = P_1P_2 \dots P_k$.

The concept of graph packing was first introduced independently by Bollobás and Eldridge [1] as well as by Sauer and Spencer [6] in the late 1970s, and it was defined as follows: Let G be a graph of order n, and let σ be a permutation from $V(G)$ to $V(K_n)$. The map $\sigma^* : E(G) \to E(K_n)$ such that $\sigma^*(xy) = \sigma(x)\sigma(y)$ is the map induced by σ. We say that there is a packing of k copies of G (into the complete graph K_n) if there exist k permutations $\sigma_i : V(G) \to V(K_n)$, where $i = 1, \dots, k$, such that $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \emptyset$ for $i \neq j$. Such a packing will be called a k-placement of G. Thus, $\sigma : V(G) \to V(K_n)$ is a 2-placement (or embedding) of G if whenever an edge xy belongs to $E(G)$, then $\sigma(x)\sigma(y)$ does not belong to $E(G)$; that is if G has a 2-placement, then G is a subgraph of its complement. A permutation σ on $V(G)$ such that $\sigma(x) \neq x$ for every x in $V(G)$ is called a fixed point free permutation.

The problem of packing paths and trees in their complements has been a long-standing fundamental inquiry in combinatorics, extensively explored in existing literature. To access an overview of this field, we refer to the survey articles of Woźniak [10] and Yap [12]. In [2], a complete description of all graphs with $v(G) = n$ and $e(G) = n - 2$ admitting a 2-placement is given. A similar result about graphs with $v(G) = n$ and $e(G) = n$ is provided in [4]. Concerning non-star trees, it is well-known that any non-star tree is contained in its own complement. This result has been improved in many ways especially in considering some additional information and conditions about embedding. An example of such a result is the following theorem contained as a lemma in [11]:

Theorem 1.1. *Let T be a non-star tree of order n with $n > 3$. Then there exists a 2-placement σ of T such that for every $x \in V(T)$,* $\mathrm{dist}(x, \sigma(x)) \leq 3$.

This theorem immediately implies the following:

Corollary 1.2. *Let T be a non-star tree of order n with $n > 3$. Then there exists an embedding σ of T such that $\sigma(T) \subset T^7$.*

In [5], Kheddouci et al. gave a better improvement in the following theorem:

Theorem 1.3. *Let T be a non-star tree and let x be a vertex of T. Then, there exists a permutation σ on $V(T)$ satisfying the following four conditions:*

1. *σ is a 2-placement of T.*
2. $\sigma(T) \subseteq T^4$.
3. $\mathrm{dist}(x, \sigma(x)) = 1$.
4. *for every neighbor y of x,* $\mathrm{dist}(y, \sigma(y)) \leq 2$.

Labeled graph packing problem is a well-known field of graph theory that has been considerably investigated. It was introduced by E. Duchêne et al. in [3]:

Definition 1.4. Consider a graph G on n vertices. Let f be a mapping from $V(G)$ into the set $\{1, 2, \ldots, p\}$, where $p \in \mathbb{N}^*$. The mapping f is called a p-labeled packing of k copies of G into K_n if there exist k permutations $\sigma_i : V(G) \to V(K_n)$, where $i = 1, \ldots, k$, such that:

1. $\sigma_i^*(E(G)) \cap \sigma_j^*(E(G)) = \emptyset$ for all $i \neq j$.
2. For every vertex v of G, we have $f(v) = f(\sigma_1(v)) = f(\sigma_2(v)) = \cdots = f(\sigma_k(v))$.

The maximum positive integer p for which G admits a p-labeled packing of k copies of G is called the labeled k-packing number of G and it is denoted by $\lambda_k(G)$.

E. Duchêne et al. also studied the labeled packing of two copies of graphs and proved the following result presented as a lemma in [3]:

Theorem 1.5. *Let G be a graph on n vertices, and let I be a maximum independent set of G. If there exists an embedding of G into K_n, then*

$$\lambda_2(G) \leq |I| + \lfloor \tfrac{n-|I|}{2} \rfloor.$$

Moreover, Tahraoui et al. in [8] gave exact values of $\lambda_2(G)$ when G is a caterpillar or a path in addition to a lower bound of $\lambda_2(T)$, where T is a non-star tree. In 2017, Tahraoui et al. in [7] improved Woźniak's bound present in [9] and they showed the following:

Theorem 1.6. *Let G be a graph with $v(G) = n$ such that $n \geq 2$ and $e(G) \leq n - 2$. Then, $\lambda_2(G) \geq \lfloor \frac{2n}{3} \rfloor$.*

In this paper, we are concerned with finding a p-labeled packing of G into G^k, and the definition of this new problem is given below:

Definition 1.7. Let f be a mapping from $V(G)$ into the set $\{1, 2, \ldots, p\}$, where $p \in \mathbb{N}^*$. The mapping f is called a p-labeled packing of G into G^k if there exists a 'permutation $\sigma : V(G) \to V(K_n)$, such that:

1. σ is a 2-placement of G.
2. $\sigma(G) \subseteq G^k$.
3. For every vertex v of G, we have $f(v) = f(\sigma(v))$.

The maximum positive integer p for which G admits a p-labeled packing of G into G^k is called the labeled packing k-power number and it is denoted by $w^k(G)$.

In Section 2, we pass by the labeled packing of a non-star tree T into T^k, $k \geq 5$, through a specific type of permutation:

Definition 1.8. Let T be a non-star tree and let x be a vertex of T. Then a fixed point free permutation σ on $V(T)$ is called a (T, x)-good 2-placement if it satisfies the following conditions:

1. σ is a 2-placement of T.
2. $\sigma(T) \subseteq T^5$.
3. $\text{dist}(x, \sigma(x)) \leq 2$.
4. $\text{dist}(y, \sigma(y)) \leq 3$ for every neighbor y of x.
5. $\text{dist}(y, \sigma(y)) \leq 4$ for every vertex y of T.

We prove then:

Theorem 1.9. *Let T be a non-star tree and let x be a vertex of T. Then there exists a (T, x)-good 2-placement.*

The above result allows us easily to find a lower bound of $w^k(T)$, $k \geq 5$, where we will prove:

Corollary 1.10. $w^k(T) \geq m_T + 1, \ k \geq 5$ *for every non-star tree T on n vertices.*

2 Labeled packing of a non-star tree T into T^k, $k \geq 5$

Before proving Theorem 1.9, we need to present some definitions. Let T be a non-star tree and let xy be an edge in T. We call a neighbor tree of y the connected component containing x in $T - \{xy\}$, and we denote it by $T_{(x,y)}$. Now, $T_{(x,y)}$ is said to be a neighbor F-tree of y if $T_{(x,y)}$ is a path of length at most two such that x is an end of $T_{(x,y)}$ whenever $T_{(x,y)}$ is a path of length two.

In order to prove Theorem 1.9, we have to pass first by the good 2-placement of paths:

Definition 2.1. Consider a path P_n, $n \geq 4$. A fixed point free permutation σ on $V(P_n)$ is called a P_n-good path 2-placement if it satisfies the following conditions:

1. σ is a 2-placement of P_n.
2. $\sigma(P_n) \subseteq P_n^5$.
3. $\text{dist}(y, \sigma(y)) \leq 2$ for every vertex y of P_n.

We are going to prove:

Theorem 2.2. *There exists a P_n-good path 2-placement for any path P_n, $n \geq 4$.*

To build the proof of the above theorem, we use the following lemma:

Lemma 2.3. *For every integer $n \geq 8$, there exists a, b, $c \in W$ such that $n = 4a + 5b + 6c$.*

Proof. The proof is by induction. Clearly, we are done for $n = 8$. Suppose it is true up to n with $n \geq 9$. If $n+1$ is a multiple of 4, 5 or 6, then we are done. Otherwise, there exist, by induction, a, b, $c \in W$ such that $n = 4a+5b+6c$. Clearly, $a+b+c \geq 2$ as $n \geq 9$. If $a \geq 1$, then $n+1 = 4(a-1)+5(b+1)+6c$. Otherwise, we will consider two cases regarding b. If $b \geq 1$, then $n+1 = 5(b-1)+6(c+1)$. If $b = 0$, then $c \geq 2$, and so $n+1 = 6c+1 = (4 \times 2)+5+6(c-2)$. □

Proof of Theorem 2.2. We proceed by induction. For each path P_n, $4 \leq n \leq 7$, we will introduce below a P_n-good path 2-placement σ:

- For $P_4 = x_1x_2x_3x_4$, $\sigma = (x_1\ x_2\ x_4\ x_3)$ is a P_4-good path 2-placement.
- For $P_5 = x_1x_2x_3x_4x_5$, $\sigma = (x_1\ x_2\ x_4\ x_5\ x_3)$ is a P_5-good path 2-placement.
- For $P_6 = x_1x_2x_3x_4x_5x_6$, $\sigma = (x_1\ x_2\ x_4\ x_6\ x_5\ x_3)$ is a P_6-good path 2-placement.
- For $P_7 = x_1x_2x_3x_4x_5x_6x_7$, $\sigma = (x_1\ x_2\ x_4\ x_6\ x_7\ x_5\ x_3)$ is a P_7-good path 2-placement.

For $n \geq 8$, let a, b, $c \in W$ such that $n = 4a+5b+6c$. Then, there exists $P_{4,1}, \ldots, P_{4,a}$, $P_{5,1}, \ldots, P_{5,b}$, $P_{6,1}, \ldots, P_{6,c}$, where $P_{i,j}$ is a subpath of P of order i such that $P_n = P_{4,1} \ldots P_{4,a}P_{5,1} \ldots P_{5,b}P_{6,1} \ldots P_{6,c}$. It is clear that there exists a $P_{i,j}$-good path 2-placement σ_i^j for $4 \leq i \leq 6$ and $1 \leq j \leq k$ where $k \in \{a, b, c\}$. Thus, $\sigma = \sigma_4^1 \ldots \sigma_4^a\sigma_5^1 \ldots \sigma_5^b\sigma_6^1 \ldots \sigma_6^c$ is a P_n-good path 2-placement. □

Regarding the proof of Theorem 1.9, an important technique, demonstrated as a lemma, is needed in order to facilitate the presentation of the proof.

Lemma 2.4. *Consider a non-star tree T containing a vertex x such that $d(x) = n \geq 2$ and $T_{(x_i,x)}$ is a neighbor F-tree of x for $i = 1, \ldots, m$, $1 \leq m < n$, where $\{x_1, \ldots, x_n\}$ are the neighbors of x. Let $T' = T - \{xx_i : i = 1, \ldots, m\}$. If there exists a $(C_{T'}(x), z)$-good 2-placement σ_0 for some z in $C_{T'}(x)$ such that $\text{dist}(x, \sigma_0(x)) \leq 2$ and $\text{dist}(y, \sigma_0(y)) \leq 2$ for any y in $N(x) \cap C_{T'}(x)$, then there exists a (T, z)-good 2-placement σ such that $\text{dist}(u, \sigma(u)) \leq \text{dist}(u, \sigma_0(u))$ for every $u \in C_{T'}(x)$.*

Proof. Let r, p and q be the number of neighbor F-trees of x that are paths of length zero, one and two respectively in the set $\{T_{(x_i,x)} : i = 1, \ldots, m\}$. In what follows we need to rename some neighbors of x for the sake of the proof. Let $T_i = T_{(x_i,x)}$ for $i = 1, \ldots, m$ such that if $r > 0$, then T_i is the vertex a_i for $i = 1, \ldots, r$, if $p > 0$, then $T_i = b_{i-r}c_{i-r}$ for $i = r+1, \ldots, p+r$ and if $q > 0$, then $T_i = d_{i-(p+r)}e_{i-(p+r)}f_{i-(p+r)}$ for $i = r+p+1, \ldots, r+p+q$. Set $T^{(0)} = C_{T'}(x)$, $T^{(1)} = T^{(0)} \cup \bigcup_{i=1}^{i=r} T_i$, $T^{(2)} = T^{(1)} \cup \bigcup_{i=1}^{i=p} T_i$, $T^{(3)} = T^{(2)} \cup \bigcup_{i=1}^{i=q} T_i$. In order to define a (T, z)-good 2-placement, we are going to extend σ_i into σ_{i+1}, where σ_{i+1} is a $(T^{(i+1)}, z)$-good 2-placement for every $i \in \{0,\ 1,\ 2\}$ and σ_3 is the desired (T, z)-good 2-placement. To construct σ_3, we need to introduce the permutations Θ, Υ and Δ. If $r > 1$, $p > 1$ and $q > 1$, define Θ over $V(\bigcup_{i=1}^{i=r} T_i)$, Υ over $V(\bigcup_{i=1}^{i=p} T_i)$ and Δ over $V(\bigcup_{i=1}^{i=q} T_i)$ respectively.

$\Theta = (a_1\, a_2\, \ldots\, a_r)$.

$\Upsilon = (b_1\, c_1\, b_2\, c_2\, \ldots\, b_p\, c_p)$.

$\Delta = (e_1\, e_2\, \ldots\, e_q)(d_1\, f_1)(d_2\, f_2) \ldots (d_q\, f_q)$.

Now, we are ready to define σ_1, then σ_2 and finally σ_3.

If $r > 1$, let $\sigma_1 = \sigma_0\ \Theta$. For the case $r = 1$, if there exists $u \in N_{T^{(0)}}(x)$ such that $\sigma_0(u) = x$, let

$$\sigma_1(v) = \begin{cases} \sigma_0(v) & \text{if } v \in V(T^{(0)}) - \{u\}, \\ a_1 & \text{if } v = u, \\ x & \text{if } v = a_1; \end{cases}$$

and if not, let

$$\sigma_1(v) = \begin{cases} \sigma_0(v) & \text{if } v \in V(T^{(0)}) - \{x\}, \\ a_1 & \text{if } v = x, \\ \sigma_0(x) & \text{if } v = a_1. \end{cases}$$

Finally, if $r = 0$, let $\sigma_1 = \sigma_0$.

In order to construct σ_2, we need as above to study three cases regarding the value of p. If $p > 1$, let $\sigma_2 = \sigma_1\ \Upsilon$. For the case $p = 1$, let

$$\sigma_2(v) = \begin{cases} \sigma_1(v) & \text{if } v \in V(T^{(1)}) - \{x\}, \\ c_1 & \text{if } v = x, \\ \sigma_1(x) & \text{if } v = b_1, \\ b_1 & \text{if } v = c_1. \end{cases}$$

Finally, if $p = 0$, let $\sigma_2 = \sigma_1$.

Now, we are ready to define σ_3. If $q > 1$, let $\sigma_3 = \sigma_2\ \Delta$. For the case $q = 1$, let

$$\sigma_3(v) = \begin{cases} \sigma_2(v) & \text{if } v \in V(T^{(2)}) - \{x\}, \\ e_1 & \text{if } v = x, \\ \sigma_2(x) & \text{if } v = d_1, \\ f_1 & \text{if } v = e_1, \\ d_1 & \text{if } v = f_1. \end{cases}$$

Finally, if $q = 0$, let $\sigma_3 = \sigma_2$.

Thus, σ_3 is a (T, x)-good 2-placement. □

Proof of Theorem 1.9. The proof is by induction on the order n of T. Since T is a non-star tree, then $n \geq 4$. For $n = 4$, P_4 is the only non-star tree, and so by Theorem 2.2, there exists a P_4-good path 2-placement.

Now, let T be a non-star tree of order n; $n \geq 5$, and suppose that the theorem holds for every non-star tree of order $m < n$. Let x be a vertex of T. If T is a path, then the result holds directly by Theorem 2.2. In what follows, T is not a path.

If there exists a vertex u in T such that $N(u) = \{a_1, \ldots, a_k\}$ with $k \geq 2$ where a_i is a leaf for all $i \in \{1, \ldots, k\}$ and $d(u) = k+1$, then consider $T' = T - \{a_1, \ldots, a_k\}$. If T' is a non-star tree, $\sigma = \sigma_i\,(a_1\ \ldots\ a_k)$ with $i \in \{1, 2\}$ is a (T, x)-good 2-placement, where σ_1 is a (T', x)-good 2-placement if $x \in T'$ and σ_2 is a (T', u)-good 2-placement by induction. If T' is a star, then T is isomorphic to the tree in Figure 1 under which we define a (T, x)-good 2-placement.

Otherwise, for every $u \in V(T)$, if $\{a_1, \ldots, a_k\} \subset N(u)$ where a_i is a leaf for all $i \in \{1, \ldots, k\}$ and $k \geq 2$, then $d(u) > k + 1$. Since T is a non-star tree, then there exists a path P in T containing x with $l(P) \geq 3$. By Theorem 2.2, there exists a P-good path 2-placement σ_0. Set $P = x_1x_2 \ldots x_r$ where $x = x_t$ for some t, $1 \leq t \leq r$. Clearly, $T - P \neq \emptyset$. For every $i \in \{1, \ldots, r\}$, set $N(x_i) = N_i \cup F_i$, where $N_i = \{w \in N(x_i) - V(P) :\ T_{(w,x_i)}$ is a non star tree$\}$ and $F_i = \{w \in N(x_i) - V(P) :$ $T_{(w,x_i)}$ is a neighbor F-tree of $x_i\}$.

Set $T_0 = P$ and $T_{i+1} = T_i \cup \left(\bigcup\limits_{w \in N(x_{i+1}) - V(P)} T_{(w,x_{i+1})} \right)$ for $i = 0, \ldots, r-1$. Now, we are going to extend σ_0 into a (T, x)-good 2-placement. This extension is done successively starting from $i = 0$ and ending at $i = r - 1$, by extending σ_i which is a (T_i, x)-good 2-placement into σ_{i+1} which is a (T_{i+1}, x)-good 2-placement, in order to reach σ_r which is the desired (T, x)-good 2-placement, and the extension will be as following: if $N_{i+1} \neq \emptyset$, then let $N_{i+1} = \{w^1_{i+1}, w^2_{i+1}, \ldots, w^{l_{i+1}}_{i+1}\}$, where $l_{i+1} \geq 1$. By induction, there exists a $(T_{(w^j_{i+1},x_{i+1})}, w^j_{i+1})$-good 2-placement, say σ^j_{i+1}, $j = 1, \ldots, l_{i+1}$, and let

$$\sigma'_{i+1}(v) = \begin{cases} \sigma_i(v) & \text{if } v \in V(T_i), \\ \sigma^j_{i+1}(v) & \text{if } v \in V(T_{(w^j_{i+1},x_{i+1})}). \end{cases}$$

Otherwise, let $\sigma'_{i+1} = \sigma_i$. Then σ'_{i+1} is a $\left(T_i \cup \left(\bigcup_{w \in N_{i+1}} T_{(w,x_{i+1})}\right), x\right)$-good 2-placement.

Now, if $F_{i+1} = \emptyset$, then let $\sigma_{i+1} = \sigma'_{i+1}$ which is a (T_{i+1}, x)-good 2-placement. Otherwise, there exists a (T_{i+1}, x)-good 2-placement, name it σ_{i+1}, by Lemma 2.4. Thus, σ_r is a (T, x)-good 2-placement. □

Proof of Corollary 1.10. Let $T' = T - \{\alpha_1, \ldots, \alpha_{m_T}\}$, where $\{\alpha_1, \ldots, \alpha_{m_T}\}$ is a maximal set of leaves that can be removed from T in such a way that the obtained tree is a non-star one. Since T' is a non-star tree, then there exists a (T', x)-good 2-placement σ' for some x in T'. We define a packing σ of T into T^k, $k \geq 5$, as follows:

$$\sigma(v) = \begin{cases} \sigma'(v) & \text{if } v \in V(T'), \\ v & \text{if } v = \alpha_i \text{ for } i = 1, \ldots, m_T. \end{cases}$$

Label α_i by i, for $i = 1, \ldots, m_T$ and label all the vertices of T' by $m_T + 1$. Hence, we obtain an $(m_T + 1)$-labeled packing of T into T^k, and so $w^k(T) \geq m_T + 1$. □

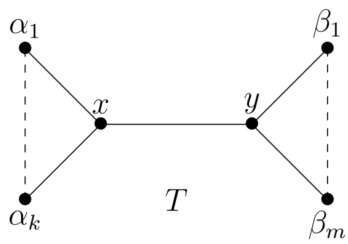

Figure 1: A non-star tree all of whose vertices are leaves except for two vertices x and y that are adjacent and $N(x) \cap N(y) = \emptyset$.

We are going to define a (T, v)-good 2-placement σ for every $v \in \{x, y, \alpha_1, \beta_1\}$:

If k is even, then consider

$$\sigma = \begin{cases} (x\,\alpha_2\,\alpha_1\,y\,\beta_1\,\ldots\,\beta_m) & \text{if } k = 2, \\ (x\,\alpha_2\,\alpha_1\,y\,\beta_1\,\ldots\,\beta_m)(\alpha_3\,\ldots\,\alpha_k) & \text{if } k > 2. \end{cases}$$

If k is odd, then consider $\sigma = (x\,\alpha_1\,y\,\beta_1\,\ldots\,\beta_m)(\alpha_2\,\ldots\,\alpha_k)$.

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