Enumerating spanning trees of a graph with edge constraints

JINSHUI GUO  WEIGEN YAN*

School of Science, Jimei University
Xiamen 361021
China
Jinshui4586@163.com  weigenyan@263.net

Abstract

Suppose that $F \cup H$ is a spanning subgraph of a complete graph $K_n$ of order $n$, where $F$ is a forest with $s$ components of orders $n_1, n_2, \ldots, n_s$ and $H$ is a subgraph of $K_n$. In this paper we prove that the number of spanning trees of $K_n$ containing all edges in $F$ and avoiding (containing no) edges in $H$ equals $n^{s-2} \left( \prod_{i=1}^{s} n_i \right) \prod_{\alpha} (n - \alpha)$, where the second product ranges over all Laplacian eigenvalues $\alpha$ of $H$; this generalizes a well-known result by Moon.

1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, and let $G^c$ be the complement of $G$. Denote the number of spanning trees of $G$ by $t(G)$. If $G$ is an edge-weighted graph, we also use $t(G)$ to denote the sum of the weights of spanning trees of $G$, where the weight of a spanning tree $T$ in $G$ is the product of weights of the edges in $T$. Enumeration of spanning trees in a graph is an important and popular topic in mathematics, physics, computer science, and so on, and has been studied extensively for a long time (see, for example, [1, 5, 6, 7, 8, 9, 11, 12, 16]).

The well-known Cayley’s formula states that a complete graph $K_n$ with $n$ vertices has $n^{n-2}$ spanning trees [2]. This result is extended to a complete multipartite graph $K_{n_1, n_2, \ldots, n_s}$ [1], and it is proved that

$$t(K_{n_1, n_2, \ldots, n_s}) = n^{s-2} \prod_{i=1}^{s} (n - n_i)^{n_i-1},$$

* Corresponding author. Partially supported by NSFC Grant (12071180).
where \( n = n_1 + n_2 + \cdots + n_s \).

For the enumeration of spanning trees of a complete graph \( K_n \) with some constraints, Moon [13] first proved that the number of spanning trees of \( K_n \) containing all edges in a spanning forest \( F = T_1 \cup T_2 \cup \cdots \cup T_c \) with \( c \) components, denoted by \( t_F(K_n) \), can be expressed by

\[
t_F(K_n) = n^{c-2} \prod_{i=1}^{c} n_i,
\]

where \( n_1, n_2, \ldots, n_c \) are the numbers of vertices of \( T_1, T_2, \ldots, T_c \), respectively.

Recently, Dong and Ge [4] extended this result to a complete bipartite graph, and obtained an interesting formula as follows. If \( F = T_1 \cup T_2 \cup \cdots \cup T_k \) is a spanning forest of a complete bipartite graph \( K_{m,n} \) with a bipartition \((X,Y)\) satisfying \( m_i = |X \cap V(T_i)| \) and \( n_i = |Y \cap V(T_i)| \) for \( 1 \leq i \leq k \), then the number of spanning trees of \( K_{m,n} \) containing edges in \( F \) can be expressed by

\[
t_F(K_{m,n}) = \frac{mn}{k} \left[ \prod_{i=1}^{k} (m_i n + n_i m) \right] \left( 1 - \sum_{j=1}^{k} \frac{m_j n_j}{m_j n + n_j m} \right),
\]

which generalizes the results in [15] by Zhang and Yan, and in [10] by Ge and Dong. Furthermore, Cheng, Chen and Yan [3] extended Eq. (2) to complete multipartite graphs.

On the other hand, Weinberg [14] first considered the enumerative problem of spanning trees of a complete graph \( K_n \) containing no edges in a subgraph \( H \) of \( K_n \). This is equivalent to enumerating spanning trees of the graph \( G = K_n - E(H) \) obtained from \( K_n \) by deleting all edges in \( H \). Weinberg [14] obtained the following formulae:

\[
t(K_n - pK_2) = n^{n-2} \left( 1 - \frac{2}{n} \right)^p
\]

and

\[
t(K_n - E(K_{1,k})) = (n - 1 - k)(n - 1)^{k-1}n^{n-k-2},
\]

where \( pK_2 \) is a matching of \( K_n \) (\( 2p \leq n \)), and \( K_{1,k} \) is a star with \( k+1 \) vertices which is a subgraph of \( K_n \) (\( k \leq n - 1 \)).

More generally, if \( H \) is a subgraph of \( K_n \) with \( s \) vertices and has Laplacian eigenvalues \( 0 = \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s \), then the Laplacian eigenvalues of \( G = K_n - E(H) \) are \( 0, \tilde{n}, \ldots, \tilde{n}, n - \alpha_2, n - \alpha_3, \ldots, n - \alpha_s \), which implies that the number of spanning trees of \( K_n \) containing no edges in \( H \) can be expressed by

\[
t(K_n - E(H)) = n^{n-s-2} \prod_{i=1}^{s} (n - \alpha_i),
\]

where the product is over all Laplacian eigenvalues of \( H \) [1]. Obviously, Eq. (5) generalizes Eqs. (3) and (4).
A natural problem is to enumerate spanning trees of $K_n$ containing all edges in $F$ and avoiding (containing no) edges in $H$, where $F$ is a forest of $K_n$ and $H$ is a subgraph of $K_n$ such that $V(F) \cap V(H) = \emptyset$ and $V(K_n) = V(F) \cup V(H)$; that is, to calculate $t_F(G)$, where $G = K_n - E(H)$.

Ge and Dong [10] first considered a similar problem, and obtained the following result. Let $M = A \cup B$ be a matching of a complete bipartite graph $K_{m,n}$ with $k$ edges, and $A \cap B = \emptyset$, $|A| = k - i$ and $|B| = i$. Then the number of spanning trees of $K_{m,n}$ containing all edges in $A$ and avoiding edges in $B$ can be expressed by

$$t_{k-i,i}(K_{m,n}) = (m + n)^{k-i-1}(mn - m - n)^{i-1}[(m + n - k)(mn - m - n) + inn]m^{n-k-1}n^{m-k-1}.$$  

In this paper, we solve the problem above, and prove mainly the following result, whose proof will be given in the next section.

**Theorem 1.1.** Let $G$ be a simple graph with $n$ vertices, which is the graph obtained from a complete graph $K_n$ by deleting all edges of a subgraph $H$ of $K_n$. Suppose that $F = T_1 \cup T_2 \cup \cdots \cup T_s$ is a subforest of $G$ with $s$ components satisfying $V(F) \cap V(H) = \emptyset$ and $V(G) = V(F) \cup V(H)$. Then the number of spanning trees of $G$ containing all edges in $F$ is

$$t_F(G) = n^{s-2} \left( \prod_{i=1}^{s} n_i \right) \prod_{\alpha} (n - \alpha),$$  

where $n_i = |V(T_i)|$ for $1 \leq i \leq s$, and the second product ranges over all Laplacian eigenvalues $\alpha$ of $H$.

**Remark 1.1.** Obviously, Eqs. (1), (3), (4) and (5) are special cases of Eq. (6) for $V(H) = \emptyset$, $H = pK_2$ (a matching with $p$ edges) and $F = (n - 2p)K_1$ ($n - 2p$ isolated vertices), $H = K_{1,k}$ and $F = (n - k - 1)K_1$, and $F = (n - s)K_1$, respectively.

**Remark 1.2.** In the theorem above, if $H$ is the vertex disjoint union of some complete bipartite graphs $K_{a_1,b_1}, K_{a_2,b_2}, \ldots, K_{a_t,b_t}$, i.e., the components of $H$ are $K_{a_1,b_1}, K_{a_2,b_2}, \ldots, K_{a_t,b_t}$, then

$$t_F(K_n - E(H)) = n^{s+2t-2} \left( \prod_{i=1}^{s} n_i \right) \prod_{j=1}^{t} (n - a_j - b_j)a_j^{b_j-1}b_j^{a_j-1}.$$  

In particular, if $H = pK_2$ is a matching of $K_n$, then

$$t_F(K_n - pK_2) = n^{s+2p-2} \left( 1 - \frac{2}{n} \right)^{p} \prod_{i=1}^{s} n_i.$$  

2 Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. Firstly, we need to introduce some lemmas as follows.
Lemma 2.1 (Matrix-Tree Theorem, [1]). Let $L_G$ be the Laplacian matrix of a graph $G$. Then
\[ t(G) = (-1)^{i+j} \det((L_G)_{ij}), \]
where $(L_G)_{ij}$ is the submatrix of $L_G$ obtained from $L(G)$ by deleting the $i$-th row and $j$-th column.

Let $G \vee H$ be the join of two vertex-disjoint graphs $G$ and $H$. That is, $G \vee H$ has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{(u, v) | u \in V(G), v \in V(H)\}$.

Given a vertex-weighted graph $G = (V(G), E(G))$ with a vertex weight function $\omega : V(G) \rightarrow \mathbb{R}$, this results in an edge-weighted graph, also denoted by $G$, in which each edge $e = uv \in E(G)$ has weight $\omega(u)\omega(v)$. Let $G^*$ be the weighted graph obtained from $G$ by replacing the induced subgraph $H$ of $G$ with \{o\} $\vee H^c$, where the weight of an edge in $G^*$ is defined as
\[
\begin{align*}
\omega(v_iv_j) &= -\omega(v_i)\omega(v_j), & \text{if } v_iv_j \in E(H^c), \\
\omega(v_iv_j) &= \omega(v_i)\omega(v_j), & \text{if } v_iv_j \in E(G) \setminus E(H), \\
\omega(ov_i) &= \sum_{v \in V(H)} \omega(v), & \text{if } v_i \in V(H).
\end{align*}
\]

Zhou and Bu [16] used the Schur complement formula to give the mesh-star transformation in vertex-weighted version as follows, which will play an important role in the proof of the main result.

Lemma 2.2 ([16]). Let $G$ be a weighted graph with vertex set $V(G)$ and edge set $E(G)$ and let $\omega : V(G) \rightarrow \mathbb{R}$ be the vertex-weighted function, where each edge $v_iv_j \in E(G)$ has weight $\omega(v_i)\omega(v_j)$. Keeping the notation above,
\[ t(G) = \frac{1}{\sum_{v \in V(H)} \omega(v)} t(G^*). \tag{7} \]

Now, we can give the proof of Theorem 1.1 as follows.
Proof of Theorem 1.1. Note that \( F \cup H \) is a spanning subgraph of \( K_n \) and \( F = T_1 \cup T_2 \cup \cdots \cup T_s \) is a subforest of \( G = K_n - E(H) \) with \( s \) components of \( n_1, n_2, \ldots, n_s \) vertices and \( H \) is a subgraph of \( K_n \) such that \( V(F) \cap V(H) = \emptyset \). Hence \( H \) has \( n - n_1 - n_2 - \cdots - n_s \) vertices. Contracting each component \( T_i \) of \( T \) in \( G = K_n - E(H) \) into a new vertex \( u_i \) for \( i = 1, 2, \ldots, s \), we get a new edge-weighted graph \( G^* \) with vertex set \( V(G^*) = \{u_1, u_2, \ldots, u_s\} \cup V(H) \) and edge set \( E(G^*) = \{u_iu_j \mid 1 \leq i < j \leq s\} \cup \{u_iv \mid 1 \leq i \leq s, v \in V(H)\} \cup E(H^c) \), and the edge weight function \( \omega \) satisfies:

\[
\omega(u_iu_j) = n_in_j \quad \text{for} \quad 1 \leq i < j \leq s,
\]

\[
\omega(u_iv) = n_i \quad \text{for} \quad 1 \leq i \leq s, v \in V(H),
\]

and \( \omega(e) = 1 \) for all edges \( e \in E(H^c) \),

where \( H^c \) is the complement of \( H \). For example, if \( n = 10 \), \( F = P_2 \cup P_3 \cup P_2 \) and \( H = P_3 \), are illustrated in Figures 1(a) and (b), and the corresponding edge-weighted graph \( G^* \) is illustrated in Figure 1(c).

Obviously, the number of spanning trees of \( K_n \) containing all edges in \( F \) and no edge in \( H \) equals the sum of weights of spanning trees in \( G^* \), that is,

\[
t_F(G) = t_F(K_n - E(H)) = t(G^*).
\]  

(8)

![G'](image)

Figure 2: The edge-weighted graph \( G' \).

Define a vertex weight function \( \omega^* : V(G^*) \rightarrow \mathbb{R} \) such that \( \omega^*(u_i) = n_i \) for \( 1 \leq i \leq s \) and \( \omega^*(v) = 1 \) for \( v \in V(H) \). Obviously, the edge weight function \( \omega \) of \( G^* \) satisfies: \( \omega(u_iu_j) = \omega^*(u_i)\omega^*(u_j) \) for any \( 1 \leq i, j \leq s \), \( \omega(u_iv) = \omega^*(u_i)\omega^*(v) \) for \( 1 \leq p \leq s \), and \( \omega(xy) = \omega^*(x)\omega^*(y) \) for any \( x, y \in V(H) \).

Let \( G' \) be the edge-weighted graph obtained from \( G^* \) by replacing \( G^* \) by \( \{o\} \cup (G^*)^c \), where the weight of each edge in \( G' \) is defined as

\[
\begin{align*}
\omega(xy) &= -\omega^*(x)\omega^*(y) = -1, & \text{for} \ xy \in E((G^*)^c), \\
\omega(u_iu_j) &= \omega^*(u_i)\omega^*(u_j) = n_in_j, & \text{for} \ 1 \leq i \neq j \leq s, \\
\omega(u_iu) &= \omega^*(u_i) \sum_{x \in V(G^*)} \omega^*(x) = n_i(\sum_{p=1}^s n_p + |V(H)|) = n_in, & \text{for} \ 1 \leq i \leq s, \\
\omega(ov) &= \omega^*(v) \sum_{x \in V(G^*)} \omega^*(x) = \sum_{p=1}^s n_p + |V(H)| = n, & \text{for} \ v \in V(H).
\end{align*}
\]
For the edge-weighted graph $G^*$ illustrated in Figure 1(c), the corresponding edge-weighted graph $G'$ is illustrated in Figure 2.

By Lemma 2.2,

$$t(G^*) = \frac{1}{\left(\sum_{x \in V(G^*)} \omega^*(x)\right)^2} t(G') = \frac{1}{n^2} t(G').$$  \hfill (9)

Note that the induced subgraph $G_1$ of $G'$ with vertex set $\{o, u_1, u_2, \ldots, u_s\}$ is an edge-weighted star in which each edge $ou_i$ has weight $nn_i$ for $i = 1, 2, \ldots, s$, and the induced subgraph $G_2$ of $G'$ with vertex set $\{o\} \cup V(H)$ is an edge-weighted graph in which each edge $ov$ has weight $n$ for each $v \in V(H)$ and each edge $e \in E(H)$ has weight $-1$. Particularly, $V(G_1) \cap V(G_2) = o$ (i.e., $o$ is a cut vertex of $G'$). Thus

$$t(G') = t(G_1)t(G_2) = n^s \left(\prod_{i=1}^{s} n_i\right) t(G_2).$$  \hfill (10)

Note that if we delete the row and column corresponding to vertex $o$ of the Laplacian matrix $L_{G_2}$ of $G_2$, then we obtain the matrix $nI - L_H$, where $I$ is the $n \times n$ identity matrix and $L_H$ is the Laplacian matrix of $H$. By Lemma 2.1,

$$t(G_2) = \det(nI - L_H) = \prod_{\alpha}(n - \alpha),$$  \hfill (11)

where the product ranges over all Laplacian eigenvalues $\alpha$ of $H$.

The theorem is immediate from Eqs. (8)–(11). \hfill $\Box$

### 3 Discussion

In this paper, by the so-called mesh-star transformation in the vertex-weighted version by Zhou and Bu [16], we obtain an enumerative formula for the number of spanning trees in a complete graph $K_n$ containing all edges in a subforest $F$ and no edge in a subgraph $H$ of $K_n$, where $F \cup H$ is a spanning subgraph of $K_n$ which satisfies $V(F) \cap V(H) = \emptyset$. This result generalizes Moon’s formula (i.e., Eq. (1)) and Weinberg’s formulae (i.e., Eqs. (3) and (4)). Note that Dong and Ge [4] generalized Moon’s formula to the case of the complete bipartite graph and Ge and Dong [10] obtained a formula for the number of spanning trees of a complete bipartite graph $K_{m,n}$ containing all edges in a matching $M_1$ of $K_{m,n}$ and avoiding all edges in a matching $M_2$ in $K_{m,n}$, where $V(M_1) \cap V(M_2) = \emptyset$. A natural problem is: if $F \cup H$ is a spanning subgraph of a complete multipartite graph $G = K_{n_1,n_2,\ldots,n_s}$ for $s \geq 2$, find a formula for the number of spanning trees of $G$ containing edges in $F$ and avoiding edges in $H$, where $F$ is a forest of $G$ and $H$ is a subgraph of $G$ such that $V(F) \cap V(H) = \emptyset$. 
Acknowledgements

We are grateful to the anonymous referees for many friendly and helpful revising suggestions and editorial comments that greatly improved the presentation of the paper. In particular, one of the referees told us Lemma 2.2, which simplifies the proof of the main result.

References


[4] F. M. Dong and J. Ge, Counting spanning trees in a complete bipartite graph which contain a given spanning forest, *J. Graph Theory* 101 (2022), 79–94.


(Received 16 June 2023; revised 21 Aug 2023)