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A minimum degree condition for the existence of S-path-systems in bipartite graphs

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Abstract

Let G be a graph, and S be a subset of $V(G)$ with even cardinality. We denote the set of the endvertices of a path P by $\mathrm{end}(P)$. A path P is an S-path if $|V(P)| \geq 2$ and $V(P) \cap S = \mathrm{end}(P)$. An l-S-path-system $\mathcal{P}$ is a set of vertex-disjoint S-paths such that $S = \bigcup_{P\in\mathcal{P}}(V(P) \cap S)$ and $|V(P)| \leq l$ for each $P \in \mathcal{P}$. In this paper, we show that if G is a bipartite graph with partite sets A and B with $\delta(G) \geq \max\{|A|, |B|\}/2$, and if S is a subset of $V(G)$ with even cardinality such that $|A\cap S|-|B\cap S| \leq 2|B\setminus S|$ and $|B \cap S| - |A \cap S| \leq 2|A\setminus S|$, then, unless $|A| = |B|$ is even and G is isomorphic to $K_{|A|/2,|A|/2} \cup K_{|A|/2,|A|/2}$, G has a 6-S-path-system $\mathcal{P}$ such that every path in $\mathcal{P}$, possibly but one, has order two or three.

1 Introduction

In this paper, we consider only finite simple graphs. Let G be a graph. We write $|G|$ for the order of G; that is, $|G| = |V(G)|$. We denote the set of the endvertices of a path P by $\mathrm{end}(P)$. Let S be a subset of $V(G)$ with even cardinality. A path P is an *S-path* if $|P| \geq 2$ and $V(P) \cap S = \mathrm{end}(P)$. An *$S$-path-system* $\mathcal{P}$ is a set of vertex-disjoint S-paths such that $S = \bigcup_{P\in\mathcal{P}}(V(P) \cap S)$. If $S = \emptyset$, then by definition, $\emptyset$ is an S-path-system. For an S-path system $\mathcal{P}$ and an integer $i \geq 2$, we let $\mathcal{P}_i = \{P \in \mathcal{P} : |P| = i\}$. For an integer $l \geq 2$, an S-path-system consisting of

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S-paths of order at most l is called an *l-S-path-system.* Thus an S-path-system $\mathcal{P}$ is an l-S-path-system if and only if $\mathcal{P}_i = \emptyset$ for all $i \geq l+1$. Note that a 2-S-path-system is simply a perfect matching on the subgraph induced by S in G.

In [2], Chiba and Yamashita pointed out that as a corollary of the following theorem, we obtain a degree sum condition for a graph to have an S-path-system.

Theorem 1.1 (Berman [1]). *Let G be a graph, and let M be a matching of G. If $d_G(x) + d_G(y) \geq |G| + 1$ for any non-adjacent vertices $x, y \in V(G)$ with $x \neq y$, then G has a cycle passing through M.*

Corollary 1.2 (Chiba and Yamashita [2]). *Let G be a graph, and let S be a subset of $V(G)$ with even cardinality. If $d_G(x)+d_G(y) \geq |G|+1$ for any non-adjacent vertices $x, y \in V(G)$ with $x \neq y$, then G has an S-path-system.*

Recently, in [6], Tsugaki and Yashima showed that the following theorem implies a refinement of Corollary 1.2, which we state as Corollary 1.4.

Theorem 1.3 (Shi [5]). *Let G be a 2-connected graph, and let S be a subset of $V(G)$. If $d_G(x) + d_G(y) \geq |G|$ for any non-adjacent vertices $x, y \in S$ with $x \neq y$, then G has a cycle which contains all the vertices of S.*

Corollary 1.4 (Tsugaki and Yashima [6]). *Let G be a graph, and let S be a subset of $V(G)$ with even cardinality. If $d_G(x)+d_G(y) \geq |G|-1$ for any non-adjacent vertices $x, y \in S$ with $x \neq y$, then G has a 3-S-path-system.*

Recall that a 2-S-path-system is a perfect matching on S, and the notion of a 3-S-path-system is a weakening of the notion of a perfect matching on S. Corollary 1.4 shows that there is a natural degree sum condition for the existence of a 3-S-path-system. This suggests the importance of the study of l-S-path-systems with small values of l.

In this paper, we focus on bipartite graphs. We denote by $G[A, B]$ a bipartite graph G with partite sets A and B. For a bipartite graph $G[A, B]$, we define

$$\sigma_{1,1}(G) = \min\{d_G(a) + d_G(b) : a \in A, b \in B, ab \notin E(G)\}$$

if $G[A, B]$ is not complete; otherwise $\sigma_{1,1}(G) = \infty$. Also we let $\delta(G)$ denote the minimum degree of G.

If a graph G has a Hamilton path, then G has an S-path-system. Thus the degree sum condition of the following result also assures us of the existence of an S-path-system in a balanced bipartite graph (in fact, a characterization of balanced bipartite graphs $G[A, B]$ such that $\sigma_{1,1}(G) \geq |B|$ and G has no Hamilton cycle is obtained in [4]).

Theorem 1.5 (Ferrara, Jacobson and Powell [4]). *Let $G[A, B]$ be a connected balanced bipartite graph. If $\sigma_{1,1}(G) \geq |B|$, then G has a Hamilton path.*

In [6], Tsugaki and Yashima showed that this degree sum condition also ensures the existence of an l-S-path-system such that l is small regardless of whether or not G is balanced.

Theorem 1.6 (Tsugaki and Yashima [6]). *Let $G[A, B]$ be a bipartite graph, and let S be a subset of $V(G)$ with even cardinality. Suppose that $|A \cap S| - |B \cap S| \leq 2|B \setminus S|$ and $|B \cap S| - |A \cap S| \leq 2|A \setminus S|$. If $\sigma_{1,1}(G) \geq \max\{|A|, |B|\}$, then one of the following statements holds:*

(i) *G has a 5-S-path-system;*
(ii) *G is balanced, and G has a 6-S-path-system $\mathcal{P}$ such that $|\mathcal{P}_4 \cup \mathcal{P}_5 \cup \mathcal{P}_6| \leq 1$; or*
(iii) *$G \cong K_{t,t} \cup K_{|B|-t,|B|-t}$ for some integer $1 \leq t \leq |B| - 1$.*

In [6], examples which show that in Theorem 1.6, for any constant α, we cannot strengthen statement (i) by adding the condition that $|\mathcal{P}_4 \cup \mathcal{P}_5| \leq \alpha$ are constructed. However, the examples constructed in [6] have a small minimum degree. Thus it is natural to conjecture that we can strengthen statement (i) if we replace the assumption that $\sigma_{1,1}(G) \geq \max\{|A|, |B|\}$ by the corresponding minimum degree condition. In this paper, we confirm this conjecture by proving the following theorem.

Theorem 1.7. *Let $G[A, B]$ be a bipartite graph, and let S be a subset of $V(G)$ with even cardinality. Suppose that $|A \cap S| - |B \cap S| \leq 2|B \setminus S|$ and $|B \cap S| - |A \cap S| \leq 2|A \setminus S|$. If $\delta(G) \geq \max\{|A|, |B|\}/2$, then one of the following statements holds:*

(i) *G is unbalanced, and G has a 5-S-path-system $\mathcal{P}$ such that $|\mathcal{P}_4 \cup \mathcal{P}_5| \leq 1$;*
(ii) *G is balanced, and G has a 6-S-path-system $\mathcal{P}$ such that $|\mathcal{P}_4 \cup \mathcal{P}_5 \cup \mathcal{P}_6| \leq 1$; or*
(iii) *$|A| = |B|$ is even, and $G \cong K_{|G|/4,|G|/4} \cup K_{|G|/4,|G|/4}$.*

Before discussing the proof of Theorem 1.7, we here insert three paragraphs concerning the sharpness of Theorem 1.7. The minimum degree condition in Theorem 1.7 is sharp in the sense that the conclusion of the theorem does not necessarily hold for bipartite graphs $G[A, B]$ with $\delta(G) \geq (\max\{|A|, |B|\} - 1)/2$. To see this, let m be a positive integer. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph with partite sets A_i and B_i ($|A_1| = m + 1$, $|A_2| = m$, $|B_1| = m$ and $|B_2| = m + 1$). Let $G[A, B]$ be the bipartite graph with partite sets $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that $E(G) = E(G_1) \cup E(G_2) \cup \{ab : a \in A_2, b \in B_1\}$. Let S be a subset of $V(G)$ such that $|S|$ is even and $A_2 \cup B_2 \subseteq S$ (see Figure 1). Then, G is a balanced bipartite graph with $\delta(G) = m = (|B| - 1)/2$ and we clearly have $|A \cap S| - |B \cap S| \leq 2|B \setminus S|$ and $|B \cap S| - |A \cap S| \leq 2|A \setminus S|$, but G has no S-path-system.

In (ii) in the conclusion of Theorem 1.7, we cannot avoid the use of a 6-S-path-system; i.e., the statement becomes false if we replace the term "6-S-path-system" by "5-S-path-system". Let m be a positive integer. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph with partite sets A_i and B_i such that $|A_i| = |B_i| = m$. Take $S \subseteq A_2 \cup B_1$ so that $|S \cap A_2|$ and $|S \cap B_1|$ are odd. Let $G[A, B]$ be the bipartite graph with partite sets $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that $E(G) =$

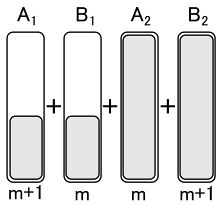

Figure 1: Sharpness of the minimum degree condition

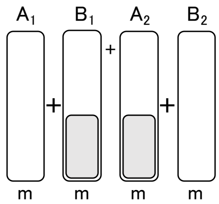

Figure 2: Necessity of the use of an S-path of order 6

$E(G_1) \cup E(G_2) \cup \{ab : a \in A_2 \backslash S, b \in B_1 \backslash S\}$ (see Figure 2). Then, G is a balanced bipartite graph such that $\delta(G) = m = |B|/2$, and we have $|A \cap S| - |B \cap S| \leq 2|B \setminus S|$ and $|B \cap S| - |A \cap S| \leq 2|A \setminus S|$, but G has no 5-S-path-system.

One might think that if we change the minimum degree condition in Theorem 1.7 into $\delta(G) \geq \max\{|A|, |B|\}/2 + \alpha$, where α is a constant, then G has a 3-S-path-system. However, this is not true. Fix a positive integer α. Let k_1, k_2, l_1 and l_2 be positive integers such that k_1 and k_2 are odd, and l_1 and l_2 are large enough and $|l_1 - l_2| \leq k_1 + k_2$. For $i = 1, 2$, let $G_i[A_i, B_i]$ be a complete bipartite graph with partite sets A_i and B_i such that $|A_1| = k_1$, $|A_2| = l_2$, $|B_1| = l_1$ and $|B_2| = k_2$. Let $G[A, B]$ be the bipartite graph with partite sets $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ such that $E(G) = E(G_1) \cup E(G_2) \cup \{ab : a \in A_2, b \in B_1\}$. Let $S = A_1 \cup B_2$ (see Figure 3). Then, $|A \cap S| - |B \cap S| = k_1 - k_2 \leq 2l_1 = 2|B \setminus S|$, $|B \cap S| - |A \cap S| = k_2 - k_1 \leq 2l_2 = 2|A \setminus S|$, and $\delta(G) = \min\{|A_2|, |B_1|\} \geq \max\{|A|, |B|\}/2 + \alpha$, but, G has no 3-S-path-system.

Theorem 1.7 follows from the following two propositions.

Proposition 1.8. *Let $G[A, B]$ be a bipartite graph, and let S be a subset of $V(G)$ such that $|S| \geq 2$ and $|S|$ is even. Suppose that $|A \cap S| - |B \cap S| \leq 2|B \setminus S|$ and $|B \cap S| - |A \cap S| \leq 2|A \setminus S|$. If $\delta(G) \geq \max\{|A|, |B|\}/2$, then there exist $s_1, s_2 \in S$ with $s_1 \neq s_2$ such that $G - \{s_1, s_2\}$ has a 3-$(S \backslash \{s_1, s_2\})$-path-system $\mathcal{P}$, and $N_G(s_i) \cap \big(V(G) \backslash \bigcup_{P \in \mathcal{P}} V(P)\big) \neq \emptyset$ for each $i = 1, 2$.*

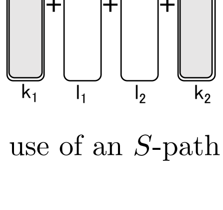

Figure 3: Necessity of the use of an S-path of order j with $j = 4, 5, 6$

Proposition 1.9. *Let $G[A, B]$ be a bipartite graph, and let S be a subset of $V(G)$ such that $|S| \geq 2$ and $|S|$ is even. Suppose that there exist vertices $s_1, s_2 \in S$ which satisfy the conclusion of Proposition 1.8. If $\sigma_{1,1}(G) \geq \max\{|A|, |B|\}$, then one of the following statements holds:*

(i) *G is unbalanced, and G has a 5-S-path-system $\mathcal{P}$ such that $|\mathcal{P}_4 \cup \mathcal{P}_5| \leq 1$;*
(ii) *G is balanced, and G has a 6-S-path-system $\mathcal{P}$ such that $|\mathcal{P}_4 \cup \mathcal{P}_5 \cup \mathcal{P}_6| \leq 1$; or*
(iii) *$G \cong K_{t,t} \cup K_{|B|-t,|B|-t}$ for some integer $1 \leq t \leq |B| - 1$.*

In Sections 2 and 3, we prove Propositions 1.8 and 1.9, respectively.

Our notation is standard, and is mostly taken from Diestel [3]. Possible exceptions are as follows. Let $G[A, B]$ be a bipartite graph. For a set $\mathcal{P}$ of paths in G, let $\mathcal{P}_{AA} = \{P \in \mathcal{P} : \text{end}(P) \subseteq A\}$, $\mathcal{P}_{BB} = \{P \in \mathcal{P} : \text{end}(P) \subseteq B\}$ and $\mathcal{P}_{AB} = \{P \in \mathcal{P} : \text{end}(P) \cap A \neq \emptyset \text{ and } \text{end}(P) \cap B \neq \emptyset\}$. For disjoint subsets X, Y of $V(G)$, let $E_G(X, Y)$ be the set of edges in G between X and Y. For a path P with a given orientation and $x \in V(P)$, we denote the successor of x on P by x^+.

2 Proof of Proposition 1.8

Let $G[A, B]$ be a bipartite graph with $\delta(G) \geq \max\{|A|, |B|\}/2$, and let S be a subset of $V(G)$ which satisfies the assumption of Proposition 1.8. Write $|S| = 2k$ ($k \geq 1$). First we prove the following lemma.

Lemma 2.1. *There exist $s_1, s_2 \in S$ such that G has a 3-($S \backslash \{s_1, s_2\}$)-path-system.*

Proof. We prove Lemma 2.1 by induction on k. By the definition of an S-path-system, we see that Lemma 2.1 is true in the case where $k = 1$. Thus let $k \geq 2$, and assume that the lemma is proved for $k - 1$. Suppose that

(C1) G has no 3-($S \backslash \{s_1, s_2\}$)-path-system for any $s_1, s_2 \in S$.

Let

$$\mathcal{F} = \{\mathcal{P} : \mathcal{P} \text{ is a 3-}(S \backslash S')\text{-path-system for some } S' \subseteq S \text{ with } |S'| = 4\}.$$

Take $s_1, s_2 \in S$ so that $s_1 \neq s_2$. By the induction hypothesis, there exist $s_3, s_4 \in S \backslash \{s_1, s_2\}$ such that $s_3 \neq s_4$ and G has a 3-($S \backslash \{s_1, s_2, s_3, s_4\}$)-path-system. This implies that $\mathcal{F} \neq \emptyset$. By the definition of $\mathcal{F}$, for each $\mathcal{P} \in \mathcal{F}$, there exists $S_{\mathcal{P}} \subseteq S$ such that $|S_{\mathcal{P}}| = 4$ and $\mathcal{P}$ is a 3-($S \backslash S_{\mathcal{P}}$)-path-system. Let $m = \min\{\sum_{P \in \mathcal{P}} |P| : \mathcal{P} \in \mathcal{F}\}$, and let

$$\mathcal{F}_{min} = \{\mathcal{P} \in \mathcal{F} : \sum_{P \in \mathcal{P}} |P| = m\}.$$

Note that for each $\mathcal{P} \in \mathcal{F}_{min}$, $\mathcal{P}_{AA} \cup \mathcal{P}_{BB} = \mathcal{P}_3$ and $\mathcal{P}_{AB} = \mathcal{P}_2$ because $\mathcal{P}$ is a 3-($S \backslash S_{\mathcal{P}}$)-path-system. For each $\mathcal{P} \in \mathcal{F}_{min}$, we have $S_{\mathcal{P}} \cap \bigcup_{P \in \mathcal{P}} V(P) = \emptyset$ by the minimality of m, and hence it follows from (C1) that $S_{\mathcal{P}}$ is an independent set. For each $\mathcal{P} \in \mathcal{F}_{min}$ and each $P \in \mathcal{P}$, write $\text{end}(P) = \{a_P, b_P\}$. We assume that we have $a_P \in A$ and $b_P \in B$ in the case where $P \in \mathcal{P}_{AB}$. Moreover, we regard P as an oriented path from a_P to b_P. For each $\mathcal{P} \in \mathcal{F}_{min}$, let $W_{\mathcal{P}} = V(G) \backslash (S_{\mathcal{P}} \cup \bigcup_{P \in \mathcal{P}} V(P))$.

By the definition of $\mathcal{F}_{min}$, we obtain the following fact.

Fact 2.2. *There exists no* $\mathcal{P} \in \mathcal{F}$ *such that* $\sum_{P \in \mathcal{P}} |P| < m$.

The rest of the proof consists of a number of claims. After stating several claims, we outline the proof in the third paragraph following Claim 2.6.

Claim 2.3.

- **(i)** *If* $\mathcal{P} \in \mathcal{F}_{min}$ *and* $s \in S_{\mathcal{P}} \cap A$*, then* $N_G(s) \cap V(P) = \emptyset$ *for each* $P \in \mathcal{P}_{BB}$.
- **(ii)** *If* $\mathcal{P} \in \mathcal{F}_{min}$ *and* $s \in S_{\mathcal{P}} \cap B$*, then* $N_G(s) \cap V(P) = \emptyset$ *for each* $P \in \mathcal{P}_{AA}$.

Proof. In view of the symmetry of A and B, it suffices to prove (i). Suppose that there exists $c \in N_G(s) \cap V(P)$ for some $\mathcal{P} \in \mathcal{F}_{min}$, $s \in S_{\mathcal{P}} \cap A$ and $P \in \mathcal{P}_{BB}$. Let $P' = sc$, $\{c'\} = \mathrm{end}(P) \backslash \{c\}$ and $S' = (S_{\mathcal{P}} \backslash \{s\}) \cup \{c'\}$. Then, $\mathcal{P}' = (\mathcal{P} \backslash \{P\}) \cup \{P'\}$ is a 3-$(S \backslash S')$-path-system with $|S'| = 4$, and $\sum_{Q \in \mathcal{P}'} |Q| = \sum_{Q \in \mathcal{P}} |Q| - |P| + |P'| = m-1$, which contradicts Fact 2.2. □

Claim 2.4. *If* $\mathcal{P} \in \mathcal{F}_{min}$*,* $s_1 \in S_{\mathcal{P}} \cap A$ *and* $s_2 \in S_{\mathcal{P}} \cap B$*, then* $|N_G(s_1) \cap V(P)| + |N_G(s_2) \cap V(P)| \leq 1$ *for each* $P \in \mathcal{P}_{AB}$.

Proof. Suppose that there exist $b_P \in N_G(s_1)$ and $a_P \in N_G(s_2)$ for some $\mathcal{P} \in \mathcal{F}_{min}$, $s_1 \in S_{\mathcal{P}} \cap A$, $s_2 \in S_{\mathcal{P}} \cap B$ and $P \in \mathcal{P}_{AB}$. Let $P_1 = s_1 b_P$ and $P_2 = s_2 a_P$. Then, $(\mathcal{P} \backslash \{P\}) \cup \{P_1, P_2\}$ is a 3-$(S \backslash (S_{\mathcal{P}} \backslash \{s_1, s_2\}))$-system, which contradicts (C1). □

By (C1), we obtain the following claim.

Claim 2.5.

- **(i)** *If* $\mathcal{P} \in \mathcal{F}_{min}$ *and* $s_1, s_2 \in S_{\mathcal{P}} \cap A$ *with* $s_1 \neq s_2$*, then* $N_G(s_1) \cap N_G(s_2) \cap W_{\mathcal{P}} = \emptyset$.
- **(ii)** *If* $\mathcal{P} \in \mathcal{F}_{min}$ *and* $s_1, s_2 \in S_{\mathcal{P}} \cap B$ *with* $s_1 \neq s_2$*, then* $N_G(s_1) \cap N_G(s_2) \cap W_{\mathcal{P}} = \emptyset$.

Claim 2.6. *We have* $|S_{\mathcal{P}} \cap A| \neq 2$ *for each* $\mathcal{P} \in \mathcal{F}_{min}$.

Proof. Suppose that $|S_{\mathcal{P}} \cap A| = 2$ for some $\mathcal{P} \in \mathcal{F}_{min}$. Write $S_{\mathcal{P}} \cap A = \{s_1, s_2\}$ and $S_{\mathcal{P}} \cap B = \{s_3, s_4\}$. By Claims 2.3 and 2.4, $\sum_{1 \leq i \leq 4} |N_G(s_i) \cap V(P)| \leq 2 \leq |P|$ for each $P \in \mathcal{P}$. By Claim 2.5, $\sum_{1 \leq i \leq 4} |N_G(s_i) \cap W_{\mathcal{P}}| \leq |W_{\mathcal{P}}|$. Since $|G| \leq 4\delta(G)$ by assumption, we get $|G| \leq \sum_{1 \leq i \leq 4} d_G(s_i) \leq \sum_{P \in \mathcal{P}} |P| + |W_{\mathcal{P}}| = |G| - |S_{\mathcal{P}}| < |G|$, a contradiction. □

Hereafter, we fix $\mathcal{P} \in \mathcal{F}_{min}$. We choose $\mathcal{P}$ so that $N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}} \neq \emptyset$ if possible. By Claim 2.6, either $|S_{\mathcal{P}} \cap A| \leq 1$ or $|S_{\mathcal{P}} \cap A| \geq 3$ holds. By the symmetry of A and B, we may assume that $|S_{\mathcal{P}} \cap A| \geq 3$.

Choose distinct $s_1, s_2 \in S_{\mathcal{P}} \cap A$ so that $|N_G(s_1) \cap W_{\mathcal{P}}| + |N_G(s_2) \cap W_{\mathcal{P}}|$ is as small as possible. Let

$$B_1 = \bigcup_{P \in \mathcal{P}_{AB} \cup \mathcal{P}_{AA}} (V(P) \cap B),$$

$$\mathcal{P}_{s_1s_2} = \{P \in \mathcal{P}_{AB} \cup \mathcal{P}_{AA} : N_G(s_1) \cap N_G(s_2) \cap V(P) \neq \emptyset\}$$

and

$$A_0 = \bigcup_{P \in \mathcal{P}_{s_1s_2}} (V(P) \cap A).$$

We here include a sketch of the proof of Lemma 2.1. We show that vertices in A_0 behave like vertices in $\mathcal{S}_{\mathcal{P}} \cap A$ (Claims 2.10 and 2.12). We then take a suitable vertex $x_0 \in A_0$ and, by using x_0, we define $A'_1 \subseteq A$ as we defined A_0 by using s_1 and s_2. From Claims 2.10 and 2.12, it follows that $|A'_1| \geq |A|/2 - 2$ (Clams 2.11 and 2.13). We observe that vertices in A'_1 also behave like vertices in $\mathcal{S}_{\mathcal{P}} \cap A$ (see the paragraph following the proof of Claim 2.13). In the case where $|\mathcal{S}_{\mathcal{P}} \cap A| = 3$, we write $\mathcal{S}_{\mathcal{P}} \cap B = \{s_4\}$ and, based on this observation, we show that $E_G(A'_1, \{s_4\}) = \emptyset$ which, in view of Claim 2.13, contradicts the minimum degree condition in Proposition 1.8 (Claim 2.16). In the case where $|\mathcal{S}_{\mathcal{P}} \cap A| = 4$, we have $(N_G(\mathcal{S}_{\mathcal{P}}) \cap W_{\mathcal{P}}) \cup \bigcup_{P \in \mathcal{P}_{BB}} (V(P) \cap B) \neq \emptyset$, and thus take $b_0 \in (N_G(\mathcal{S}_{\mathcal{P}}) \cap W_{\mathcal{P}}) \cup \bigcup_{P \in \mathcal{P}_{BB}} (V(P) \cap B)$ and, again based on the above observation, we can show that $E_G(A'_1, \{b_0\}) = \emptyset$ (Claims 2.14 and 2.15), which contradicts the minimum degree condition (see the last paragraph of the proof of the lemma).

Now by Claim 2.5 and the choice of s_1 and s_2, we obtain the following claim.

Claim 2.7. $|N_G(s_1) \cap W_{\mathcal{P}}| + |N_G(s_2) \cap W_{\mathcal{P}}| \leq 2|W_{\mathcal{P}} \cap B|/|S_{\mathcal{P}} \cap A|$.

Claim 2.8. $\max\{|A|, |B|\} - 2|W_{\mathcal{P}} \cap B|/|S_{\mathcal{P}} \cap A| - |B_1| \leq |N_G(s_1) \cap N_G(s_2) \cap B_1| \leq |A_0|$.

Proof. By Claim 2.3, $|N_G(s_1) \cap V(P)| + |N_G(s_2) \cap V(P)| = 0$ for each $P \in \mathcal{P}_{BB}$. Hence by Claim 2.7,

$$\begin{aligned} |N_G(s_1) \cap N_G(s_2) \cap B_1| &\geq |N_G(s_1) \cap B_1| + |N_G(s_2) \cap B_1| - |B_1| \\ &\geq 2\delta(G) - 2|W_{\mathcal{P}} \cap B|/|S_{\mathcal{P}} \cap A| - |B_1| \\ &\geq \max\{|A|, |B|\} - 2|W_{\mathcal{P}} \cap B|/|S_{\mathcal{P}} \cap A| - |B_1|. \end{aligned}$$

Since $|V(P) \cap B| \leq |V(P) \cap A|$ for each $P \in \mathcal{P}_{s_1s_2}$, we also have $|N_G(s_1) \cap N_G(s_2) \cap B_1| \leq |A_0|$. □

Claim 2.9. $N_G(s_1) \cap N_G(s_2) \cap B_1 \neq \emptyset$.

Proof. Suppose that $N_G(s_1) \cap N_G(s_2) \cap B_1 = \emptyset$. Since $|S_{\mathcal{P}} \cap A| \geq 3$, we clearly have $2|W_{\mathcal{P}}|/|S_{\mathcal{P}} \cap A| \leq 2|W_{\mathcal{P}}|/3$. Hence it follows from Claim 2.8 that

$$\begin{aligned} 0 = |N_G(s_1) \cap N_G(s_2) \cap B_1| &\geq \max\{|A|, |B|\} - 2|W_{\mathcal{P}} \cap B|/3 - |B_1| \\ &\geq \max\{|A|, |B|\} - |W_{\mathcal{P}} \cap B| - |B_1| \\ &\geq \max\{|A|, |B|\} - |B|. \end{aligned}$$

Consequently $|B| \geq |A|$ and $W_{\mathcal{P}} \cap B = \emptyset$, which further implies $B = B_1$, and hence $|S_{\mathcal{P}} \cap B| = 0$ and $|S_{\mathcal{P}} \cap A| = 4$. Since $|V(P) \cap A| \geq |V(P) \cap B|$ for each $P \in \mathcal{P}_{AB} \cup \mathcal{P}_{AA}$, we now get $|A| = |A \backslash S_{\mathcal{P}}| + 4 \geq |B_1| + 4 = |B| + 4$, which contradicts the assertion that $|B| \geq |A|$. □

For each $x \in A_0$, let $P_x \in \mathcal{P}_{s_1s_2}$ be the path such that $x \in V(P_x)$.

Claim 2.10. *We have $N_G(x) \cap N_G(y) \cap W_{\mathcal{P}} = \emptyset$ for any $x, y \in A_0$ with $x \neq y$.*

Proof. Let $x, y \in A_0$ with $x \neq y$. By symmetry, we may assume that (i) $P_x, P_y \in \mathcal{P}_{AB}$, (ii) $P_x \in \mathcal{P}_{AB}$ and $P_y \in \mathcal{P}_{AA}$, (iii) $P_x, P_y \in \mathcal{P}_{AA}$ and $P_x \neq P_y$, or (iv) $P_x, P_y \in \mathcal{P}_{AA}$ and $P_x = P_y$.

Recall that for each $P \in \mathcal{P}$, $\mathrm{end}(P) = \{a_P, b_P\}$ and P is oriented from a_P to b_P and, in the case where $P \in \mathcal{P}_{AB}$, we have $a_P \in A$ and $b_P \in B$.

We may assume that $y = a_{P_y}$ in the case where (ii) holds, $x = a_{P_x}$ and $y = a_{P_y}$ in the case where (iii) holds, and $x = a_{P_x}$ in the case where (iv) holds. Let

$$\mathcal{P}' = \begin{cases} (\mathcal{P} \backslash \{P_x, P_y\}) \cup \{s_1 b_{P_x}, s_2 b_{P_y}\} & \text{if (i) holds,} \\ (\mathcal{P} \backslash \{P_x, P_y\}) \cup \{s_1 b_{P_x}, s_2 y^+ b_{P_y}\} & \text{if (ii) holds,} \\ (\mathcal{P} \backslash \{P_x, P_y\}) \cup \{s_1 x^+ b_{P_x}, s_2 y^+ b_{P_y}\} & \text{if (iii) holds,} \\ (\mathcal{P} \backslash \{P_x\}) \cup \{s_1 x^+ s_2\} & \text{if (iv) holds.} \end{cases}$$

Then, $\mathcal{P}'$ is a 3-$(S \backslash ((S_{\mathcal{P}} \backslash \{s_1, s_2\}) \cup \{x, y\}))$-path-system such that $\mathcal{P}' \in \mathcal{F}_{min}$, $W_{\mathcal{P}'} \cap B = W_{\mathcal{P}} \cap B$ and $x, y \in S_{\mathcal{P}'} \cap A$. Hence applying Claim 2.5 (i) to $\mathcal{P}'$, we get $N_G(x) \cap N_G(y) \cap W_{\mathcal{P}} = \emptyset$. □

Note that $A_0 \neq \emptyset$ by Claims 2.8 and 2.9. Choose $x_0 \in A_0$ so that $|N_G(x_0) \cap W_{\mathcal{P}}|$ is as small as possible.

Claim 2.11. $|N_G(x_0) \cap W_{\mathcal{P}}| \leq 2$.

Proof. If $W_{\mathcal{P}} \cap B = \emptyset$, then $|N_G(x_0) \cap W_{\mathcal{P}}| = 0 \leq 2$. Thus we may assume that $W_{\mathcal{P}} \cap B \neq \emptyset$, which implies that $|B| - 2|W_{\mathcal{P}} \cap B|/|S_{\mathcal{P}} \cap A| - |B_1| > |B| - |W_{\mathcal{P}} \cap B| - |B_1| \geq 0$. Hence, in view of the minimality of $|N_G(x_0) \cap W_{\mathcal{P}}|$, it follows from Claims 2.8 and 2.10 that

$$\begin{aligned} |N_G(x_0) \cap W_{\mathcal{P}}| &\leq |W_{\mathcal{P}} \cap B|/|A_0| \\ &\leq |W_{\mathcal{P}} \cap B|/(|B| - 2|W_{\mathcal{P}} \cap B|/|S_{\mathcal{P}} \cap A| - |B_1|). \end{aligned} \tag{1}$$

If $|S_{\mathcal{P}} \cap A| = 4$, then by (1), we obtain

$$\begin{aligned} |N_G(x_0) \cap W_{\mathcal{P}}| &\leq |W_{\mathcal{P}} \cap B|/(|B| - |W_{\mathcal{P}} \cap B|/2 - |B_1|) \\ &\leq |W_{\mathcal{P}} \cap B|/(|W_{\mathcal{P}} \cap B|/2) = 2. \end{aligned}$$

If $|S_{\mathcal{P}} \cap A| = 3$, then by (1), we obtain

$$\begin{aligned} |N_G(x_0) \cap W_{\mathcal{P}}| &\leq |W_{\mathcal{P}} \cap B|/(|B| - 2|W_{\mathcal{P}} \cap B|/3 - |B_1|) \\ &\leq |W_{\mathcal{P}} \cap B|/(|W_{\mathcal{P}} \cap B|/3 + |S_{\mathcal{P}} \cap B|) \\ &< |W_{\mathcal{P}} \cap B|/(|W_{\mathcal{P}} \cap B|/3) = 3. \end{aligned}$$

□

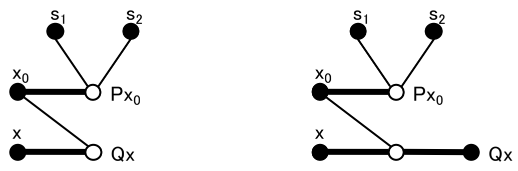

Figure 4: Definition of Q_x (the case where $x \neq x_0$ and $P_{x_0} \in \mathcal{P}_{AB}$)

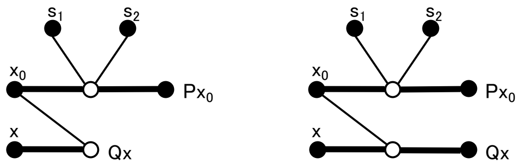

Figure 5: Definition of Q_x (the case where $x \neq x_0$ and $P_{x_0} \in \mathcal{P}_{AA}$)

Claim 2.12. *$E_G(\{x_0\}, V(P)) = \emptyset$ for each $P \in \mathcal{P}_{BB}$.*

Proof. Note that if $P_{x_0} \in \mathcal{P}_{AA}$, then we may assume that $x_0 = a_{P_{x_0}}$. Under this assumption, let $P' = s_1 b_{P_{x_0}}$ or $P' = s_1 a^+_{P_{x_0}} b_{P_{x_0}}$ according as $P_{x_0} \in \mathcal{P}_{AB}$ or $P_{x_0} \in \mathcal{P}_{AA}$. Then, $\mathcal{P}' = (\mathcal{P} \backslash \{P_{x_0}\}) \cup \{P'\}$ is a 3-$S\backslash((S_{\mathcal{P}} \backslash \{s_1\}) \cup \{x_0\})$-path-system such that $\mathcal{P}' \in \mathcal{F}_{min}$ and $\mathcal{P}'_{BB} = \mathcal{P}_{BB}$ and $x_0 \in S_{\mathcal{P}'} \cap A$. Hence applying Claim 2.3 (i) to $\mathcal{P}'$, we see that $E_G(\{x_0\}, V(P)) = \emptyset$ for each $P \in \mathcal{P}_{BB}$. □

Let

$$\mathcal{P}_{x_0} = \{P \in \mathcal{P}_{AB} \cup \mathcal{P}_{AA} : N_G(x_0) \cap V(P) \neq \emptyset\}.$$

We have $P_{x_0} \in \mathcal{P}_{x_0}$. Let

$$A'_1 = \bigcup_{P \in \mathcal{P}_{x_0}} (V(P) \cap A).$$

For each $x \in A'_1$, let $Q_x \in \mathcal{P}_{x_0}$ be the path such that $x \in V(Q_x)$ (see Figures 4 through 6; in these figures, bold lines indicate P_{x_0} and Q_x). Note that $Q_{x_0} = P_{x_0}$.

Claim 2.13. *$|A'_1| \geq |A|/2 - 2$.*

Proof. Since $|V(P) \cap A| \geq |V(P) \cap B|$ for each $P \in \mathcal{P}_{x_0}$, it follows from Claims 2.11 and 2.12 that $|A'_1| \geq |\mathcal{P}_{x_0}| = d_G(x_0) - |N_G(x_0) \cap W_{\mathcal{P}}| \geq |A|/2 - 2$. □

For each $i = 1, 2$ and each $x \in A'_1$, we define $\mathcal{P}(i, x) \in \mathcal{F}_{min}$ as follows.

Case I. $P_{x_0} \in \mathcal{P}_{AB}$.

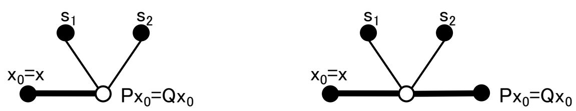

Figure 6: Definition of Q_x (the case where $x = x_0$)

Note that if $Q_x \in \mathcal{P}_{AA}$, then we may assume that $x = a_{Q_x}$. Under this assumption, let

$$\mathcal{P}(i,x)=\begin{cases}(\mathcal{P}\backslash\{Q_x\})\cup\{s_i b_{P_{x_0}}\} & \text{if } Q_x=P_{x_0},\\ (\mathcal{P}\backslash\{P_{x_0},Q_x\})\cup\{s_i b_{P_{x_0}},x_0 b_{Q_x}\} & \text{if } Q_x\in\mathcal{P}_{AB} \text{ and } Q_x\neq P_{x_0},\\ (\mathcal{P}\backslash\{P_{x_0},Q_x\})\cup\{s_i b_{P_{x_0}},x_0 x^+ b_{Q_x}\} & \text{if } Q_x\in\mathcal{P}_{AA}.\end{cases}$$

Case II. $P_{x_0} \in \mathcal{P}_{AA}$.

If $Q_x = P_{x_0}$, then we may assume that $x = a_{P_{x_0}}$; if $Q_x \in \mathcal{P}_{AA}$ and $Q_x \neq P_{x_0}$, then we may assume that $x_0 = a_{P_{x_0}}$ and $x = a_{Q_x}$; if $Q_x \in \mathcal{P}_{AB}$, then we may assume that $x_0 = a_{P_{x_0}}$. Let

$$\mathcal{P}(i,x)=\begin{cases}(\mathcal{P}\backslash\{Q_x\})\cup\{s_i x^+ b_{P_{x_0}}\} & \text{if } Q_x=P_{x_0},\\ (\mathcal{P}\backslash\{P_{x_0},Q_x\})\cup\{s_i x_0^+ b_{P_{x_0}},x_0 x^+ b_{Q_x}\} & \text{if } Q_x\in\mathcal{P}_{AA} \text{ and } Q_x\neq P_{x_0},\\ (\mathcal{P}\backslash\{P_{x_0},Q_x\})\cup\{s_i x_0^+ b_{P_{x_0}},x_0 b_{Q_x}\} & \text{if } Q_x\in\mathcal{P}_{AB}.\end{cases}$$

Note that $\mathcal{P}(i,x) \in \mathcal{F}_{min}$, $S_{\mathcal{P}(i,x)} = (S_{\mathcal{P}}\backslash\{s_i\}) \cup \{x\}$, $W_{\mathcal{P}(i,x)} = W_{\mathcal{P}}$ and $\mathcal{P}(i,x)_{BB} = \mathcal{P}_{BB}$.

Claim 2.14. *$E_G(A_1', N_G(S_{\mathcal{P}} \cap A) \cap W_{\mathcal{P}}) = \emptyset$.*

Proof. Let $x \in A_1'$ and $y \in S_{\mathcal{P}} \cap A$. By the symmetry of s_1 and s_2, we may assume that $y \neq s_1$. Note that $\mathcal{P}(1,x) \in \mathcal{F}_{min}$ and $x, y \in S_{\mathcal{P}(1,x)} \cap A$. Hence by Claim 2.5 (i), $N_G(x) \cap N_G(y) \cap W_{\mathcal{P}(1,x)} = \emptyset$. Since $W_{\mathcal{P}(1,x)} = W_{\mathcal{P}}$, this implies that $E_G(\{x\}, N_G(y) \cap W_{\mathcal{P}}) = \emptyset$. Since $x \in A_1'$ and $y \in S_{\mathcal{P}} \cap A$ are arbitrary, we get $E_G(A_1', N_G(S_{\mathcal{P}} \cap A) \cap W_{\mathcal{P}}) = \emptyset$. □

Claim 2.15. *$E_G(A_1', V(P)) = \emptyset$ for each $P \in \mathcal{P}_{BB}$.*

Proof. Let $x \in A_1'$ and $P \in \mathcal{P}_{BB}$. Then, since $\mathcal{P}(1,x) \in \mathcal{F}_{min}$, $\mathcal{P}(1,x)_{BB} = \mathcal{P}_{BB}$ and $x \in S_{\mathcal{P}(1,x)} \cap A$, it follows from Claim 2.3 (i) that $N_G(x) \cap V(P) = \emptyset$. Hence $E_G(A_1', V(P)) = \emptyset$ for each $P \in \mathcal{P}_{BB}$. □

Claim 2.16. *$S_{\mathcal{P}} \subseteq A$.*

Proof. Suppose not. Write $(S_{\mathcal{P}}\backslash\{s_1, s_2\}) \cap A = \{s_3\}$ and $(S_{\mathcal{P}}\backslash\{s_1, s_2\}) \cap B = \{s_4\}$. For each $x \in A_1'$, $xs_4 \notin E(G)$ because $x, s_4 \in S_{\mathcal{P}(1,x)}$ and $S_{\mathcal{P}(1,x)}$ is an independent set. Hence $E_G(A_1', \{s_4\}) = \emptyset$. By (C1), $N_G(s_4) \cap \{s_1, s_2, s_3\} = \emptyset$. Hence by Claim 2.13, $d_G(s_4) \leq |A| - |A_1'| - 3 \leq |A|/2 - 1$, which contradicts the minimum degree condition. □

Claim 2.17. *If $N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}} = \emptyset$, then $E_G(A_1', W_{\mathcal{P}}) = \emptyset$.*

Proof. Suppose that $N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}} = \emptyset$ and $E_G(A_1', W_{\mathcal{P}}) \neq \emptyset$. Then, there exists $x \in A_1'$ such that $N_G(x) \cap W_{\mathcal{P}} \neq \emptyset$. Since $W_{\mathcal{P}(1,x)} = W_{\mathcal{P}}$ and $x \in S_{\mathcal{P}(1,x)}$, this implies that $N_G(S_{\mathcal{P}(1,x)}) \cap W_{\mathcal{P}(1,x)} \neq \emptyset$. Hence $N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}} \neq \emptyset$ by the choice of $\mathcal{P}$, a contradiction. □

Claim 2.18. *If $W_{\mathcal{P}} \cap B \neq \emptyset$, then $N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}} \neq \emptyset$.*

Proof. Suppose that $W_{\mathcal{P}} \cap B \neq \emptyset$ and $N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}} = \emptyset$, and take $b \in W_{\mathcal{P}} \cap B$. Since $N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}} = \emptyset$, it follows from Claim 2.17 that $E_G(A'_1, W_{\mathcal{P}}) = \emptyset$. Hence $d_G(b) \leq |A| - |A'_1 \cup (S_{\mathcal{P}} \cap A)| \leq |A|/2 - 2$ by Claims 2.13 and 2.16, which contradicts the minimum degree condition. □

Suppose that $W_{\mathcal{P}} \cap B = \emptyset$ and $\mathcal{P}_{BB} = \emptyset$. Then, $\mathcal{P} = \mathcal{P}_{AB} \cup \mathcal{P}_{AA}$, and it follows from Claim 2.16 that $B = \bigcup_{P \in \mathcal{P}} (V(P) \cap B)$. Let $P \in \mathcal{P}$. If $P \in \mathcal{P}_{AA}$, then $|V(P) \cap A \cap S| = 2$, $|V(P) \cap B \cap S| = 0$ and $|V(P) \cap (B \backslash S)| = 1$; if $P \in \mathcal{P}_{AB}$, then $|V(P) \cap A \cap S| = 1$, $|V(P) \cap B \cap S| = 1$ and $|V(P) \cap (B \backslash S)| = 0$. In either case, $|V(P) \cap A \cap S| - |V(P) \cap B \cap S| = 2|V(P) \cap (B \backslash S)|$. Since P is arbitrary, we obtain $|(A \cap S) \backslash S_{\mathcal{P}}| - |B \cap S| = 2|B \backslash S|$. Since $|A \cap S| = |(A \cap S) \backslash S_{\mathcal{P}}| + 4$ by Claim 2.16, it follows that $|A \cap S| - |B \cap S| = 2|B \backslash S| + 4$, which contradicts the assumption that $|A \cap S| - |B \cap S| \leq 2|B \backslash S|$. Thus either $W_{\mathcal{P}} \cap B \neq \emptyset$ or $\mathcal{P}_{BB} \neq \emptyset$. By Claim 2.18, this implies that there exists $b_0 \in \big(N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}}\big) \cup \bigcup_{P \in \mathcal{P}_{BB}} (V(P) \cap B)$. If $b_0 \in N_G(S_{\mathcal{P}}) \cap W_{\mathcal{P}}$, then $|N_G(b_0) \cap S_{\mathcal{P}}| = 1$ by Claim 2.5 (i); if $b_0 \in \bigcup_{P \in \mathcal{P}_{BB}} (V(P) \cap B)$, then $N_G(b_0) \cap S_{\mathcal{P}} = \emptyset$ by Claim 2.3 (i). Hence by Claims 2.13, 2.14, 2.15 and 2.16, $|N_G(b_0)| \leq |A| - |A'_1| - 3 \leq |A|/2 - 1$, which contradicts the minimum degree condition. This completes the proof of Lemma 2.1. □

By Lemma 2.1, there exist $s_1, s_2 \in S$ such that G has a 3-($S \backslash \{s_1, s_2\}$)-path-system $\mathcal{P}$. Choose such s_1, s_2 and $\mathcal{P}$ so that

(T1) $\sum_{P \in \mathcal{P}} |P|$ is as small as possible, and
(T2) $\sum_{i=1,2} |N_G(s_i) \backslash \bigcup_{P \in \mathcal{P}} V(P)|$ is as large as possible, subject to (T1).

By (T1), $s_1, s_2 \notin \bigcup_{P \in \mathcal{P}} V(P)$. For each $P \in \mathcal{P}$, let a_P, b_P be as in the proof of Lemma 2.1. We show that $N_G(s_i) \backslash \bigcup_{P \in \mathcal{P}} V(P) \neq \emptyset$ for each $i \in \{1, 2\}$. Suppose that for some $i \in \{1, 2\}$, we have $N_G(s_i) \backslash \bigcup_{P \in \mathcal{P}} V(P) = \emptyset$. By the symmetry of s_1 and s_2, we may assume that $N_G(s_1) \backslash \bigcup_{P \in \mathcal{P}} V(P) = \emptyset$. By the symmetry of A and B, we may assume that $s_1 \in A$. By (T1), we obtain the following claim.

Claim 2.19. *We have $N_G(s_1) \cap V(P) = \emptyset$ for each $P \in \mathcal{P}_{BB}$.*

Let

$$B_1 = \bigcup_{P \in \mathcal{P}_{AB} \cup \mathcal{P}_{AA}} (V(P) \cap B).$$

By Claim 2.19 and the minimum degree condition, $|N_G(s_1) \cap B_1| = |N_G(s_1)| \geq |A|/2$. Let

$$\mathcal{P}_{s_1} = \{P \in \mathcal{P}_{AB} \cup \mathcal{P}_{AA} : N_G(s_1) \cap V(P) \neq \emptyset\},$$

$$A_0 = \bigcup_{P \in \mathcal{P}_{s_1}} (V(P) \cap A).$$

Then, $|A_0| \geq |\mathcal{P}_{s_1}| = |N_G(s_1) \cap B_1| \geq |A|/2$. Let $x \in A_0$, and let $P_x \in \mathcal{P}_{s_1}$ be the path such that $x \in V(P_x)$. Note that if $P_x \in \mathcal{P}_{AA}$, then we may assume that $x = a_{P_x}$.

Under this assumption, let

$$\mathcal{P}' = \begin{cases} (\mathcal{P}\backslash\{P_x\}) \cup \{s_1 b_{P_x}\} & \text{if } P_x \in \mathcal{P}_{AB}, \\ (\mathcal{P}\backslash\{P_x\}) \cup \{s_1 x^+ b_{P_x}\} & \text{if } P_x \in \mathcal{P}_{AA}. \end{cases}$$

Then $\mathcal{P}'$ is a 3-$(S\backslash\{x, s_2\})$-path-system such that

$$\sum_{P\in\mathcal{P}'} |P| = \sum_{P\in\mathcal{P}} |P|, \quad V(G)\backslash \bigcup_{P\in\mathcal{P}'} V(P) = ((V(G)\backslash \bigcup_{P\in\mathcal{P}} V(P))\backslash\{s_1\}) \cup \{x\}$$

and $\mathcal{P}'_{BB} = \mathcal{P}_{BB}$. Note that $s_1 \notin N_G(s_2)$ because $N_G(s_1)\backslash \bigcup_{P\in\mathcal{P}} V(P) = \emptyset$. This implies that $|N_G(s_2)\backslash \bigcup_{P\in\mathcal{P}'} V(P)| \geq |N_G(s_2)\backslash \bigcup_{P\in\mathcal{P}} V(P)|$.

Hence $N_G(x)\backslash \bigcup_{P\in\mathcal{P}'} V(P) = \emptyset$ by (T2) and, applying Claim 2.19 to $\mathcal{P}'$, we see that $N_G(x) \cap V(P) = \emptyset$ for each $P \in \mathcal{P}_{BB}$. Since $x \in A_0$ is arbitrary, it follows that $N_G(A_0) \subseteq B_1$.

Suppose that $B = B_1$. Then, $\mathcal{P}_{BB} = \emptyset$ and $s_2 \in A$. Since $|V(P)\cap A\cap S| - |V(P)\cap B\cap S| = 2|V(P)\cap(B\backslash S)|$ for each $P \in \mathcal{P}_{AA} \cup \mathcal{P}_{AB}$, it follows that $|A\cap S| - |B\cap S| = |(A\cap S)\backslash\{s_1, s_2\}| - |B\cap S| + 2 = 2|B\backslash S| + 2$, which contradicts the assumption that $|A\cap S| - |B\cap S| \leq 2|B\backslash S|$. Thus $B\backslash B_1 \neq \emptyset$. Take $b_0 \in B\backslash B_1$ (it is possible that $b_0 = s_2$). Then, $b_0 \in B\backslash \bigcup_{P\in\mathcal{P}} V(P)$ or $b_0 \in \bigcup_{P\in\mathcal{P}_{BB}}(V(P) \cap B)$. If $b_0 \in B\backslash \bigcup_{P\in\mathcal{P}} V(P)$, then $s_1 \notin N_G(b_0)$ by the assumption that $N_G(s_1)\backslash \bigcup_{P\in\mathcal{P}} V(P) = \emptyset$; if $b_0 \in \bigcup_{P\in\mathcal{P}_{BB}}(V(P)\cap B)$, then $s_1 \notin N_G(b_0)$ by Claim 2.19. Hence $s_1 \notin N_G(b_0)$. Since $N_G(A_0) \subseteq B_1$, we also have $N_G(b_0)\cap A_0 = \emptyset$. Therefore, $|N_G(b_0)| \leq |A| - |A_0| - |\{s_1\}| \leq |A| - |A|/2 - 1$, which contradicts the assumption that $\delta(G) \geq \max\{|A|, |B|\}/2$. This completes the proof of Proposition 1.8. □

3 Proof of Proposition 1.9

In this section, we present a proof of Proposition 1.9. The proof is almost the same as that of Theorem 1.6 described in [6]. However, for the convenience of the reader, we here include a rather detailed description of the proof of Proposition 1.9.

Proof of Proposition 1.9. Let $G[A, B]$ be a bipartite graph with $\sigma_{1,1}(G) \geq \max\{|A|, |B|\}$, and let S be a subset of $V(G)$ which satisfies the assumption of Proposition 1.9. Suppose that

(D1) G satisfies neither (i) nor (ii) of Proposition 1.9.

The following claim immediately follows from the assumption of Proposition 1.9.

Claim 3.1. *There exist $s_1, s_2 \in S$ with $s_1 \neq s_2$ such that $G - \{s_1, s_2\}$ has a 3-$(S\backslash\{s_1, s_2\})$-path-system $\mathcal{P}$ and $N_G(s_i)\backslash \bigcup_{P\in\mathcal{P}} V(P) \neq \emptyset$ for each $i \in \{1, 2\}$.*

Let $s_1, s_2, \mathcal{P}$ be as in Claim 3.1. Take $t_i \in N_G(s_i)\backslash \bigcup_{P\in\mathcal{P}} V(P)$ for each $i \in \{1, 2\}$. By (D1), $\{s_1, t_1\} \cap \{s_2, t_2\} = \emptyset$ and $E_G(\{s_1, t_1\}, \{s_2, t_2\}) = \emptyset$. For each $i \in \{1, 2\}$, write $\{s_i, t_i\} = \{x_i, y_i\}$ with $x_i \in A$ and $y_i \in B$.

Before proving the next claim, we outline the proof of Proposition 1.9. We first prove inequalities concerning the number of edges between $\{x_1, y_1, x_2, y_2\}$ and $V(P)$ with $P \in \mathcal{P}$ (Claims 3.2 and 3.3). Based on Claim 3.3, we show that the vertices of G can be classified into several types (Claim 3.6). Using (D1), we show the nonexistence of edges between some two types of vertices described in Claim 3.6 (Claims 3.7 through 3.10). Finally we derive the desired conclusion from Claims 3.9 and 3.10 and the degree condition in the proposition.

Claim 3.2. *Let $P \in \mathcal{P}$.*

(i) *For each $i \in \{1, 2\}$, we have $N_G(x_i) \cap V(P) = \emptyset$ or $N_G(y_{3-i}) \cap V(P) = \emptyset$.*

(ii) *If $N_G(x_1) \cap V(P) \neq \emptyset$ and $N_G(x_2) \cap V(P) \neq \emptyset$, then $|(N_G(x_1) \cup N_G(x_2)) \cap V(P)| = 1$.*

(iii) *If $N_G(y_1) \cap V(P) \neq \emptyset$ and $N_G(y_2) \cap V(P) \neq \emptyset$, then $|(N_G(y_1) \cup N_G(y_2)) \cap V(P)| = 1$.*

Proof. We prove (i), (ii) and (iii) simultaneously. If $N_G(x_i) \cap V(P) \neq \emptyset$ and $N_G(y_{3-i}) \cap V(P) \neq \emptyset$ for some $i \in \{1, 2\}$, or $N_G(x_1) \cap V(P) \neq \emptyset$ and $N_G(x_2) \cap V(P) \neq \emptyset$ and $|(N_G(x_1) \cup N_G(x_2)) \cap V(P)| = 2$, or $N_G(y_1) \cap V(P) \neq \emptyset$ and $N_G(y_2) \cap V(P) \neq \emptyset$ and $|(N_G(y_1) \cup N_G(y_2)) \cap V(P)| = 2$, then there exist disjoint S-paths Q, Q' such that $V(Q) \cup V(Q') \subseteq V(P) \cup \{x_1, y_1, x_2, y_2\}$, $\min\{|Q|, |Q'|\} \leq 3$ and $\max\{|Q|, |Q'|\} \leq 4$, and hence $\mathcal{P}' = (\mathcal{P} \backslash \{P\}) \cup \{Q, Q'\}$ is a 4-S-path-system such that $|\mathcal{P}'_4| \leq 1$, which contradicts (D1). □

For each $P \in \mathcal{P}_2$, write $V(P) = \{a_P, b_P\}$ with $a_P \in A$ and $b_P \in B$. For $i = 1, 2$, let

$$\mathcal{P}(x_i, y_i) = \{P \in \mathcal{P} : V(P) \subseteq N_G(x_i) \cup N_G(y_i)\}.$$

Also let

$$\mathcal{P}_2(x_1, x_2) = \{P \in \mathcal{P}_2 : b_P \in N_G(x_1) \cap N_G(x_2)\},$$
$$\mathcal{P}_2(y_1, y_2) = \{P \in \mathcal{P}_2 : a_P \in N_G(y_1) \cap N_G(y_2)\}.$$

It follows from Claim 3.2 (i) that $\mathcal{P}(x_1, y_1)$, $\mathcal{P}(x_2, y_2)$, $\mathcal{P}_2(x_1, x_2)$ and $\mathcal{P}_2(y_1, y_2)$ are pairwise disjoint.

Claim 3.3. *For each $P \in \mathcal{P}$, $\sum_{i=1,2}(|N_G(x_i) \cap V(P)| + |N_G(y_i) \cap V(P)|) \leq |P|$. Further, if equality holds, then $P \in \mathcal{P}(x_1, y_1) \cup \mathcal{P}(x_2, y_2) \cup \mathcal{P}_2(x_1, x_2) \cup \mathcal{P}_2(y_1, y_2)$.*

Proof. Let $P \in \mathcal{P}$. By Claim 3.2 (i), we have $N_G(x_1) \cap V(P) = \emptyset$ or $N_G(y_2) \cap V(P) = \emptyset$. By symmetry, we may assume that $N_G(x_1) \cap V(P) = \emptyset$. By Claim 3.2 (i), we also have $N_G(x_2) \cap V(P) = \emptyset$ or $N_G(y_1) \cap V(P) = \emptyset$.

First assume that $N_G(x_2) \cap V(P) = \emptyset$. Since $|N_G(y_1) \cap V(P)| + |N_G(y_2) \cap V(P)| \leq 2$ by Claim 3.2 (iii), we get

$$\begin{aligned} &\sum_{i=1,2}(|N_G(x_i) \cap V(P)| + |N_G(y_i) \cap V(P)|) \\ =\ & |N_G(y_1) \cap V(P)| + |N_G(y_2) \cap V(P)| \leq 2 \leq |P|. \end{aligned}$$

If equality holds, then $P \in \mathcal{P}_2$, and hence $|N_G(y_1) \cap V(P)| = 1$ and $|N_G(y_2) \cap V(P)| = 1$, which implies that $P \in \mathcal{P}_2(y_1, y_2)$.

Next assume that $N_G(y_1) \cap V(P) = \emptyset$. Then,

$$\begin{aligned} & \sum_{i=1,2} (|N_G(x_i) \cap V(P)| + |N_G(y_i) \cap V(P)|) \\ = \; & |N_G(x_2) \cap V(P)| + |N_G(y_2) \cap V(P)| \\ \leq \; & |V(P) \cap B| + |V(P) \cap A| = |P|. \end{aligned}$$

If equality holds, then $N_G(x_2) \cap V(P) = V(P) \cap B$ and $N_G(y_2) \cap V(P) = V(P) \cap A$, and hence $V(P) \subseteq N_G(x_2) \cup N_G(y_2)$, which implies $P \in \mathcal{P}(x_2, y_2)$. □

Let $W = V(G) \setminus \bigcup_{P \in \mathcal{P}} V(P)$ and, for each $i \in \{1, 2\}$, let $W(x_i) = N_G(x_i) \cap W$ and $W(y_i) = N_G(y_i) \cap W$ (note that $y_i \in W(x_i)$ and $x_i \in W(y_i)$). Recall that $\{x_i, y_i\} = \{s_i, t_i\}$. The following claim follows from the definition of $\mathcal{P}(x_i, y_i)$, $W(x_i)$ and $W(y_i)$.

Claim 3.4. *Let $i \in \{1, 2\}$.*

(i) *For each $P \in \mathcal{P}(x_i, y_i)$ and each $z \in V(P)$, there exists an S-path Q such that $|Q| \leq 3$, $V(Q) \subseteq \{x_i, y_i\} \cup V(P)$, $z \notin V(Q)$ and $|V(Q) \cap V(P)| = 1$.*

(ii) *For each $z \in W(x_i) \cup W(y_i)$, there exists a path R such that $|R| \leq 3$, $V(R) \subseteq \{x_i, y_i, z\}$ and $\mathrm{end}(R) = \{s_i, z\}$.*

Claim 3.5. *We have $W(x_1) \cap W(x_2) = \emptyset$ and $W(y_1) \cap W(y_2) = \emptyset$.*

Proof. Suppose that $W(x_1) \cap W(x_2) \neq \emptyset$ or $W(y_1) \cap W(y_2) \neq \emptyset$, and take $z \in (W(x_1) \cap W(x_2)) \cup (W(y_1) \cap W(y_2))$. By Claim 3.4 (ii), there exist paths R, R' such that $V(R) \subseteq \{x_1, y_1, z\}$, $\mathrm{end}(R) = \{s_1, z\}$, $V(R') \subseteq \{x_2, y_2, z\}$ and $\mathrm{end}(R) = \{s_2, z\}$. Then, $R \cup R'$ is an S-path with $|R \cup R'| \leq 5$, which contradicts (D1). □

Recall that $x_1 y_2, x_2 y_1 \notin E(G)$. Hence by Claims 3.3 and 3.5,

$$\begin{aligned} |G| & \leq 2 \max\{|A|, |B|\} \\ & \leq 2\sigma_{1,1}(G) \\ & \leq (d_G(x_1) + d_G(y_2)) + (d_G(x_2) + d_G(y_1)) \\ & \leq \sum_{P \in \mathcal{P}} |P| + |W| = |G|. \end{aligned}$$

Hence equality holds throughout, which, together with Claim 3.3, implies the following claim.

Claim 3.6.

(i) $\sigma_{1,1}(G) = |A| = |B|$.

(ii) $\mathcal{P} = \mathcal{P}(x_1, y_1) \cup \mathcal{P}(x_2, y_2) \cup \mathcal{P}_2(x_1, x_2) \cup \mathcal{P}_2(y_1, y_2)$.

(iii) $W = \bigcup_{i=1,2} (W(x_i) \cup W(y_i))$.

For $i = 1, 2$, let $V_i = W(x_i) \cup W(y_i) \cup \bigcup_{P \in \mathcal{P}(x_i, y_i)} V(P)$. Note that $V(G) = V_1 \cup V_2 \cup \bigcup_{P \in \mathcal{P}_2(x_1, x_2) \cup \mathcal{P}_2(y_1, y_2)} V(P)$ by Claim 3.6 (ii), (iii).

Claim 3.7. *$E_G(W(x_1) \cup W(y_1), W(x_2) \cup W(y_2)) = \emptyset$.*

Proof. Suppose not, and let zz' be an edge with $z \in W(x_1) \cup W(y_1)$ and $z' \in W(x_2) \cup W(y_2)$. By Claim 3.4 (ii), there exist paths R, R' such that $V(R) \subseteq \{x_1, y_1, z\}$, $\mathrm{end}(R) = \{s_1, z\}$, $V(R') \subseteq \{x_2, y_2, z'\}$, $\mathrm{end}(R') = \{s_2, z'\}$. Concatenating R and R' with the edge zz', we get an S-path Q in $V(G) \backslash (\bigcup_{P \in \mathcal{P}} V(P))$ having order at most 6. Then, $\mathcal{P}' = \mathcal{P} \cup \{Q\}$ is a 6-S-path-system such that $|\mathcal{P}'_4 \cup \mathcal{P}'_5 \cup \mathcal{P}'_6| \leq 1$. Since G is balanced by Claim 3.6 (i), this contradicts (D1). □

Claim 3.8. *Let $i \in \{1, 2\}$. Then, $E_G(V(P), W(x_{3-i}) \cup W(y_{3-i})) = \emptyset$ for each $P \in \mathcal{P}(x_i, y_i)$.*

Proof. Let $P \in \mathcal{P}(x_i, y_i)$. Suppose that $E_G(V(P), W(x_{3-i}) \cup W(y_{3-i})) \neq \emptyset$, and let zz' be an edge with $z \in V(P)$ and $z' \in W(x_{3-i}) \cup W(y_{3-i})$. By Claim 3.4 (i), there exists an S-path Q such that $V(Q) \subseteq \{x_i, y_i\} \cup V(P)$, $z \notin V(Q)$ and $|V(Q) \cap V(P)| = 1$. By Claim 3.4 (ii), there exists a path R such that $V(R) \subseteq \{x_{3-i}, y_{3-i}, z'\}$ and $\mathrm{end}(R) = \{s_{3-i}, z'\}$. Write $S \cap V(P) = \{u, v\}$ with $V(Q) \cap V(P) = \{u\}$. Since $z \in V(P) \backslash \{u\}$ and $zz' \in E(G)$, we can extend R to an S-path Q' such that $V(Q') \subseteq V(R) \cup (V(P) \backslash \{u\})$ and $\mathrm{end}(Q') = \{s_{3-i}, v\}$. Note that $|V(Q')| \leq 5$. Hence $\mathcal{P}' = (\mathcal{P} \backslash \{P\}) \cup \{Q, Q'\}$ is a 5-S-path-system such that $|\mathcal{P}_4 \cup \mathcal{P}_5| \leq 1$, which contraicts (D1). □

Claim 3.9. *$E_G(V_1, V_2) = \emptyset$.*

Proof. In view of Claims 3.7 and 3.8, it suffices to show that $E_G(V(P), V(P')) = \emptyset$ for each $P \in \mathcal{P}(x_1, y_1)$ and each $P' \in \mathcal{P}(x_2, y_2)$. Let $P \in \mathcal{P}(x_1, y_1)$ and $P' \in \mathcal{P}(x_2, y_2)$. Suppose that $E_G(V(P), V(P')) \neq \emptyset$, and let zz' be an edge with $z \in V(P)$ and $z' \in V(P')$. By Claim 3.4 (i), there exist S-paths Q, Q' such that $V(Q) \subseteq \{x_1, y_1\} \cup V(P)$, $z \notin V(Q)$, $|V(Q) \cap V(P)| = 1$, $V(Q') \subseteq \{x_2, y_2\} \cup V(P')$, $z' \notin V(Q')$ and $|V(Q') \cap V(P')| = 1$. Write $S \cap V(P) = \{u, v\}$ and $V(Q) \cap V(P) = \{u\}$. Also write $S \cap V(P') = \{u', v'\}$ and $V(Q') \cap V(P') = \{u'\}$. Since $zz' \in E(G)$, there exists an S-path Q'' with $V(Q'') \subseteq (V(P) \backslash \{u\}) \cup (V(P') \backslash \{u'\})$ and $\mathrm{end}(Q'') = \{v, v'\}$. Then, $\mathcal{P}' = (\mathcal{P} \backslash \{P, P'\}) \cup \{Q, Q', Q''\}$ is a 4-S-path-system such that $|\mathcal{P}_4| \leq 1$, which contradicts (D1). □

Claim 3.10.

- **(i)** *For each $P \in \mathcal{P}_2(x_1, x_2)$, $N_G(a_P) \subseteq \{b_R : R \in \mathcal{P}_2(x_1, x_2)\}$.*
- **(ii)** *For each $P \in \mathcal{P}_2(y_1, y_2)$, $N_G(b_P) \subseteq \{a_R : R \in \mathcal{P}_2(y_1, y_2)\}$.*

Proof. By the symmetry of A and B, it suffices to prove (i). Let $P \in \mathcal{P}_2(x_1, x_2)$.

First we show that $N_G(a_P) \cap (V_1 \cup V_2) = \emptyset$. Suppose that $N_G(a_P) \cap (V_1 \cup V_2) \neq \emptyset$, and take $z \in N_G(a_P) \cap (V_1 \cup V_2)$. By the symmetry of $\{x_1, y_1\}$ and $\{x_2, y_2\}$, we may assume that $z \in V_1$. Note that by the definition of $\mathcal{P}_2(x_1, x_2)$, there exists an S-path Q such that $V(Q) \subseteq \{x_2, y_2, b_P\}$ and $\mathrm{end}(Q) = \{s_2, b_P\}$. Assume for the moment that $z \in W(x_1) \cup W(y_1)$. By Claim 3.4 (ii), there exists a path R such that $V(R) \subseteq \{x_1, y_1, z\}$ and $\mathrm{end}(R) = \{s_1, z\}$. Since $a_P z \in E(G)$, we can extend R to an S-path Q' such that $V(Q') = V(R) \cup \{a_P\}$ and $\mathrm{end}(Q') = \{s_1, a_P\}$. Then, $\mathcal{P}' = (\mathcal{P}\backslash\{P\}) \cup \{Q, Q'\}$ is a 4-S-path-system such that $|\mathcal{P}'_4| \leq 1$, which contradicts (D1). Thus $z \in \bigcup_{P' \in \mathcal{P}(x_1,y_1)} V(P')$. Let $P' \in \mathcal{P}(x_1, y_1)$ be the path such that $z \in V(P')$. By Claim 3.4 (i), there exists an S-path Q' such that $V(Q') \subseteq \{x_1, y_1\} \cup V(P')$, $z \notin V(Q')$ and $|V(Q') \cap V(P')| = 1$. Write $S \cap V(P') = \{u, v\}$ with $V(Q') \cap V(P') = \{u\}$. Since $a_P z \in E(G)$, there exists an S-path Q'' such that $V(Q'') \subseteq \{a_P\} \cup (V(P')\backslash\{u\})$ and $\mathrm{end}(Q'') = \{a_P, v\}$. Then, $(\mathcal{P}\backslash\{P, P'\}) \cup \{Q, Q', Q''\}$ is a 3-S-path-system, which contradicts (D1). Consequently $N_G(a_P) \cap (V_1 \cup V_2) = \emptyset$.

Next we show that $N_G(a_P) \cap \bigcup_{P' \in \mathcal{P}_2(y_1,y_2)} V(P') = \emptyset$. Suppose that there exists $P' \in \mathcal{P}_2(y_1, y_2)$ such that $N_G(a_P) \cap V(P') \neq \emptyset$. Then, $a_P b_{P'} \in E(G)$. Let Q be as above. There also exists an S-path Q' such that $V(Q') \subseteq \{x_1, y_1, a_{P'}\}$ and $\mathrm{end}(Q') = \{s_1, a_{P'}\}$. Then, $(\mathcal{P}\backslash\{P, P'\}) \cup \{Q, Q', a_P b_{P'}\}$ is a 3-S-path-system, which contradicts (D1). Thus $N_G(a_P) \cap \bigcup_{P' \in \mathcal{P}_2(y_1,y_2)} V(P') = \emptyset$.

Therefore, $N_G(a_P) \subseteq B\backslash(V_1 \cup V_2 \cup \bigcup_{P' \in \mathcal{P}_2(y_1,y_2)} V(P')) = \{b_R : R \in \mathcal{P}_2(x_1, x_2)\}$. □

Now suppose that $\mathcal{P}_2(x_1, x_2) \cup \mathcal{P}_2(y_1, y_2) \neq \emptyset$. We may assume that $\mathcal{P}_2(x_1, x_2) \neq \emptyset$. Take $Q \in \mathcal{P}_2(x_1, x_2)$. By Claim 3.10 (i), $d_G(a_Q) \leq |\mathcal{P}_2(x_1, x_2)|$. On the other hand, it follows from Claims 3.9 and 3.10 (i) that $d_G(y_1) \leq |A| - |V_2 \cap A| - |\{a_P : P \in \mathcal{P}_2(x_1, x_2)\}| = |A| - |V_2 \cap A| - |\mathcal{P}_2(x_1, x_2)|$. Recall that $x_2 \in V_2 \cap A$. Hence $d_G(y_1) < |A| - |\mathcal{P}_2(x_1, x_2)|$. Consequently $d_G(a_Q) + d_G(y_1) < |A|$. Since $a_Q y_1 \notin E(G)$ by Claim 3.10 (i), this contradicts the assumption that $\sigma_{1,1}(G) \geq |B|$ $(= |A|)$. Therefore, $\mathcal{P}_2(x_1, x_2) \cup \mathcal{P}_2(y_1, y_2) = \emptyset$. Since $V_1, V_2 \neq \emptyset$, it now follows from Claim 3.9 that G is disconnected. In view of the assumption that $\sigma_{1,1}(G) \geq |B|$, this forces $G \cong K_{t,t} \cup K_{|B|-t,|B|-t}$ for some $1 \leq t \leq |B| - 1$. This completes the proof of Proposition 1.9. □

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