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Root distributions in Moebius–Kantor complexes

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Abstract

We study the distribution of roots of rank 2 in nonpositively curved 2-complexes with Moebius–Kantor links. For every face in such a complex, the parity of the number of roots of rank 2 in a neighbourhood of the face is a well-defined geometric invariant determined by the root distribution. We study the relation between the root distribution and the parity distribution. We prove that there exist parity distributions in flats which are disallowed in Moebius–Kantor complexes. This contrasts with the fact that every root distribution can be realized. We classify the root distributions associated with an even parity distribution (i.e., such that every face is even) on a flat plane. We prove that there exists up to isomorphism a unique even simply connected Moebius–Kantor complex — namely, the Pauli complex.

1 Introduction

Let T denote the equilateral triangle lattice in the Euclidean plane $\mathbb{R}^2$. We shall call the choice for every vertex in the lattice of one of the three simplicial directions in T a *root distribution* on T.

Graphically, one may represent a root distribution by a field of segments of length 2ε (for some small fixed $\varepsilon > 0$) which takes a vertex x to the segment $(x - \varepsilon, x + \varepsilon)$ containing x in the direction specified at x. Figure 1 shows an example of a root distribution in a simplicial ball of radius 3.

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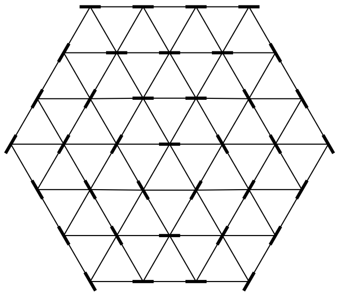

Figure 1

The classification of root distributions subject to local constraints is a combinatorial problem similar to certain problems arising from physics (see §1.6). In the present paper, we study a connection between root distributions and a class of discrete groups acting on nonpositively curved spaces called the Moebius–Kantor complexes.

Definition 1.1. A *Moebius–Kantor complex* is a 2-complex with triangle faces whose links are isomorphic to the Moebius–Kantor graph.

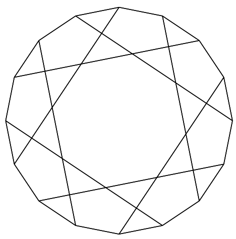

Figure 2

The Moebius–Kantor graph is shown in Figure 2. It is the unique cubic symmetric graph with 16 vertices.

Every Moebius–Kantor complex can be viewed as a nonpositively curved 2-complex, in which every face is isometric to an equilateral triangle with sides of length 1 (by the Gromov link condition, see [7, II.5.1]).

In order to state our main results, we shall begin by reviewing a few definitions which are directly relevant to the study of Moebius–Kantor complexes.

1.1 Roots

Let Δ be a nonpositively curved 2-complex, $x \in \Delta$ be a vertex, and L_x the link at x, endowed with the angular metric. We call a *root* at x an isometric embedding $\alpha\colon [0,\pi] \hookrightarrow L_x$ such that $\alpha(0)$ is a link vertex of degree greater than 2. Every root has a *rank* $\mathrm{rk}(\alpha)$, which is a rational number in $[1,2]$ defined by

$$\mathrm{rk}(\alpha) := 1 + \frac{N(\alpha)}{q_\alpha}$$

where

$$N(\alpha) := |\{\beta \in \Phi_x \mid \alpha \neq \beta, \alpha(0) = \beta(0), \alpha(\pi) = \beta(\pi)\}|,$$

Φ_x denotes the set of roots at x and, for a root α, q_α denotes the degree of $\alpha(0)$ minus 1:

$$q_\alpha := \mathrm{val}(\alpha(0)) - 1.$$

We refer to [2, §4] for more details on these definitions. In a Moebius–Kantor complex, the rank of a root can be either $\frac{3}{2}$ or 2. The reader can verify that both types of roots can be found in the obvious Hamiltonian cycle of length 16 shown in the above drawing of the Moebius–Kantor graph (see Figure 2). Furthemore, this graph being 2-transitive, it is obvious that there can be only two distinct orbits of roots.

1.2 Parity in Moebius–Kantor complexes

Let Δ be a Moebius–Kantor complex. Let f be a face in Δ. The parity of f is defined as follows (cf. [4, §2]). Let $\tilde{f}$ denote the equilateral triangle formed by the union of f and three faces in Δ corresponding to a choice of a face not equal to f adjacent to every side of f. We call $\tilde{f}$ a *large triangle* containing f. For every vertex x of f, the triangle $\tilde{f}$ determines a root α_x at x in Δ.

Definition 1.2. We call the *parity* of f the parity of the number of roots α_x which are of rank 2, when x runs over the three vertices of the face f.

This is a well-defined invariant, i.e., the parity of f does not depend on the choice of the large triangle $\tilde{f}$ containing f, by [4, Lemma 2.1].

1.3 Root distributions

Let Π be a flat plane in a simply connected Moebius–Kantor complex Δ, i.e., the image of an isometric embedding $\mathbb{R}^2 \to \Delta$, where $\mathbb{R}^2$ is endowed with the Euclidean metric. Every flat is tessellated by equilateral triangles with sides of length 1.

Let $x \in \Pi$ be a vertex. Let $\ell \subset \Pi$ be a simplicial line through x. Every root $\alpha \in \Phi_x$ whose image is included in Π and whose extremities belong to ℓ is has the same rank. We call ℓ a simplicial direction at x, and the common value $\mathrm{rk}(\alpha)$ the rank of ℓ. Observe that precisely two of the three simplicial directions at x in Π are of rank 2 (a proof of this observation can be found in [5, §4.2, Prop. 41]).

Definition 1.3. The *root distribution of* Π is the map δ which associates to every vertex $x \in \Pi$ the unique simplicial direction at x which is not of rank 2.

Thus, one may represent a root distribution δ as a field of segments of length 2ε (for some small fixed $\varepsilon > 0$) as described above.

We call *abstract root distribution* on $\mathbb{R}^2$ the choice, for every vertex x of the tessellation by equilateral triangle, of a simplicial direction $\delta(x)$ at x.

Definition 1.4. An abstract root distribution is *realized* in a simply connected Moebius–Kantor complex Δ if it is the root distribution of some flat Π in Δ.

We shall first prove that every abstract root distribution can be realized in some Moebius–Kantor complex:

Proposition 1.5. *Every abstract root distribution can be realized in a simply connected Moebius–Kantor complex.*

We refer to §2 for the proof. This result does not hold for abstract parity distributions (see Corollary 1.8).

In the terminology of §1.6, this shows that every abstract root distribution on the tessellation T can be realized by a magnetic field associated with a simply connected Moebius–Kantor complex.

1.4 Parity prescriptions in a flat

Every abstract root distribution δ on $\mathbb{R}^2$ induces a parity distribution p on the triangle faces of the tessellation. If f is a face, then the parity $p(f) \in \{0,1\} = \{\text{even}, \text{odd}\}$ is the parity of the image of f under an isometric embedding $\mathbb{R}^2 \to \Delta$ which realizes δ in some Moebius–Kantor complex Δ (which exists by Proposition 1.5). Since the value of $p(f)$ does not depend on the choice of the large triangle $\tilde{f}$, one may always assume that $\tilde{f}$ is included in the image of $\mathbb{R}^2 \to \Delta$. Thus, $p(f)$ depends only on the abstract root distribution δ. It computes the parity of the number of sides of $\tilde{f}$ which are not in the simplicial directions selected by δ.

Definition 1.6. We call p the *parity distribution of* δ.

An *abstract parity distribution* on $\mathbb{R}^2$ is the choice, for every face f of the tessellation by equilateral triangle, of value $p(f) \in \{0,1\}$.

We aim to show (in contrast with Proposition 1.5) that there exist abstract parity distributions which are not the parity distribution of any flat in any simply connected Moebius–Kantor complex. This will be an immediate corollary of the following result.

Theorem 1.7. *There exists an abstract parity distribution which is not the parity distribution of an abstract root distribution.*

We refer to §3 for the proof.

An abstract parity distribution is said to be *realized* in a Moebius–Kantor complex Δ if it is the induced parity distribution of some flat in Δ.

Corollary 1.8. *There exists an abstract parity distribution on* $\mathbb{R}^2$ *which is not realized in a simply connected Moebius–Kantor complex.*

Proof. The parity distribution given by Theorem 3.1 is not realized in a Moebius-Kantor complex Δ, for it would be the parity distribution of the root distribution induced by Δ. □

On the other hand, we can prove the following result.

Proposition 1.9. *Every parity distribution on a ball of radius 1 (a hexagon) in $\mathbb{R}^2$ can be realized by an abstract root distribution, and therefore can be realized in a Moebius–Kantor complex.*

This is established in §4.

1.5 Parity prescriptions in a Moebius-Kantor complex

By Theorem 3.1, there exists an abstract parity distribution on $\mathbb{R}^2$ which cannot be realized in a Moebius–Kantor complex. This should be compared with the results in §5 of [4]. In the latter paper, it is shown that the parity can be "freely prescribed" in Moebius–Kantor complexes, and therefore, that any abstract parity distribution can always "be realized" in some Moebius–Kantor complex.

The difference is as follows. In the case of a parity prescription as conceived in [4], the parity distribution can only be pre-assigned *in stages* in successive spheres of the complex under construction. Namely, the parity can be realized in an inductive manner by constructing successive spheres appropriately, as opposed to being prescribed *in advance* as a 0-1 valued function on a plane. In fact, for a prescription theorem in the case of Moebius–Kantor complexes, there does not exist an a priori defined "substratum space" Δ that would serve as domain, for an abstract parity function $f\colon \Delta \to \{0,1\}$ which is to be realized.

More precisely, let us state here the free prescription (in stages) theorem (from [4, §5]) for the parity in a Moebius–Kantor complex. Let B_n denote a ball of radius n centered at a vertex in a simply connected Moebius–Kantor complex, and $S_n := B_n \setminus (B_{n-1})^\circ$ denote the closed simplicial sphere of radius n, where $(B_{n-1})^\circ$ denotes the interior of B_{n-1}. One can realize an arbitrary parity function on S^n in a simply connected Moebius–Kantor complex by iterating the following construction.

Theorem 1.10. *Given an arbitrary function $p\colon S_n \to \{0,1\}$ defined on the 2-skeleton of S_n, there exists a ball B_{n+1} in a simply connected Moebius–Kantor complex, containing B_n, such that the parity of the faces of S_n in B_{n+1} is given by p.*

Free construction theorems of this sort originated in the theory of buildings, in particular in works of Ronan and Tits (see for instance [9, 10, 11, 12] and references therein).

It was mentioned in [4] that the proof of Theorem 1.10 appears to provide additional free parameters in the constructions, i.e., that the parity function p should not determine the isomorphism type of B_{n+1} in general.

That these additional degrees of freedom ultimately exist for a specific p depends on the balance between the root distribution at the vertices in the boundary of B_n, and the parity distribution on the faces of S_n.

A direct computation can be made to compare, very roughly, the degree of freedom for the parity in relation with the degree of freedom for the roots. Namely, let f_n denote the number of faces in S_n. There can be two types of vertices in the boundary of B_n, those of degree 3 and those of degree 4. Let a_n and b_n denote their respective number. Then the relative degree of freedom is computed as follows: there are 2^{f_n} choices for the parity in S_n, and at most $8^{a_n}16^{b_n}$ choices for the roots at the boundary of B_n; the quantities a_n, b_n, f_n are related by the equality:

$$f_{n+1} = 18a_n + 15b_n;$$

the quantities a_n and b_n can be computed explicitly by the recurrence relation:

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

Therefore,

$$2^{f_n} < 8^{a_n}16^{b_n}.$$

Since we do not need them except for a general estimate of the degrees of freedom, we leave to the reader the exercise of verifying these relations.

A Moebius–Kantor complex is said to be *even* if every face is even. In [4, Remark 5.3(1)], it was suggested to study the case of even complexes, and that—in view of the above estimates—there "ought to exist" uncountably many non isomorphic constructions of such complexes. It is this problem that lead us to study root distributions and the relation between root distributions and parity distributions.

One of the main results of the present paper shows that there exists a unique simply connected construction of an even complex *up to isomorphism*, i.e., that the free constructions exhibited in [4] are in fact all isomorphic in the case of even simply connected Moebius–Kantor complexes. In the remainder of this introduction, we shall state this uniqueness result more precisely, together with some applications which resolve other questions that we left open in earlier investigations (namely, [5] and [6]).

1.6 An interpretation of root distributions

Root distributions are somewhat analogous in nature to standard combinatorial objects arising from physics. For example, the following are two well-known models which lead to the study of fields of segments positioned at the vertices of a lattice in $\mathbb{R}^2$.

1. The dimer model studies the set of configurations of all monomers (molecules occuping a single vertex) and dimers (molecules occupying two adjacent vertices) covering the vertex set of a Eulidean square lattice.
2. The Ising model studies the set of configuration of possible directions (say, up or down) at the vertices of a Euclidean square lattice, in which the edges are seen as bonds interacting in the neighbouring directions.

Although our root distributions originated in geometric group theory, one can nevertheless imagine an abstract physics model in which the segments of length 2ε

represent magnets in a given simplicial orientation, and that the local constraints on the configuration space of magnets are produced by an ambient magnetic field applied to the plane. Such a magnetic field, for instance, can be created by positioning a generator at the center of every triangle of the plane. The distributions studied in the present paper are of this type: they are created by fixing one magnetic generator at the center of every triangle. The generators can be of two types, even and odd, corresponding to their action on the neighbouring magnets. The effect of a generator is to fix the parity of the number of magnets which are not parallel to the opposite side of the triangle on which the generator sits.

Thus, a magnetic field in this case is entirely determined by a function on the set of triangles of T with values in $\mathbb{Z}/2\mathbb{Z}$, which we called a parity distribution. Every embedding of T as a flat in a Moebius–Kantor complex induces a corresponding parity distribution, and can be seen as a way to create a magnetic field on T. As shown above, these fields can induce an arbitrary root distribution, by choosing the complex and the embedding appropriately. In the next section, we study the case of a field with vanishing parity distribution, in which every magnetic generator is set to even. As we shall see, these fields are all induced from some embedding in the unique even simply connected Moebius–Kantor complex.

1.7 Even root distributions

We shall first describe the relation between root distributions and parity distributions in the even case.

The even parity distribution on $\mathbb{R}^2$ is the contant distribution $p \equiv 0$ in which every face is even. A root distribution is said to be even if its parity distribution is the even parity distribution.

We say that two abstract root distributions δ and δ' on $\mathbb{R}^2$ are isomorphic if there exists a simplicial isometry α of $\mathbb{R}^2$ such that $\delta \circ \alpha = \delta'$.

Theorem 1.11. *The even root distributions can be classified up to isomorphism.*

A more precise version of this result, stated below, contains an explicit classification of the distributions (see Theorem 7.7).

The proof can be done in two ways, either directly, or as a consequence of [6] and the results below. We provide both arguments in the present paper. The second approach is shorter since it only requires two additions (Propositions 1.12 and 1.13) to the results of [6]. The direct approach is given in §7.

In [6], a Moebius–Kantor complex Δ_P called the *Pauli complex* was introduced. It is defined using a triangle of groups construction associated with the group generated by the Pauli matrices in SU(2). The fundamental group of this triangle of groups is a group G_P which is developable and therefore acts properly isometrically on a CAT(0) complex, the complex Δ_P. (More details on this construction are given in §5 below.)

The first observation is the following result.

Proposition 1.12. *The Pauli complex Δ_P is even.*

This proposition is established in §5 below; it relies on basic relations verified by the Pauli matrices.

In order to deduce Theorem 1.11 from [6] (in particular, Theorem 5.7 therein), it is sufficient to prove the following result (see §5).

Proposition 1.13. *Every even root distribution can be realized in the Pauli complex.*

One can reformulate this proposition by saying that the map $\Pi \mapsto \delta_\Pi$ which to a flat Π of the Pauli complex associates its unique root distribution is surjective onto the set of even root distributions. In [6], the flats of the Pauli complex were classified by means of ring puzzles. The map $\Pi \mapsto \delta_\Pi$ can be reinterpreted as a map from ring puzzles to root distributions. It follows by Theorem 5.7 in [6] that the Pauli ring puzzles can be classified (i.e., listed explicitly); therefore, so can the even root distributions.

This proves that one can deduce Theorem 1.11, and indeed, have an explicit classification of the even root distributions, from the classification of Pauli puzzles given in Theorem 5.7 in [6]. In the terminology of [6], the classification comprises:

1. a unique even root distribution associated with the Pauli M-puzzles; and,
2. an infinite family of pairwise non-isomorphic root distributions associated with the Pauli T-puzzles;

every even root distribution is isomorphic to a root distribution in case (1) or (2). This statement is the more precise version of Theorem 1.11 we were looking for. As mentioned above, a more direct approach to this result, which classifies the even root distributions, is given in §7.

1.8 Even Moebius–Kantor complexes

Several examples of even Moebius–Kantor complexes were given in [5]. In fact, it was shown there that there exist precisely four pairwise non isomorphic even Moebius–Kantor complexes having a single vertex. In the notation of [5], these complexes are V_0^1, V_0^2, $\check{V}_0^2$, and V_4^1; we refer to [5] for their description (see also [4, Prop. 3.2]).

Theorem 1.14. *Suppose that Δ and Δ' are even simply connected Moebius–Kantor complexes, and let $x \in \Delta$ and $x' \in \Delta'$ be two vertices. Let $n \geq 1$, and let $\varphi_n: B_n(x) \to B_n(x')$ be an isomorphism between the balls of radius n around x and x' in Δ and Δ', respectively. Then there exists a unique isomorphism $\varphi: \Delta \to \Delta'$ such that $\varphi_{|B_n(x)} = \varphi_n$.*

This result is established in §6. The following are immediate corollaries from Theorem 1.14.

Corollary 1.15. *Every even simply connected Moebius–Kantor complex is isometrically isomorphic to the Pauli complex.*

In fact, we will prove Theorem 1.14 by cross referencing the additional structure found in the Pauli complex with that of an arbitrary even simply connected Moebius–Kantor complex.

Corollary 1.15 shows that the Pauli complex is unique in a family of nonpositively curved complexes with "prescribed local data". We refer the reader to [11], [1] and [8], for well-known unique construction results of nonpositively curved complexes in which the local data are prescribed. In these results, the manner to prescribe the local data depends on the complex under consideration; for Moebius–Kantor complexes, the link type is clearly not enough local data to ensure uniqueness, while assuming in addition the parity to be uniformly even, is, by the first corollary.

The second corollary follows by choosing an appropriate isomorphism $\varphi_1\colon B_1(x) \to B_1(y)$ where x and y are two points in the Pauli complex:

Corollary 1.16. *The symmetry group of the unique even simply connected Moebius–Kantor is flag transitive.*

(We recall that a flag is a triple (x, e, f) where x is a vertex, e an edge, and f a face, such that $x \in e \in f$.)

The following two corollaries were left open in [5] and [6], respectively.

Corollary 1.17. *The universal covers of V_0^1, V_0^2, $\check{V}_0^2$, and V_4^1, are pairwise isometrically isomorphic.*

We recall that G_P denotes the fundamental group of the complex of groups defining the Pauli complex (see §1.7 and §5).

Corollary 1.18. *The group G_P contains a torsion free subgroup of finite index.*

2 Abstract root distributions

Proposition 2.1. *Every abstract root distribution can be realized in a simply connected Moebius–Kantor complex.*

Proof. Let Π be a flat plane endowed with a root distribution. Fix a vertex $x \in \Pi$. Let H_n denote the hexagon of center x and radius n in Π. The proof proceeds by induction to construct a ball B_n of radius n in a Moebius–Kantor complex, containing H_n, and realizing the root distribution.

For $H_1 \subset B_1$, it suffices to embed a 6-cycle with marked roots in a Moebius–Kantor graph respecting the rank of every root. Suppose $H_n \subset B_n$ has been constructed. Let $\tilde{B}_n$ be obtained by adding a pair of faces for every edge in the boundary S_n of B_n, together with H_{n+1}, to B_n. For every vertex in S_n, the link is either a tree (of diameter at most 5) or the union of such a tree and the 1-neighbourhood of a 6-cycle (adding at most two edges to the tree). In the first case (which occurs at every vertex not in H_n), the tree can be completed arbitrarily into a Moebius–Kantor graph. In the second case (which occurs at every vertex in H_n), the roots are pre-positioned in the 6-cycle, and there exists a completion which respects the roots. These two assertions are readily verified; the first one is obvious, and the second one follows by embedding the 6-cycle first in a root preserving manner. We note that this argument is specific to the Moebius–Kantor graph.

Finally, the links can be completed to form B_{n+1} from $\tilde{B}_n$, and the direct limit of the B_n's constructed in a such a way provides one (among uncountably many non pairwise isomorphic) Moebius–Kantor complex realizing the root distribution. □

Every root distribution defines uniquely the parity of its faces.

Example 2.2. A root distribution with parallel roots defines an even flat; however, the same flat can also be obtained with different root distributions.

We shall prove in the forthcoming section that there exists a parity distribution on a flat plane which is not realized by a root distribution. The notion of abstract root (and parity) distribution extends to simplicial subsets of $\mathbb{R}^2$ (for example, strips or balls of radius n centered at a vertex x) in the obvious way. Every parity (respectively root) distributions on a simplicial subset of $\mathbb{R}^2$ is the restriction of a parity (respectively root) distribution on $\mathbb{R}^2$.

Observe that every parity distribution on a bi-infinite strip of height 1 can be realized in a flat in a simply connected Moebius–Kantor complex. Indeed, it suffices to choose appropriately a root distribution on its boundary to realize the parity, extend the partial distribution in a flat, and apply Proposition 2.1. The same holds true for balls of radius 1 as we shall see in §4.

3 Abstract parity distributions

Theorem 3.1. *There exists an abstract parity distribution which is not the parity distribution of an abstract root distribution.*

Proof. We shall start with an even face, say x, and define a parity distribution stepwise building outward from the face x in such a way that the root distribution is analytically determined, until a contradiction arises.

Consider the following root distribution:

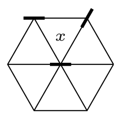

Figure 3

In Figure 3, we have represented a root distribution graphically as mentioned in the introduction (see §1.3).

In Figure 4, an extension of this root distribution is shown. The triangles labelled with label “1” indicate the odd faces, every other face is even.

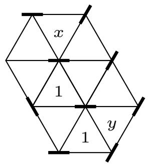

Figure 4

It is obvious that the face with label y, which we define to be even, is odd according to the root distribution. This provides a contradiction.

We may now extend the parity distribution by an order 3 symmetry as follows:

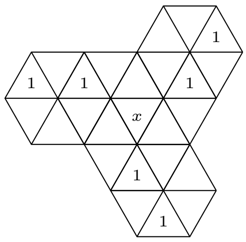

Figure 5

It follows that for every parity distribution on $\mathbb{R}^2$ containing the above distribution, the twelve (= 3×4) root distributions on x in which the number of roots of rank 2 is 2, are disallowed. (Indeed, by symmetry, one may assume that the root distribution is oriented downwards, as represented in the first figure above, then by analyticity using the prescribed face parities, one obtain a last face y which is supposed to be even but is odd according to the distribution.)

Finally, the remaining even root distribution on x:

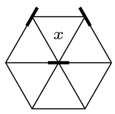

Figure 6

is also disallowed, for the same reason (i.e., it extends as the initial root distribution). This concludes the proof of the proposition. □

4 Parity prescription in the neighbourhood of a vertex

Proposition 4.1. *Every parity distribution on a ball of radius* 1 *(hexagon) can be realized by an abstract root distribution.*

Proof. Let B be a ball of radius 1 around a vertex. We distinguish three cases: either B contains at most two odd faces, at most two even faces, or precisely three odd faces and three even faces.

The first two cases consists of five subcases each, which correspond to the number of ways to assign a parity to B with at most two even (or odd) faces. In these cases, the root distributions can be chosen as follows:

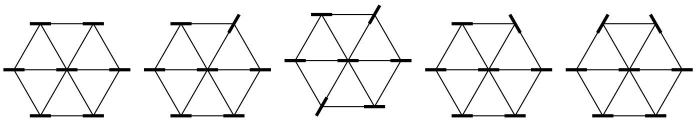

Figure 7: Root distributions with at most 2 odd faces

It is easily seen that the above root distributions exhaust the parity distributions with at most two odd faces. Similarly:

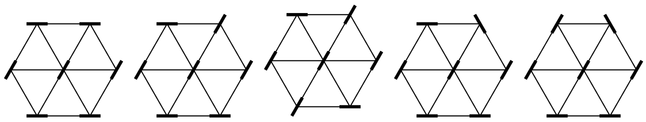

Figure 8: Root distributions with at most 2 even faces

The last case, with 3 even faces and 3 odd faces, consists of three subcases, and for each of these cases, a root distribution exists:

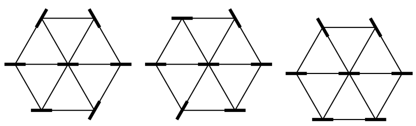

Figure 9: Root distributions with exactly three odd faces

This concludes the proof of the proposition. □

5 The Pauli complex Δ_P

The Pauli complex is a simply connected Moebius–Kantor complex Δ_P obtained by a standard graph of groups construction ([7, Chap. III.C]) using a triangle of groups associated with the group P generated by the three Pauli matrices in SU(2):

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The direct limit G_P of this triangle of groups acts on Δ_P in a flag transitive way. The complex Δ_P is obtained by developing around a fundamental face f using the group P.

The complex Δ_P has an important property, which was used in [6] to classify the flat planes that it contains: every face in Δ_P can be assigned with a Pauli matrix X, Y or Z in an equivariant way [6, §3]. Under this assignment, the link edges are naturally labeled, and the link is a Moebius–Kantor graph isomorphic to the Cayley graph of P with respect to $\{X, Y, Z\}$. The 6-cycles in the link (which correspond to rings in ring puzzles) are given by the following relations between the Pauli matrices

$$XYXZYZ = \mathrm{Id}$$

$$YZYXZX = \mathrm{Id}$$

$$ZXZYXY = \mathrm{Id}$$

in P. A root in Δ_P corresponds to a path of length 3 in the link with an edge labelling in $\{X, Y, Z\}$.

Lemma 5.1. *A root is of rank 2 if and only if it contains the three labels X, Y, Z.*

It is clear how to label the Moebius–Kantor graph in such a way that the above property is true.

Proof. Since the Moebius–Kantor graph is 2-transitive, it is enough to prove the lemma for the roots having labels XYZ and XYX. It is clear that a root with labels XYZ can be extended in two ways using the 6-cycles $XYZXZY$ and $XYZYXZ$, and therefore is of rank 2. A root with labels XYX can be extended in a single way using a 6-cycle, namely, $XYXZYZ$. The word $XYXY$ is not a subword of a relation of length 6 for the group P. □

This implies:

Proposition 5.2. *The Pauli complex Δ_P is an even Moebius–Kantor complex.*

Proof. Since Δ_P is flag transitive, it is enough to prove that the fundamental face is even. We assume that the face f is labelled with the matrix Y, and extend f into an equilateral triangle $\tilde{f}$ using three faces with the same labels, say X. Then the three roots associated with f have labels XYX. By the previous lemma, these roots are not of rank 2. Therefore, Δ_P is even. □

As shown in the introduction, in order to deduce Theorem 1.11 from [6], it is sufficient to prove the following result.

Proposition 5.3. *Every even root distribution can be realized in the Pauli complex.*

One argument would be to modify the proof of Proposition 2.1, to ensure that the Moebius–Kantor complex is even, and apply Theorem 1.14.

We shall proceed as follows.

Proposition 5.4. *Let* Π *be a flat plane endowed with an even root distribution, and consider a labelling by* X *and* Y*, of two consecutive faces in* Π*. There exists a unique labelling of the faces of* Π *with labels in* $\{X, Y, Z\}$ *such that the* 6*-cycles in the links correspond to the relations* $XYXZYZ = \mathrm{Id}$, $YZYXZX = \mathrm{Id}$, *and* $ZXZYXY = \mathrm{Id}$. *In other words, there exists a unique Pauli puzzle associated with* Π *and the labelling of two consecutive faces.*

Proposition 5.3 follows from Proposition 5.4 and the fact that every Pauli puzzle is realized in the Pauli complex (Theorem 3.3 in [6]).

Proof. Uniqueness is clear. We prove existence by induction on the radius n of the ball B_n centered at one of the two common vertices of the labelled consecutive triangles.

Suppose that there exists a labelling of B_n. For every vertex x in the boundary of B_n, at least two faces of the ball $B_1(x)$ of radius 1 at x are labelled, and therefore there exists a unique labelling of $B_1(x)$ which is consistent with the parity at x.

Let $[x, y]$ be an edge in the boundary of B_n, and let t and t' denotes the two triangles adjacent to $[x, y]$, where t is the triangle included in B_n.

We have to show that the labelling of t' from $B_1(x)$ coincides with the labelling from $B_1(y)$. However, since two of the three faces adjacent to t are included in B_n and therefore labelled, only one of the two possible labels of t' ensure, by Lemma 5.1, that t is an even face (compare Lemma 6.2). □

6 Every even complex is isomorphic to Δ_P

Let Δ be a simply connected even complex, $x_0 \in \Delta$ be a vertex, B_n be the ball of center x_0 and radius n in Δ, and $S_n := \partial B_n$ be the sphere of center x_0 and radius n. We fix an isometric embedding

$$\varphi_1 : B_1 \to \Delta_P.$$

Using φ_1, we may label every face in B_1 with one of the Pauli matrices X, Y, Z.

Theorem 6.1. *There exists a unique isometric embedding* $\varphi : \Delta \to \Delta_P$ *such that* $\varphi_{|B_1} = \varphi_1$.

Proof. We prove by induction that there exists a unique isometric embedding $\varphi_n : B_n \to \Delta_P$ such that $\varphi_{n|B_1} = \varphi_1$. The theorem follows immediately.

Suppose that $\varphi_n : B_n \to \Delta_P$ is an isometric isomorphism onto its image. Using φ_n, we may label every face in B_n with one of the Pauli matrices X, Y, Z. Let us construct φ_{n+1}. This construction will also establish Theorem 1.14.

For every vertex $x \in S_n$, there exists a unique isometric embedding $\varphi_x : L_x \to \Delta$ whose restriction to B_n coincides with φ_n.

For every edge $[x, y]$ in S_n, let $h_{x,y}^k$, $k = 0, 1$, denote the height, perpendicular to $[x, y]$, in the two faces containing $[x, y]$ which are not contained in B_n. We write

$H^k_{x,y}$ and $H^k_{y,x}$ for the two half faces, separated by $h^k_{x,y}$, respectively containing x and y. Thus, $H^k_{x,y} \cup H^k_{y,x}$, $k = 0, 1$, are the two faces on $[x, y]$ not contained in B_n.

For every $x \in S_n$ we may label, using φ_x, every half face $H^k_{x,y}$, $k = 0, 1$, with one of the Pauli matrices X, Y, Z.

We claim (Lemma 6.2):

> for every $[x, y]$ and $k = 0, 1$, the labels of the half faces $H^k_{x,y}$ and $H^k_{y,x}$ coincide.

This shows that there exists a well-defined isometric embedding $\varphi_{n+1} : B_{n+1} \to \Delta$ extending φ_n, which coincides with φ_x on L_x for every $x \in S_n$. It takes $H^k_{x,y} \cup H^k_{y,x}$ to the unique face in Δ_P which contains $\varphi_n([x, y])$, and whose label is the common label of $H^k_{x,y}$ and $H^k_{y,x}$. The uniqueness of such an embedding is clear. □

We now prove the claim.

Lemma 6.2. *For every $[x, y]$ and $k = 0, 1$, the labels of the half faces $H^k_{x,y}$ and $H^k_{y,x}$ coincide.*

Proof. Consider the face $F = [x, y, z]$ in B_n containing $[x, y]$, and two faces F_x and F_y in B_n, adjacent to $[z, x]$ and $[z, y]$ respectively, and having identical labels, say X. Let $k, l \in \{0, 1\}$ denote the two indices such that $H^k_{x,y}$ and $H^l_{y,x}$ have label X. We must show that $k = l$.

Since the labels of the faces F_x, F_y, and half faces $H^k_{x,y}$ and $H^l_{y,x}$, coincide, the rank of the roots corresponding three roots at x, y and z is $\frac{3}{2}$ (Lemma 5.1).

Since F is an even face, the number of roots of rank $\frac{3}{2}$ in a large triangle containing F is odd. Since the root at z is of rank $\frac{3}{2}$, the roots at x and y must have the same rank. Therefore, $k = l$. For the corresponding face in Δ, the large triangle adjoins two roots of rank $\frac{3}{2}$ at x and y for a total of 3 such roots. □

7 Classifying the even root distributions

In this section, we provide a direct approach to the classification of even root distributions in a flat, which rely only on the root distribution rather than the Pauli complex.

We will in fact start with an arbitrary even simply connected Moebius–Kantor complex, not a priori known to be isomorphic to Δ_P, and show how its flats can be classified by studying the root distributions. The same proof applies to classify the abstract even root distributions.

In §2 of [3], a notion of w-block in an $\mathrm{Aut}(F_2)$-puzzle was introduced. It is a geometric configuration of shapes having the property of being *forward analytic* in $\mathrm{Aut}(F_2)$ puzzles. Informally, this refers to the fact that this block has a specific direction in which it determines the puzzle uniquely, at least, in a cone. In fact, the w-block can be viewed as a sort of ‘glider’ with the property of being ‘self-reproducing’, in the chosen direction (implying analyticity).

Our main claim is that a configuration similar to the w-block in $\mathrm{Aut}(F_2)$-puzzles, exists in every even Moebius–Kantor puzzle, and leads to a similar 'forward analytic' behaviour that determines the puzzle analytically in a cone. The configuration is a trapezoid, and the following lemma establishes analyticity.

Let Δ be an even simply connected Moebius–Kantor complex.

Lemma 7.1. *Consider a trapezoid t of height 1 with a base of length 3, such that the roots on the base have rank 2, and the root on the top, rank 3/2. If t belongs to a flat Π, then t admits a unique extension into a sector of Π of base t and rank 2 boundary.*

The trapezoid is shown in Figure 10; the two roots at the bottom are of rank 2, that on top of rank 3/2. Analyticity occurs in the upward direction in the figure.

Proof. The rank 3/2 root in t extends in a unique way (i.e., in Π). Since Δ is even, the slanted sides of length 2 are of rank 3/2. This follows from the fact that the inner vertices are both of rank 2. Thus, t extends uniquely into a 2-strip S_2 of length 3 in Π. Since Δ is even, the two upper roots in Π are of rank 3/2, and therefore extend uniquely into Π. Again, the slanted sides of length 2 are of rank 3/2, and S_2 extends uniquely into Π into a partial cone S_3 of height 3. By induction, the upper roots in S_n are always of rank 3/2, which provides a uniquely defined sector $S \coloneqq \bigcup S_i$ of base t in Π. $\square$

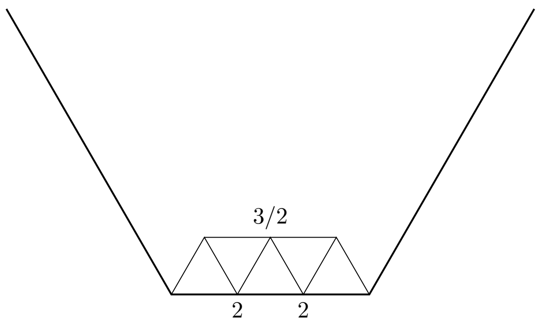

Figure 10: The figure shows the sector S and its rank 2 boundary

For even Moebius–Kantor puzzles, the following strong property holds.

Lemma 7.2. *The sector S (in the notation of Lemma 7.1) belongs to a unique flat called the t-flat and denoted Π_t.*

Proof. Let $S \subset \Pi$ be a copy of S in a flat, and let $\bar{t}$ denote the reflection of t with respect to its rank 2 boundary. If the lower boundary of $\bar{t}$ is of rank 3/2, we have an opposite copy of t in Π, and Lemma 7.1 shows that $\bar{t}$ extends to a copy $\overline{S}$ of S in Π. By convexity, there exists at most one flat of Δ containing $S \cup \overline{S}$. It is easy to check that such a flat indeed exists. (It is represented below.)

Let us show that the lower boundary of $\bar{t}$ must be of rank 3/2. Otherwise, it is of rank 2. Since the central triangle in t is even, the 2-triangle T (which is formed of 4 equilateral triangles) of base the lower base of $\bar{t}$ admits two sides of rank 3/2 or two sides of rank 2. Since the center triangle in $\bar{t}$ is even, this proves that the lower side of T is of rank 3/2. $\square$

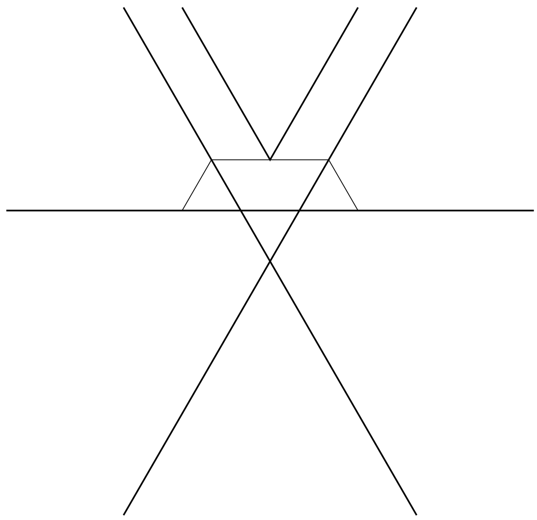

Figure 11: The t-flat Π_t; three intersecting rank 2 (in dark) geodesics are drawn, and one of the trapezoid t is shown; the symmetries S_3 of the center triangle extend to the t-flat.

Remark 7.3. The flat Π_t corresponds to one of the Pauli M-puzzles in [6], where $M \in \{X, Y, Z\}$. The root distribution of Π_t was shown in Figure 1 (see §1).

By a strip of rank 2, we mean a simplicial flat strip whose boundary roots are of rank 2.

Lemma 7.4. *If Δ is even, the minimal strips of rank 2 are of height 1.*

Proof. Suppose that there exists a minimal strip Σ with rank 2 boundaries. If it is not of height 1, then there exists a root of rank 3/2 at distance 1 from the boundary. Since the boundary is of rank 2, Σ contains a trapezoid t. By Lemma 7.1 and Lemma 7.2, Σ must be included in the t-flat. However, the latter does not contain such a strip. $\square$

By a geodesic of rank 2 in a flat, we mean a simplicial straight line containing exclusively roots of rank 2.

Lemma 7.5. *If a flat Π contains a geodesic l of rank 2, then either Π coincides with the t-flat Π_t, or it is a union of minimal strips of rank 2 parallel to l.*

Proof. Let l be a rank 2 geodesic. If Π is not a union of minimal strips of rank 2 parallel to l, then there exists a parallel geodesic $l' \parallel l$ of rank 2, and a parallel geodesic at distance 1 from l' which contains a root of rank 3/2. Thus, Π contains the trapezoid t which implies $\Pi = \Pi_t$. □

We shall now consider a 'glider' (i.e., a configuration which is forward analytic in flats) which behaves similarly to the w-block in $\mathrm{Aut}(F_2)$ puzzles. Furthermore, if $\Pi \neq \Pi_t$ then it will also be backward analytic.

Lemma 7.6. *Let Δ be an even simply connected Moebius–Kantor complex. Consider a trapezoid t' of height 1 with a base of length 3, such that precisely one of the two roots on the base have rank 2, and such that the top root has rank 3/2. Every flat containing t' contains a geodesic of rank 2.*

Proof. Suppose that t' belongs to a flat Π. We first show that t' admits a unique upward extension into a half-strip $S^\uparrow$ transverse to t' of height 3 with rank 2 boundaries. By symmetry, we may assume that the bottom rank 2 root is on the right. Consider the unique upward extension of t' (whose boundary is of rank 3/2). Since the left top face of the trapezoid is even, the righthand side of the extension is of rank 3/2, and therefore admits a unique extension into Π. This implies that the top root of the resulting extension is of rank 3/2. Therefore, a copy of t' sits on top of it; by induction, we find a half strip $S^\uparrow$ piling up infinitely many copies of t'.

Next we prove that if $\Pi \neq \Pi_t$ then $S^\uparrow$ extends uniquely into a strip S of height 3 with rank 2 boundary. Consider a trapezoid t'' extending $S^\uparrow$ downwards. Since the faces of t' are even, the right bottom root of t'' is of rank 2. If the left bottom root of t'' is even, then t'' is isomorphic to t, so $\Pi = \Pi_t$ by Lemmas 7.1 and 7.2. Therefore t'' is isomorphic to t'; by induction, we find a strip S containing $S^\uparrow$ and piling up infinitely many copies of t'.

To prove the lemma, we may assume that $\Pi \neq \Pi_t$; note then that the strip $S \subset \Pi$ is a union of strips with rank 2 boundaries. □

Theorem 7.7. *A flat plane in an even simply connected Moebius–Kantor complex is either a flat with the root distribution of Π_t, or a union of strips of height 1 and rank 2.*

Proof. Let Δ be an even Moebius–Kantor complex and Π be a flat in Δ. The previous results show that the conclusions hold if Π contains t (see Lemma 7.1) or t' (see Lemma 7.6). Otherwise, Π contains neither t nor t'. This implies the following statement:

> If τ is a trapezoid of base of length 3 and top root of rank 3/2, then the bottom roots are of rank 3/2.

By connectedness, this statement in turn implies that the roots of rank 3/2 are aligned in Π. There exists a unique configuration with this property. Since the roots of rank 3/2 are aligned in Π and Δ is a Moebius–Kantor complex, the two transverse directions must be rank 2. Thus, Π is a union of strips of height 1 and rank 2. □

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