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RS-complete cycle types

Agastya Goel   Simon Rubinstein-Salzedo

Euler Circle, Mountain View
CA 94040, U.S.A.
goel.agastya@gmail.com   simon@eulercircle.com

Abstract

We characterize the class of cycle types that can achieve all Young tableau shapes (except the trivial ones with a single row or a single column) under the Robinson–Schensted (RS) correspondence, a property that we call RS-completeness. We prove that for even n, cyclic permutations comprise the only cycle type that is RS-complete. For odd n, cyclic permutations and almost cyclic permutations (which have a cycle of length $n-1$) are the only RS-complete cycle types.

1 Introduction

The Robinson–Schensted (RS) correspondence, described in [10, 11] and generalized in [5], is a way of mapping a sequence (typically, one representing a permutation) to a pair of Young tableaux. Both tableaux have the same shape, which we call the *RS shape* of the sequence. See [2] for a thorough description of the RS correspondence and its main application, to the representation theory of the symmetric group; other applications in combinatorics and group theory can be found in [6, 8, 1].

This correspondence encodes certain permutation statistics in easily visible ways. The one most directly relevant to our work is that the length of the longest increasing (respectively, decreasing) subsequence of the permutation is the same as the length of the first row (respectively, column) of its RS shape, described in [11]. Further, this can be extended to the length of the largest possible union of k increasing/decreasing subsequences, as shown in [3]. Given that some of the most interesting properties of the RS correspondence relate to the shape of the resulting tableaux, much work has been done in the past relating certain classes of permutations, such as Boolean permutations [4] and pattern-avoiding permutations [7, 13, 14, 9], to their RS shapes. In this paper, we consider classes of permutations based on their cycle type.

We start with the following natural question: *what classes of permutations can produce all possible RS shapes?* The RS shapes that consist of a single row or a single column can only be achieved using the identity permutation and the reverse permutation respectively, so we leave these two trivial cases aside. Then we define

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Class of Symbols	Objects Represented	Examples
Lowercase Greek Letters	Sequences (e.g., permutations, shapes)	σ, ϕ
Lowercase Latin Letters	Integers	i, w
Capital Fraktur Latin Letters	Young tableaux	$\mathfrak{T}$
Capital Latin Letters	Constructions of Sequences and Tableaux	B, P
Capital Greek Letters	Properties of Insertion Paths	Δ, Λ

Table 1: Common notational conventions used in this paper.

a class of permutations over $\{1, 2, \ldots, n\}$ to be *RS-complete* if permutations in this class can generate all remaining RS shapes of that size.

Involutions are easily visible under the RS correspondence: the involutions are exactly the permutations for which the two RS tableaux are the same [12]. This means that the class of involutions is RS-complete. Our focus in this paper will be on permutations with a *fixed* cycle type. Specifically, we show that cyclic permutations, which have the simple cycle type (n), are RS-complete, and for even n, this is the the only RS-complete cycle type. For odd n, there is just one more RS-complete cycle type, namely "almost cyclic" permutations with cycle type $(n-1, 1)$, i.e., those that are $(n-1)$-cycles. Thus we obtain a complete characterization of RS-complete cycle types.

In Section 2, we define the RS correspondence and RS-completeness formally and establish notation that we will use for the rest of the paper. A brief summary of notational conventions can be found in Table 1. In Section 3, we show that cyclic permutations are RS-complete. Section 4 shows that almost cyclic permutations are RS-complete for odd n. In Section 5, we show that no other fixed cycle type can be RS-complete, thus yielding a complete characterization of RS-complete cycle types.

In Section 4, we also prove an additional connection between almost cyclic permutations and the class of realizable RS shapes when n is even: that the class of realizable shapes includes everything except the two trivial shapes and the shape containing just two rows of size $\frac{n}{2}$ each. An interesting problem for future work is to characterize the class of RS shapes achievable by any given cycle type.

2 Preliminaries

Definition 2.1. A *Young diagram* is a subset of cells of a grid, with the cells in each row being left-justified (i.e., the row is filled from left to right), and the row lengths being nonincreasing from top to bottom. A *Young tableau* is obtained by taking a Young diagram and filling each cell with a positive integer. We then say a Young tableau is *semistandard* if its rows are weakly increasing from left to right and its columns are strictly increasing from top to bottom. If there are n cells in our tableau and each cell is filled with a distinct integer in $\{1, 2, \ldots, n\}$ in addition to being semistandard, we say that the tableau is a *standard Young tableau*, or *SYT*.

Note that the above definition uses the English notation for Young diagrams. In

(a) The SYT $((1,4,5),(2,6),(3))$.   (b) The corresponding shape (3, 2, 1).

Figure 1: A standard Young tableau and its shape.

this paper, we are concerned primarily with studying the shapes of Young tableaux. These shapes correspond to the shape of the underlying Young diagram.

Definition 2.2. The *shape* of a Young tableau is its list of row lengths, sorted in nonincreasing order.

Thus the shape of a Young tableau with n elements is a partition of n. We can write a Young tableau as a sequence of sequences. For example, the SYT in Figure 1a corresponds to the sequence $((1,4,5),(2,6),(3))$, and Figure 1b represents its shape. For a Young tableau $\mathfrak{T}$, let $\mathfrak{T}_r$ represent the r^{th} row of $\mathfrak{T}$, let $\mathfrak{T}_{r,c}$ represent the c^{th} element in $\mathfrak{T}_r$, and let $\mathrm{sh}(\mathfrak{T})$ represent its shape.

Definition 2.3. For a sequence γ, we let $|\gamma|$ denote the length of the sequence, and we implicitly let $\gamma_i = 0$ for $i > |\gamma|$. Then, given two sequences γ^1 and γ^2, we let $\gamma^3 = \gamma^1 + \gamma^2$ denote the sequence of length $\max(|\gamma^1|, |\gamma^2|)$ where $\gamma^3_i = \gamma^1_i + \gamma^2_i$ for $1 \leq i \leq \max(|\gamma^1|, |\gamma^2|)$.

For example, $(3,2,1) + (1,1) = (4,3,1)$.

Given any sequence σ, we use σ_i to refer to the i^{th} element, and $\sigma[\ell, r]$ to refer to the subsequence $(\sigma_\ell, \ldots, \sigma_r)$. Additionally, we use (k^n) to denote the sequence consisting of n k's.

Definition 2.4. We say that a sequence σ is a *permutation* if it contains every element in $\{1, 2, \ldots, n\}$ exactly once.

Finally, we define the *Robinson–Schensted correspondence* [10, 11, 5] (or simply the RS correspondence), which is the main topic of the paper. This is defined as two functions P and Q, which take in a sequence as input. P returns a tableau containing the elements of the sequence, and Q returns one containing the elements $\{1, 2, \ldots, n\}$, where n is the length of the sequence. We will not use Q in this paper, and we are further only interested in SYTs. Thus we will only consider the first tableau in the pair, and only the case where the integers in the sequence are distinct.[1]

Definition 2.5. The *Robinson–Schensted correspondence* denoted by P maps a sequence σ of distinct integers to a Young tableau. It can be defined recursively as

[1]We will primarily be interested in applying this correspondence to permutations; however, in the process of constructing these correspondences, we will also have to consider subsequences of permutations, hence the use of sequences of distinct integers.

follows: First, define $P(\sigma[1,1])$ to be the SYT with only one row which contains exactly one cell, filled with the element σ_1. Now, assume that $\mathfrak{T} = P(\sigma[1,c])$ has already been computed. To compute $P(\sigma[1,c+1])$, we use the displacement procedure. If the element to insert is a, and a is larger than the last element in the first row of $\mathfrak{T}$, put a at the end of the first row and terminate. Else, let b be the leftmost element greater than a in the first row. Replace b with a and insert b into the next row recursively.

Example. Consider the permutation

$$\sigma = (2,5,1,4,6,3). \tag{1}$$

Then $P(\sigma[1,4])$ is the following tableau:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline \end{array}\,.$$

When we add 6, it is the largest element in the first row, so we simply add it to the end of the first row and stop. Hence $P(\sigma[1,5])$ is

$$\begin{array}{|c|c|c|}\hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}\,.$$

When we next add 3, it displaces 4 from the first row, which displaces 5 from the second row, which in turn starts a new row, and we get the following tableau as $P(\sigma)$:

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 6 \\ \hline 2 & 4 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}\,.$$

We also use the following theorem about SYTs.

Theorem 2.6 (Schensted [11])**.** *The length of the first row of $P(\sigma)$ is equal to the longest increasing subsequence (LIS) of σ, and the length of the first column of $P(\sigma)$ is equal to the longest decreasing subsequence (LDS) of σ.*

Given the concrete connection between the LIS/LDS and the RS shape of a sequence, it is natural to ask which class of permutations can generate a given class of shapes. From Schensted's Theorem, it immediately follows that the trivial shapes (n) and (1^n) can only be generated by the identity permutation and the reverse permutation, respectively. This motivates the definition of *RS-complete* classes of permutations. Here S_n represents the n^{th} symmetric group (i.e., all permutations of $\{1,2,\ldots,n\}$).

Definition 2.7. A set of permutations $S \subseteq S_n$ is said to be *RS-complete* if any partition of n, other than the trivial partitions (n) and (1^n), is $\mathrm{sh}(P(\sigma))$ for some $\sigma \in S$.

Definition 2.8. A cycle type $\lambda = (n_1, n_2, \ldots, n_k)$, where $n_1 \geq n_2 \geq \cdots \geq n_k$ and $\sum_{i=1}^{k} n_i = n$, is said to be *RS-complete* if the set of all permutations with cycle type λ is RS-complete.

Definition 2.9. A permutation σ is said to be a *cycle* or a *cyclic permutation* if its cycle type is $(|\sigma|)$.

Definition 2.10. For a sequence σ, we call its *associated directed graph* (or just its *directed graph*) the directed graph $G = (V, E)$ where

$$V = \{1, 2, \ldots, |\sigma|\} \cup \{\sigma_1, \sigma_2, \ldots, \sigma_{|\sigma|}\}$$

and

$$E = \{(1, \sigma_1), (2, \sigma_2), \ldots, (|\sigma|, \sigma_{|\sigma|})\}.$$

For example, the directed graphs of $(6, 5, 4, 2, 1)$ and $(7, 6, 5, 3, 2, 1)$ are shown in Figure 2.

Definition 2.11. A sequence of positive integers σ of length n is said to be a *path* if there exists some permutation σ' of size n such that for all $1 \leq i < n$, $\sigma_{\sigma'_i} = \sigma'_{i+1}$, and $\sigma_{\sigma'_n} > n$. More explicitly, we say that σ is a path from σ'_1 to $\sigma_{\sigma'_n}$.

Equivalently, a sequence forms a path if its directed graph also forms a path. Thus the two sequences (and their directed graphs) shown in Figure 2 are paths.

Definition 2.12. The consecutive sequence $\sigma[\ell, r]$ is a *shifted path* if there is some permutation σ' of $\{\ell, \ldots, r\}$ such that for all $\ell \leq i < r$, $\sigma_{\sigma'_i} = \sigma'_{i+1}$. More explicitly, we say that the subsequence is a shifted path from σ'_ℓ to $\sigma_{\sigma'_r}$.

Informally, a shifted path is a part of a permutation that is a path when considered in the context of the permutation's graph.

We now introduce the idea of an insertion path.

Definition 2.13. The *insertion path* of an element x that is being inserted into a Young tableau $\mathfrak{T}$ is the sequence of positions modified when that element is inserted. Since the row indices are consecutive, we can simply write the sequence of column indices. We use $\Delta_{\mathfrak{T}}(x)$ to denote this sequence, and we drop the subscript $\mathfrak{T}$ when we are inserting the element σ_i into $P\left(\sigma[1, i-1]\right)$ and σ is clear from context. Further, we let $\Lambda_{\mathfrak{T}}(x)$ be the sequence of displaced elements, so $(\Lambda_{\mathfrak{T}}(x))_i = \mathfrak{T}_{i, \Delta(x)_i}$, for $1 \leq i < |\Delta(x)|$. Again, we will often drop the subscript.

In the example containing equation (1), the insertion path of element 6 is (3), the insertion path of element 3 is $(2, 2, 1)$, and $\Lambda(3) = (4, 5)$. Note that $\Lambda(3)$ is increasing. Insertion paths will prove useful in analyzing the properties of SYTs as they are being constructed.

We now define tail-monotone permutations, where the largest elements of the permutation are in decreasing order towards the end, with one possible exception. These sequences will be useful for incremental construction of SYTs with a desired shape.

Definition 2.14. Let n and k be positive integers. A permutation σ of length $m = n + k$ is said to be (n, k)-*tail-monotone* if the following are true:

- The subsequence $\sigma[n+1, n+k]$ is a descending sequence.
- $\sigma_{n+1} = m$.
- There is at most one $i \leq n$ such that $\sigma_i > n$.

Theorem 2.15. *Let σ be an (n, k)-tail-monotone permutation. Then*

$$\mathrm{sh}(P(\sigma)) = \mathrm{sh}(P(\sigma[1, n])) + \mathrm{sh}(P(\sigma[n+1, n+k])) = \mathrm{sh}(P(\sigma[1, n])) + (1^k).$$

That is, the shape of $P(\sigma)$ is the shape of $P(\sigma[1, n])$, with an extra cell in each of the first k rows. For example, if $\sigma = (4, 1, 2, 7, 3, 8, 6, 5)$, which is $(5, 3)$-tail-monotone, then $P(\sigma[1, 5])$ is

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 7 \\ \cline{1-2}\end{array},$$

and $P(\sigma)$ is

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 5 \\ \hline 4 & 6 & 8 \\ \cline{1-3} 7 \\ \cline{1-1}\end{array}.$$

Observe that the second tableau is not simply obtained by adding an element from the tail to each row. We will prove Theorem 2.15 using insertion paths.

Lemma 2.16. *Let σ be an (n, k)-tail-monotone permutation. For all z such that $1 < z \leq k$, we must have $|\Delta(\sigma_{n+z})| \geq z$.*

We first present a weaker version of this lemma.

Lemma 2.17. *Let σ be an (n, k)-tail-monotone permutation. For all z such that $1 < z \leq k$, we must have $|\Delta(\sigma_{n+z})| \geq |\Delta(\sigma_{n+z-1})|$, and for all $1 \leq r \leq |\Delta(\sigma_{n+z-1})|$, we must have $\Delta(\sigma_{n+z})_r \leq \Delta(\sigma_{n+z-1})_r$.*

Proof. We first prove that $\Delta(\sigma_{n+z})_r \leq \Delta(\sigma_{n+z-1})_r$ is equivalent to $\Lambda(\sigma_{n+z})_r \leq \Lambda(\sigma_{n+z-1})_r$. Consider the tableaux $\mathfrak{T}^{n+z-2} = P(\sigma[1, n+z-2]), \mathfrak{T}^{n+z-1} = P(\sigma[1, n+z-1])$. Then we may note that $\mathfrak{T}_r^{n+z-2}$ and $\mathfrak{T}_r^{n+z-1}$ only differ at the position $\Delta(\sigma_{n+z-1})_r$. All values to the right of this position in $\mathfrak{T}^{n+z-1}$ must be greater than $\Lambda(\sigma_{n+z-1})_r$, while elements at positions less than or equal to $\Delta(\sigma_{n+z-1})_r$ must be less than or equal to $\Lambda(\sigma_{n+z-1})_r$. Thus $\Delta(\sigma_{n+z})_r \leq \Delta(\sigma_{n+z-1})_r$ iff $\Lambda(\sigma_{n+z})_r \leq \Lambda(\sigma_{n+z-1})_r$.

We proceed via induction on r. Specifically, for each $1 \leq r \leq |\Delta(\sigma_{n+z-1})|$, we show that $|\Lambda(\sigma_{n+z})| \geq r$ and $\Lambda(\sigma_{n+z})_r \leq \Lambda(\sigma_{n+z-1})_r$. For our base case, consider $r = 1$. Clearly, $|\Lambda(\sigma_{n+z})| \geq 1$. Since $\sigma_{n+z} < \sigma_{n+z-1}$, σ_{n+z} displaces an element with a lower (or equal) index than σ_{n+z-1} displaces in the first row. Thus $\Lambda(\sigma_{n+z})_1 \leq \Lambda(\sigma_{n+z-1})_1$.

For the inductive step, we assume that the hypothesis is true for $1 \leq r < |\Delta(\sigma_{n+z-1})|$ and prove it for $r+1$. Since $\Lambda(\sigma_{n+z})_r \leq \Lambda(\sigma_{n+z-1})_r$, the insertion of σ_{n+z} cannot terminate at row r, and so $|\Lambda(\sigma_{n+z})| \geq r+1$. Now, $\Lambda(\sigma_{n+z})_r \leq \Lambda(\sigma_{n+z-1})_r$ is equivalent to saying the element displaced to row $r+1$ by σ_{n+z} is less than or equal to the element displaced by σ_{n+z-1}. A smaller element will displace a smaller element, so $\Lambda(\sigma_{n+z})_{r+1} \leq \Lambda(\sigma_{n+z-1})_{r+1}$ as well. □

We may now prove Lemma 2.16.

Proof. Since $\sigma_{n+1} = n+k$, we must have $|\Delta(\sigma_{n+1})| = 1$. Then, by induction, it suffices to show that $|\Delta\left(\sigma_{n+z}\right)| > |\Delta\left(\sigma_{n+z-1}\right)|$ for all $1 < z \leq k$. Let $r = |\Delta(\sigma_{n+z-1})|$. Then, by Lemma 2.17, $\Delta(\sigma_{n+z})_r \leq \Delta(\sigma_{n+z-1})_r$. Therefore, the insertion of σ_{n+z} does not terminate at row r, and $|\Delta(\sigma_{n+z})| > r = |\Delta(\sigma_{n+z-1})|$ as desired. □

Lemma 2.18. *Let σ be an (n,k)-tail-monotone permutation. For all z such that $1 \leq z \leq k$, we must have $|\Delta(\sigma_{n+z})| \leq z$.*

Proof. Consider two cases. First, assume that there does not exist an element $1 \leq p \leq n$ with $\sigma_p > n$. In this case, the only elements that may be on the insertion path of σ_{n+z} are $\sigma_{n+1}, \sigma_{n+2}, \ldots, \sigma_{n+z}$, and so $|\Delta(\sigma_{n+z})| \leq z$.

In the second case, consider the unique element $1 \leq p \leq n$ with $\sigma_p > n$. Let σ_p be in the x^{th} row of $P\left(\sigma[1,n]\right)$. Then, for $1 \leq z < x$, we may use the argument from the first case to conclude that $|\Delta(\sigma[1, n+z])| = z$. Furthermore, $\sigma_{n+1} = n+k$ must be displaced in all insertions not involving row x. Thus $\sigma_{n+1} = n+k$ is in the $(x-1)^{\text{st}}$ row of $P\left(\sigma[1, n+x-1]\right)$. Hence, σ_p cannot be displaced in the insertion of σ_{n+x}, since the element that would have to displace it is $\sigma_{n+1} = n+k$. Thus $|\Delta(\sigma[1, n+x])| = x$.

Finally, for $x < z \leq k$, we may proceed via induction. Since no element of $P\left(\sigma[1,n]\right)_z$ is initially greater than or equal to σ_{n+z}, and for all $1 \leq z' < z$, we have $|\Delta\left(\sigma_{n+z'}\right)| < z$, it must be that no element of $P\left(\sigma[1, n+z-1]\right)_z$ is greater than or equal to σ_{n+z}. Hence, no element in row z can be displaced in the insertion of σ_{n+z}, and so $|\Delta\left(\sigma_{n+z}\right)| \leq z$ as desired. □

We can now prove Theorem 2.15.

Proof of Theorem 2.15. By Lemmas 2.16 and 2.18, $|\Delta(\sigma_{n+z})| = z$ for all $1 \leq z \leq k$, meaning that our final shape is

$$\mathrm{sh}(P(\sigma[1,n])) + (1^k).$$ □

3 Cyclic Permutations

In this section, we prove the following theorem:

Theorem 3.1. *The cycle type (n) is RS-complete.*

Our proof is constructive. To construct a cyclic permutation with a given shape, we start with one of two base constructions: the first construction gives all shapes where all boxes are either in the first row or the first column (namely, hook shapes), and the second one gives all shapes with two columns which are not hook shapes. Then we apply another construction to successively add columns, until we have the desired shape.

First, let us begin with the following construction, which is the main building block for the two-column base case as well as for adding columns.

Definition 3.2. For an integer $n \geq 1$, let $L(n)$ be defined as follows:

$$L(n) = \left(n, n-1, \ldots, \left\lceil \frac{n}{2} \right\rceil + 1, \left\lceil \frac{n}{2} \right\rceil - 1, \ldots, 2, 1\right).$$

Note that the length of $L(n)$ is $n-1$ since it is missing the element $\lceil n/2 \rceil$.

For example, $L(6) = (6, 5, 4, 2, 1)$, and $L(7) = (7, 6, 5, 3, 2, 1)$.

Lemma 3.3. *Consider the directed graph $G = (V, E)$, where $V = \{1, 2, \ldots, n\}$, and $E = \{(i, L(n)_i) : 1 \leq i < n\}$. Then G forms a path from $\left\lceil \frac{n}{2} \right\rceil$ to n.*

Proof. Note that $n = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$. Partition the vertices of G into two parts, with nodes $1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ on the left, and nodes $\left\lfloor \frac{n}{2} \right\rfloor + 1, \left\lfloor \frac{n}{2} \right\rfloor + 2, \ldots, n$ on the right. Let $\ell_i = i$ for $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, and let $r_i = n + 1 - i$ for $1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$.

For all j where $1 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor$, we have that $L(n)_j = n + 1 - j$. Thus, for all $1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$, $L(n)_{\ell_i} = n + 1 - \ell_i = n + 1 - i = r_i$. Similarly, for all j where $\left\lfloor \frac{n}{2} \right\rfloor < j < n$, $L(n)_j = n - j$. Thus, for $2 \leq i \leq \left\lceil \frac{n}{2} \right\rceil$, $L(n)_{r_i} = n - r_i = i - 1 = \ell_{i-1}$.

Thus, if n is even, our graph forms a path in the order

$$\left\lceil \frac{n}{2} \right\rceil = \ell_{\frac{n}{2}}, r_{\frac{n}{2}}, \ell_{\frac{n}{2}-1}, \ldots, \ell_1, r_1 = n,$$

and if n is odd, our graph forms a path in the order

$$\left\lceil \frac{n}{2} \right\rceil = r_{\frac{n+1}{2}}, \ell_{\frac{n-1}{2}}, r_{\frac{n-1}{2}}, \ldots, \ell_1, r_1 = n.$$

In either case, our graph forms a path from $\left\lceil \frac{n}{2} \right\rceil$ to n. □

For example, the directed graphs corresponding to $L(6)$ and $L(7)$ (as defined in Lemma 3.3) are shown in Figure 2.

We also need to introduce a key operation on sequences.

Definition 3.4. Given a sequence σ, let $I(\sigma)$ be the sequence such that $|I(\sigma)| = |\sigma|$ and for all $1 \leq i \leq |\sigma|$, $I(\sigma)_i = \sigma_i + 1$. Define $I^k(\sigma)$ to denote σ for $k = 0$ and $I\left(I^{k-1}(\sigma)\right)$ for $k > 0$, where k is an integer.

In the following constructions, we use $\oplus$ to denote the concatenation operator on sequences. For example, $(2, 4) \oplus (1, 3) = (2, 4, 1, 3)$.

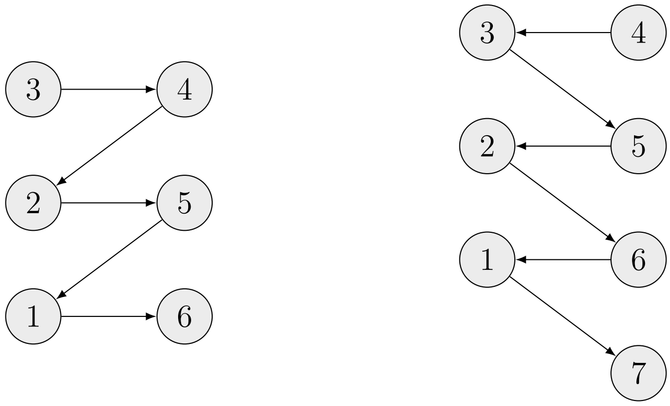

Figure 2: The directed graphs corresponding to $L(6) = (6, 5, 4, 2, 1)$ and $L(7) = (7, 6, 5, 3, 2, 1)$, on the left and right respectively.

Definition 3.5. For two integers $1 < m < n$, define $B(m, n)$ to be the following concatenation of sequences:

$$B(m,n) = (2, 3, \dots, m-1) \oplus \left(\left\lceil \frac{n+m-1}{2} \right\rceil \right) \oplus I^{m-1}\left(L(n-m+1)\right) \oplus (1).$$

Lemma 3.6. *For any two integers $1 < m < n$, the sequence $B(m, n)$ is a permutation of length n.*

Proof. We wish to show that every integer from 1 to n appears exactly once in $B(m, n)$. Let us reorder $B(m, n)$ into four monotone sequences for clarity: (1), $(2, 3, \dots, m-1)$, $I^{m-1}(L(n-m+1))$, and $\left(\left\lceil \frac{n+m-1}{2} \right\rceil\right)$. Clearly, the first two sequences give us the elements $\{1, 2, \dots, m-1\}$. By definition, $L(n-m+1)$ contains the elements $\{1, 2, \dots, \left\lceil \frac{n-m-1}{2} \right\rceil, \left\lceil \frac{n-m+3}{2} \right\rceil, \dots, n-m, n-m+1\}$. Hence, $I^{m-1}(L(n-m+1))$ contains the elements

$$\begin{aligned} &\left\{ m, m+1, \dots, \left\lceil \frac{n+m-3}{2} \right\rceil, \left\lceil \frac{n+m+1}{2} \right\rceil, \dots, n-1, n \right\} \\ &\quad = \{m, m+1, \dots, n\} \setminus \left\{ \left\lceil \frac{n+m-1}{2} \right\rceil \right\}. \end{aligned}$$

Thus our final two sets taken together yield each value in $\{m, m+1, \dots, n\}$, showing that $B(m, n)$ is a permutation. ☐

Lemma 3.7. *The permutation $B(m, n)$ is cyclic, and* $\mathrm{sh}\left(P\left(B(m,n)\right)\right) = (m, 1^{n-m})$.

Proof. First we prove cyclicity. Clearly, $B(m, n)[1, m-1]$ forms a path from 1 to $\left\lceil \frac{n+m-1}{2} \right\rceil$. Then note that positions m through $n-1$ of $B(m, n)$ are the elements of $L(n-m+1)$ with both indices and values increased by $m-1$. Hence, we may apply Lemma 3.3 to conclude that $B(m, n)[m, n-1]$ forms a shifted path from $\left\lceil \frac{n+m-1}{2} \right\rceil$ to n. Additionally, $B(m, n)[n]$ is an edge from n to 1. Since, by Lemma 3.6, $B(m, n)$

is a permutation, the union of the paths corresponding to $B(m,n)[1,m-1]$ (from 1 to $\lceil\frac{n+m-1}{2}\rceil$), $B(m,n)[m,n-1]$ (from $\lceil\frac{n+m-1}{2}\rceil$ to n), and $B(m,n)[n]$ (an edge from n to 1) is a cycle, and the permutation is cyclic.

To deduce its shape, we can use Schensted's Theorem. Note that $B(m,n)$ is unimodal (i.e., it is increasing up to some point and decreasing after that point). Further note that the maximum element occurs at position m. Hence, the LIS is $B(m,n)[1,m]$ with length m, and the LDS is $B(m,n)[m,n]$ with length $n-m+1$. Applying Schensted's theorem, the shape of the corresponding tableau must be $\mathrm{sh}\,(B(m,n)) = (m, 1^{n-m})$, as desired. $\square$

For example, when $(m,n)=(5,7)$, we have $B(m,n)=(2,3,4,6,7,5,1)$. This permutation corresponds to the SYT

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 3 & 4 & 5 & 7 \\ \hline 2 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array},$$

which has the desired shape.

We also require use of the following construction.

Definition 3.8. For integers m,n such that $1 < m \leq \frac{n}{2}$, define $B'(m,n)$ to be the following concatenation of sequences:

$$B'(m,n) = \left(\left\lceil\frac{n+m}{2}\right\rceil\right) \oplus I\left(L\left(m-1\right)\right) \oplus (1) \oplus I^m\left(L\left(n-m\right)\right) \oplus \left(\left\lceil\frac{m+1}{2}\right\rceil\right).$$

Lemma 3.9. *For any two integers m,n such that $1 < m \leq \frac{n}{2}$, the sequence $B'(m,n)$ is a permutation of length n.*

Proof. Once again, consider $B'(m,n)$ in terms of the sets of elements that it contains: (1), $I\left(L(m-1)\right)$, $\left(\lceil\frac{m+1}{2}\rceil\right)$, $I^m\left(L(n-m)\right)$, and $\left(\lceil\frac{n+m}{2}\rceil\right)$. Writing these as sets, we obtain the following set of elements for the union of the first three:

$$\begin{aligned}&\{1\} \cup \left\{2,3,\ldots,\left\lceil\frac{m+1}{2}\right\rceil - 1, \left\lceil\frac{m+1}{2}\right\rceil + 1, \ldots, m-1, m\right\} \cup \left\{\left\lceil\frac{m+1}{2}\right\rceil\right\}\\ &\quad = \{1,2,\ldots,m\}.\end{aligned}$$

Following a similar process for our last two sequences, we obtain the set

$$\begin{aligned}&\left\{m+1, m+2, \ldots, \left\lceil\frac{n+m}{2}\right\rceil - 1, \left\lceil\frac{n+m}{2}\right\rceil + 1, \ldots, n-1, n\right\} \cup \left\{\left\lceil\frac{n+m}{2}\right\rceil\right\}\\ &\quad = \{m+1, m+2, \ldots, n\}.\end{aligned}$$

Thus, overall, we have the set $\{1,2,\ldots,n\}$, as desired. $\square$

Lemma 3.10. *The permutation $B'(m,n)$ is cyclic, and*

$$\mathrm{sh}\,(P\,(B'(m,n))) = (2^m, 1^{n-2m}).$$

Proof. First we prove cyclicity. We can prove this the same way as in the proof of Lemma 3.7. Since $B'(m,n)[2,m-1] = I\,(L(m-1))$ and $B'(m,n)[m+1,n-1] = I^m\,(L(n-m))$, by Lemma 3.3, these two subsequences form shifted paths, from $\lceil \frac{m+1}{2} \rceil$ to m and from $\lceil \frac{n+m}{2} \rceil$ to n respectively. Thus $B'(m,n)[1,m]$ and $B'(m,n)[m+1,n]$ are shifted paths from $\lceil \frac{m+1}{2} \rceil$ to $\lceil \frac{n+m}{2} \rceil$ and from $\lceil \frac{n+m}{2} \rceil$ to $\lceil \frac{m+1}{2} \rceil$ respectively, so $B'(m,n)$ is cyclic, as desired.

To determine its shape, consider the LIS and LDS of $B'(m,n)$. Since $B'(m,n)$ is a concatenation of two descending sequences, it has an LIS of 2. The LDS can be computed by splitting $B'(m,n)$ into four blocks: $\left(\lceil \frac{n+m}{2} \rceil\right)$, $I\,(L\,(m-1)) \oplus (1)$, $I^m\,(L\,(n-m))$, and $\left(\lceil \frac{m+1}{2} \rceil\right)$. The second and third blocks form descending sequences, but all elements in the second block are less than all elements in the third block. Hence, a decreasing sequence can only contain elements from at most one of these two blocks. If we do not use the third block, we obtain a descending sequence of length m, while not using the second block yields a descending sequence of length $n-m$. Since $m \leq \frac{n}{2}$, our LDS is of length $n-m$. Hence by Schensted's theorem, our shape is $(2^m, 1^{n-2m})$ as claimed. □

Example. When $(m,n) = (3,7)$, $B'(m,n) = (5,3,1,7,6,4,2)$. This permutation corresponds to the SYT

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline 7 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}\,,$$

which has the desired shape.

Together, $B(m,n)$ and $B'(m,n)$ establish useful base cases which will help prove the main theorem. Now, consider the following construction.

Definition 3.11. Given a cyclic permutation σ of length n and an an integer $k > 1$, we define $A(\sigma,k)$ to be the sequence of length $n+k$ obtained by modifying σ as follows: first, replace n in σ with $n + \lceil \frac{k}{2} \rceil$, and then append $I^n\,(L(k)) \oplus (n)$.

Lemma 3.12. *For a cyclic permutation σ and an integer $k > 1$, $A(\sigma,k)$ is a permutation.*

Proof. $A(\sigma,k)[1,n]$ contains the elements $\{1,2,\dots,n-1\} \cup \{n + \lceil \frac{k}{2} \rceil\}$ and $A(\sigma,k)[n+1,n+k]$ contains $\{n,n+1,\dots,n+k\} \setminus \{n+\lceil \frac{k}{2} \rceil\}$. Hence, the elements in $A(\sigma,k)$ are

$$\{1,2,\dots,n-1\} \cup \left\{n + \left\lceil \frac{k}{2} \right\rceil\right\} \cup \{n,n+1,\dots,n+k\} \setminus \left\{n + \left\lceil \frac{k}{2} \right\rceil\right\} = \{1,2,\dots,n+k\}.$$

Thus $A(\sigma,k)$ is a permutation. □

Lemma 3.13. *The permutation $A(\sigma,k)$ is cyclic, and*

$$\mathrm{sh}\,(P\,(A(\sigma,k)) = \mathrm{sh}\,(P\,(\sigma)) + (1^k).$$

Proof. In terms of cyclicity, we see that $A(\sigma, k)$ is a union of three shifted paths, one from n to $n + \lceil \frac{k}{2} \rceil$, one from $n + \lceil \frac{k}{2} \rceil$ to $n + k$, and one from $n + k$ to n (a single edge). Thus $A(\sigma, k)$ is cyclic, as desired. Further, $A(\sigma, k)$ is (n, k)-tail-monotone and therefore Theorem 2.15 immediately implies $\mathrm{sh}\left(P\left(A(\sigma, k)\right)\right) = \mathrm{sh}\left(P(\sigma)\right) + (1^k)$. □

Example. Consider the permutation $\sigma = (4, 1, 5, 3, 2)$, which is $B'(2, 5)$. Then $P(\sigma)$ is the SYT

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1} \end{array} .$$

Further, $A(\sigma, 3) = (4, 1, 7, 3, 2, 8, 6, 5)$. Then $P(A(\sigma, 3))$ is the SYT

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 6 & 8 \\ \hline 4 & 7 \\ \cline{1-2} \end{array} ,$$

which has the shape specified in Lemma 3.13.

We are now ready to prove our main result, Theorem 3.1.

Proof of Theorem 3.1. Consider a shape γ with at least two rows and at least two columns. Now, we proceed via induction, assuming that for all nontrivial shapes $\gamma' \neq \gamma$ with $\gamma_i \geq \gamma'_i$ for all i, we can construct a cyclic permutation with shape γ'.

Case 1 $\gamma_2 = 1$. In this case, since $\gamma_1 > 1$, we are done using the B construction.

Case 2 $\gamma_1 = \gamma_2 = 2$. In this case, we are done using the B' construction.

Case 3 $\gamma_1 > 2, \gamma_2 > 1$. In this case, consider the shape γ' such that $\gamma'_i = \gamma_i - 1$ for $1 \leq i \leq |\gamma|$. Then, $\gamma'_1 > 1, \gamma'_2 > 0, \gamma' \neq \gamma$, and for all i, $\gamma_i \geq \gamma'_i$. Thus, by inductive hypothesis, there exists some cyclic permutation σ' with $\mathrm{sh}\left(P(\sigma')\right) = \gamma'$. We conclude by letting $\sigma = A(\sigma', |\gamma|)$, and noting that $\mathrm{sh}\left(P(\sigma)\right) = \gamma' + (1^{|\gamma|}) = \gamma$ as desired. □

4 Almost Cyclic Permutations

In this section we show that the cycle type $(n-1, 1)$ is RS-complete for odd n, and that this cycle type can achieve all but one nontrivial RS shapes for even n.

Definition 4.1. A permutation is defined to be *almost cyclic* if its cycle type is $(n-1, 1)$.

Theorem 4.2. *When n is odd, all RS shapes apart from $(1, 1, \ldots, 1)$ and (n) can be achieved using only almost cyclic permutations. When n is even, all RS shapes apart from $(1, 1, \ldots, 1)$, (n), and $\left(\frac{n}{2}, \frac{n}{2}\right)$ can be achieved using only almost cyclic permutations.*

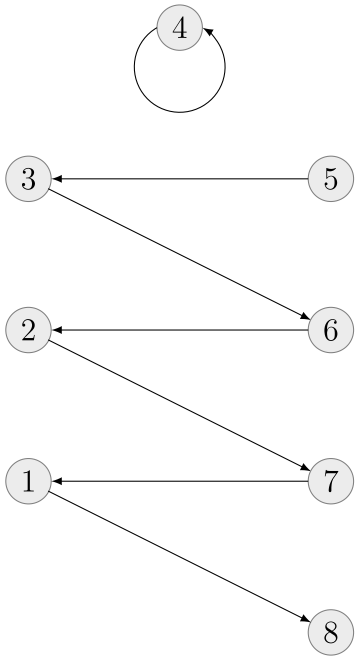

Figure 3: The directed graph corresponding to the sequence $(8) \oplus L'(7) = (8, 7, 6, 4, 3, 2, 1)$.

Remark 4.3. When n is even, the special case where $\gamma = (\frac{n}{2}, \frac{n}{2})$ and $n > 2$ is, in fact, unachievable using any permutation with cycle type containing a 1 (and thus unachievable using almost cyclic permutations). However, this claim is reserved for Lemma 5.5 and is not part of Theorem 4.2.

The proof of Theorem 4.2 follows a path similar to the proof of Theorem 3.1. We will introduce several new base cases and continue to use construction A. First, we need the following variation on $L(n)$.

Definition 4.4. Given any integer $n > 1$, let p be 0 when n is even and 1 when n is odd, and let w denote $\lfloor \frac{n}{2} \rfloor + 2p$. We define $L'(n)$ to be the sequence

$$L'(n) = (n, n-1, \ldots, w+1, w-1, \ldots, 2, 1).$$

Observe that this sequence is missing the element w, hence is of length $n - 1$.

Lemma 4.5. *Let n, t be two positive integers such that $n > 1$ and t is not in $L'(n)$. Let σ denote the sequence $(t) \oplus L'(n)$. Consider the directed graph $G = (V, E)$, where $V = \{1, 2, \ldots, n, t\}$ and $E = \{(i, \sigma_i) : 1 \leq i \leq n\}$. Then G consists of a loop (a cycle of size 1) at $\lfloor \frac{n}{2} \rfloor + 1$ and a path from w to t.*

Note that if $t = w$ as defined in Definition 4.4, we may identify the node corresponding to t with the node corresponding to w, creating a cycle.

Proof of Lemma 4.5. We separate the proof of Lemma 4.5 into two cases.

Case 1 n is even. In this case, $w = \frac{n}{2}$, as defined in Definition 4.4. Then $\sigma_{w+1} = n+2-(w+1) = w+1$. Hence, $w+1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$ forms a loop. With this in mind, partition the remaining nodes in G into two parts, with nodes 1 through w on the left and nodes $w+2$ through n, as well as t, on the right. For $1 < j \leq w$, $\sigma_j = n+2-j$, and for $w+2 \leq j \leq n$, $\sigma_j = n+1-j$. For $1 \leq i < w$, let $\ell_i = w+1-i$ and $r_i = w+1+i$. Further, let $\ell_w = 1$ and $r_w = t$. Then, for $1 \leq i < w$,

$$\sigma_{\ell_i} = n+2-\ell_i = n+2-(w+1-i) = w+1+i = r_i,$$

and

$$\sigma_{r_i} = n+1-r_i = n+1-(w+1+i) = w-i = \ell_{i+1}.$$

Additionally, note that $\sigma_{\ell_w} = \sigma_1 = t = r_w$. Therefore, the remaining nodes form a path in the order $w = \ell_1, r_1, \ell_2, \ldots, \ell_w, r_w = t$.

Case 2 n is odd. In this case, $w = \frac{n+3}{2}$, as defined in Definition 4.4. Then $\sigma_{w-1} = n+1-(w-1) = w-1$. Hence, $w-1 = \left\lfloor \frac{n}{2} \right\rfloor + 1$ forms a loop. With this in mind, partition the remaining nodes of G into two parts, with nodes 1 through $w-2$ on the left and nodes w through n, as well as t, on the right. For $1 \leq i \leq w-2$, let $\ell_i = w-1-i$ and $r_i = w-1+i$. Further, let $r_{w-1} = t$. Then, for $1 < j \leq w-2$, $\sigma_j = n+2-j$, and for $w \leq j \leq n$, $\sigma_j = n+1-j$. Then, for $1 \leq i \leq w-3$,

$$\sigma_{\ell_i} = n+2-\ell_i = n+2-(w-1-i) = w+i = r_{i+1},$$

and, for $1 \leq i \leq w-2$,

$$\sigma_{r_i} = n+1-r_i = n+1-(w-1+i) = w-1-i = \ell_i.$$

Additionally, note that $\sigma_{\ell_{w-2}} = \sigma_1 = t = r_{w-1}$. Therefore, the remaining nodes form a path in the order $w = r_1, \ell_1, r_2, \ldots, \ell_{w-2}, r_{w-1} = t$. □

For example, $(8) \oplus L'(7) = (8,7,6,4,3,2,1)$, which corresponds to the directed graph shown in Figure 3.

Definition 4.6. Given positive integers $(m,n) \neq (2,4)$ such that $1 < m \leq \frac{n}{2}$, let p be 1 if m is odd and 0 otherwise, and let w denote $\left\lfloor \frac{m}{2} \right\rfloor + 2p$. Then define $D(m,n)$ to be the following concatenation of sequences:

$$D(m,n) = \left(\left\lceil \frac{n+m}{2} \right\rceil \right) \oplus L'(m) \oplus I^m \left(L(n-m) \right) \oplus (w).$$

Lemma 4.7. *For $1 < m \leq \frac{n}{2}$, $D(m,n)$ is a permutation.*

Proof. Take w and p to have values as defined in Definition 4.6. Then we once again consider sets of elements. We have the sets $\{\lceil \frac{n+m}{2} \rceil\}$, $\{1,2,\ldots,w-1,w+1,\ldots,m-1,m\}$, $\{m+1,m+2,\ldots,\lceil \frac{n+m}{2} \rceil - 1, \lceil \frac{n+m}{2} \rceil + 1, \ldots, n-1, n\}$, and $\{w\}$. The union of the second and the fourth sets is $\{1,2,\ldots,m\}$ (since $1 \leq w \leq m$ for $m > 1$), and the union of the first and third sets is $\{m+1,m+2,\ldots,n\}$. Thus, taken together, the union of all of our sets is $\{1,2,\ldots,n\}$. □

Lemma 4.8. *The permutation $D(m,n)$ is almost cyclic with the loop not at position n, and*

$$\mathrm{sh}\left(P\left(D(m,n)\right)\right) = (2^m, 1^{n-2m}).$$

Proof. First consider cyclicity. Since $D(m,n)[1,m] = \left(\left\lceil \frac{n+m}{2} \right\rceil\right) \oplus L'(m)$, it forms a path from w (as defined in Definition 4.6) to $t = \left\lceil \frac{n+m}{2} \right\rceil$ and a loop at $\left\lfloor \frac{m}{2} \right\rfloor + 1$ by Lemma 4.5. Similarly, $D(m,n)[m+1,n-1] = I^m\left(L(n-m)\right)$, and forms a shifted path from $\left\lceil \frac{n+m}{2} \right\rceil$ to n by Lemma 3.3. Since $D(m,n)[n]$ is an edge from n to w, $D(m,n)$ forms a cycle and a loop, giving the desired cycle type of $(n-1,1)$. By Lemma 4.5, the loop is at position $\left\lfloor \frac{n}{2} \right\rfloor + 1 \neq n$.

We can, once again, use the LIS and LDS of $D(m,n)$ to determine its shape. Since $D(m,n)$ is composed of two descending sequences, it has an LIS of 2. To determine its LDS, we split it into four blocks: $\left(\left\lceil \frac{n+m}{2} \right\rceil\right)$, $L'(m)$, $I^m\left(L(n-m)\right)$, and (w). Once again, the second and third blocks form descending sequences, but all elements of the second block are less than all elements of the third block. Hence, a decreasing sequence can only contain elements from at most one of these two blocks. If we do not use the second block, we get a descending sequence of length $n-m$.

Now, assume that we do not use the third block. If $m > 2$, then $w > 1$ and hence the second block ends with a 1. In this case, using the second block yields an LDS of length m, which is not longer than $n-m$. However, if $m = 2$, then we get an LDS of length $m+1$; but since $n > 4$ when $m = 2$, the third block still provides an LDS of length $n - m \geq m+1$.

In all cases, our LDS is of length $n-m$. Then, by Schensted's theorem, we have two columns and $n-m$ rows. The only valid shape is then $(2^m, 1^{n-2m})$ as claimed. □

Example. If $(m,n) = (3,7)$, $D(m,n) = (5,2,1,7,6,4,3)$. This sequence corresponds to the following tableau:

1	3
2	4
5	6
7	

which has the shape specified in Lemma 4.8.

Definition 4.9. Given a positive integer $n > 3$, define $D'(n)$ to be the following concatenation of sequences:

$$D'(n) = (w) \oplus L'(n)$$

where w is defined as in Definition 4.4.

Lemma 4.10. *For all $n > 3$, $D'(n)$ is a permutation.*

Proof. The set of elements contained in $L'(n)$ is $\{1,2,\ldots,n\} \setminus \{w\}$, hence the set of elements contained in $D'(n)$ is $\{1,2,\ldots,n\}$. Since $L'(n)$ is of length $n-1$, the sequence $D'(n)$ forms a permutation. □

Lemma 4.11. *The permutation $D'(n)$ is almost cyclic with the loop not at position n, and* $\mathrm{sh}\,(P\,(D'(n))) = (2, 1^{n-2})$.

Proof. We can see that $D'(n)$ is almost cyclic by Lemma 4.5. Since $n > 3$, we have $D'(n)_2 = n > D'(n)_1$, and so $D'(n)$ is $(1, n-1)$-tail-monotone. Then the shape follows from Theorem 2.15. By Lemma 4.5, the loop is at position $\lfloor \frac{n}{2} \rfloor + 1 \neq n$. $\square$

Lemma 4.12. *If σ is an almost cyclic permutation with $\sigma_n \neq n$, and $k > 1$ is an integer, then $A(\sigma, k)$ is almost cyclic with shape* $\mathrm{sh}\,(P(\sigma)) + (1^k)$.

Proof. In terms of cyclicity, we see that $A(\sigma, k)$ is a union of a fixed point and three shifted paths: one from n to $n+\lceil \frac{k}{2} \rceil$, one from $n+\lceil \frac{k}{2} \rceil$ to $n+k$, and one from $n+k$ to n (a single edge). Thus $A(\sigma, k)$ is almost cyclic, as desired. Further, $A(\sigma, k)$ is (n, k)-tail-monotone and therefore Theorem 2.15 immediately implies $\mathrm{sh}\,(P\,(A(\sigma, k))) = \mathrm{sh}\,(P(\sigma)) + (1^k)$. $\square$

We can now prove Theorem 4.2.

Proof of Theorem 4.2. Let us consider any shape γ with at least two rows, at least two columns, and not of the form $(\frac{n}{2}, \frac{n}{2})$.

Case 1 $\gamma = (2, 1^{n-2})$. If $n = 3$, the permutation $\sigma = (1, 3, 2)$ has the desired shape and cycle type, while if $n > 3$, we may use $\sigma = D'(n)$.

Case 2 $\gamma_1 > \gamma_2$, and $\gamma_1 > 2$. In this case, we can consider the sequence γ' where $\gamma'_1 = \gamma_1 - 1$, and $\gamma'_i = \gamma_i$ for $i > 1$. This sequence clearly represents a shape with at least two columns and at least two rows, and thus by Theorem 3.1, we can create a cyclic permutation σ with the shape γ'. Then the almost cyclic permutation $\sigma \oplus (|\sigma| + 1)$ has shape $\gamma' + (1)$, which is γ.

For the remaining cases, in which $\gamma_1 = \gamma_2 \geq 2, \gamma_3 > 0$, we prove a slightly stronger version of Theorem 4.2, in which we show that there is an almost cyclic permutation σ with shape γ such that $\sigma_{|\sigma|} \neq |\sigma|$ (i.e., σ does not have a loop at position n). We now proceed via induction, assuming that for all nontrivial shapes $\gamma' \neq \gamma$ with $\gamma'_1 = \gamma'_2 \geq 2, \gamma'_3 > 0$, and $\gamma_i \geq \gamma'_i$ for all i, we can construct an almost cyclic permutation σ' with shape γ', such that σ' does not have a loop at position $|\sigma'|$.

Case 3 $\gamma_1 = \gamma_2 = 2, \gamma_3 > 0$. In this case, we may use D to obtain an almost cyclic permutation with the desired shape.

Case 4 $\gamma_1 = \gamma_2 > 2, \gamma_3 > 0$. Let q be the largest position with $\gamma_1 = \gamma_q$. Then let γ' be the shape with $\gamma'_i = \gamma_i - 1$ for $i \leq q$ and $\gamma'_i = \gamma_i$ for $i > q$. γ' is nonincreasing and represents a shape with at least two rows and columns since $\gamma'_2 \geq 2$. Furthermore, we have $\gamma'_1 = \gamma'_2 = \gamma_1 - 1 \geq 2$ and $\gamma'_3 > 0$, since either $q = 2$ (and therefore $\gamma'_3 = \gamma_3 > 0$), or $q > 2$ (and therefore $\gamma'_3 = \gamma'_1 \geq 2$). Hence, by the inductive hypothesis, there exists an almost cyclic permutation σ' with shape γ' and loop not at position $|\sigma'|$. Then, by Lemma 4.12, we are finished by setting $\sigma = A(\sigma', q)$. $\square$

5 The Converse

In this section, we complete the proof of our main theorem:

Theorem 5.1. *For even n, the only RS-complete cycle type is (n). For odd n, only the cycle types (n) and $(n-1,1)$ are RS-complete.*

We have already proven, in Sections 3 and 4, that the cycle type (n) is RS-complete for all n and the cycle type $(n-1,1)$ is RS-complete for odd n. In this section, we show that no other cycle type is RS-complete.

Let us start by describing the cycle types of permutations that can achieve the shape $(n-1,1)$.

Lemma 5.2. *For $n \geq 2$, all permutations that can achieve the shape $(n-1,1)$ have cycle types of the form $(k, 1^{n-k})$, for some $1 < k \leq n$.*

Proof. Consider any permutation σ with the desired shape. Then, by Schensted's Theorem, the LIS of σ has length $n-1$. Thus all but one element are in the same relative order as the identity permutation. Hence, we form σ by performing a cyclic shift on some subrange $[i,j]$ of the identity permutation, where $1 \leq i < j \leq n$. Here, we either move element i to position j and move elements $i+1,\ldots,j$ to one position lower, or we move element j to position i and move elements $i,\ldots,j-1$ to one position higher. Then the subrange $[i,j]$ forms a cycle while the rest of the elements are fixed points, so the resulting cycle type would be $(j-i+1, 1^{n+i-j-1})$. □

Lemma 5.2 significantly restricts the structure of any RS-complete cycle type. We now show that this restricted class does not contain any RS-complete cycle types other than the ones that we have already identified.

Lemma 5.3. *Given a permutation $\sigma \in S_n$ such that $|\mathrm{sh}(P(\sigma))| = 2$ (i.e., the RS shape of σ has two rows) and an integer $1 \leq i \leq n$ such that $\sigma_i = i$ (i.e., i is a fixed point of the permutation), we must have $\sigma_j < i$ for all $1 \leq j < i$.*

Proof. Let σ_j be the largest element for $1 \leq j < i$. By the pigeonhole principle, if $\sigma_j > i$, there is some other j' such that $i < j' \leq n$ and $\sigma_{j'} < i$. This would then create a descending subsequence (j, i, j') of length 3. Then, by Schensted's Theorem, $|\mathrm{sh}(P(\sigma))| \geq 3$, which is a contradiction. □

Definition 5.4. Given a positive integer f and a sequence σ that does not contain f, define $R(\sigma, f)$ to be the sequence σ' of the same size as σ constructed as follows:

$$\sigma'_i = \begin{cases} \sigma_i & \text{if } \sigma_i < f \\ \sigma_i - 1 & \text{if } \sigma_i > f. \end{cases}$$

Lemma 5.5. *Any permutation σ with $|\mathrm{sh}(P(\sigma))| = 2$ whose cycle type contains at least r 1's must have $\mathrm{sh}(P(\sigma))_1 \geq \mathrm{sh}(P(\sigma))_2 + r$.*

Proof. Let us proceed by induction. Let our base case be $r = 0$. Clearly the result holds in this case. Now, for $r > 0$, assume the result holds for $r - 1$. Then consider any i for which $\sigma_i = i$. Then, by Lemma 5.3, for all $1 \leq j \leq n$, if $j < i$, $\sigma_j < \sigma_i$, and if $j > i$, $\sigma_j > i$. Let $\sigma' = R(\sigma[1, i-1] \oplus \sigma[i+1, n], i)$. By our inductive hypothesis, $\operatorname{sh}(P(\sigma))_1 \geq \operatorname{sh}(P(\sigma))_2 + r - 1$. Now, consider the LIS (denoted by ϕ) of σ that does not contain i. Note that this has the same length as the LIS of σ'. Then, since $\sigma_j < i$ for $j < i$ and $\sigma_j > i$ for $j > i$, if we insert σ_i into ϕ, the resulting sequence will still be increasing. Hence, the length of the LIS of σ is one greater than the length of the LIS of σ', and so, by Schensted's theorem, $\operatorname{sh}(P(\sigma))_1 = \operatorname{sh}(P(\sigma'))_1 + 1 \geq \operatorname{sh}(P(\sigma'))_2 + r = \operatorname{sh}(P(\sigma))_2 + r$, completing the induction. □

Finally, we can prove Theorem 5.1.

Proof of Theorem 5.1. We separate the proof of Theorem 5.1 into two cases.

Case 1 n is even. In this case, by Lemma 5.2, all permutations, apart from cyclic ones, that have a shape of $(n-1, 1)$ have a cycle type containing at least one fixed point. Thus, by Lemma 5.5, such a cycle type cannot have a permutation with shape $(\frac{n}{2}, \frac{n}{2})$, as desired.

Case 2 n is odd. In this case, by Lemma 5.2, all permutations, apart from cyclic and almost cyclic ones, that have a shape of $(n-1, 1)$ have a cycle type containing at least two fixed points. Thus, by Lemma 5.5, such a cycle type cannot have a permutation with shape $(\frac{n+1}{2}, \frac{n-1}{2})$. □

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