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Product throttling for power domination

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Abstract

The product power throttling number of a graph is defined to study product throttling for power domination. The domination number of a graph is an upper bound for its product power throttling number. It is established that the two parameters are equal for certain families including paths, cycles, complete graphs, unit interval graphs, and grid graphs (on the plane, cylinder, and torus). Families of graphs for which the product power throttling number is less than the domination number are also exhibited. Graphs with extremely high or low product power throttling number are characterized and bounds on the product power throttling number are established.

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1 Introduction

Many graph search processes observe all vertices starting with an initial set of vertices through a process consisting of rounds or discrete time steps. Throttling minimizes the sum or product of the resources used to accomplish a task (number of initial vertices) and the time (number of rounds) needed to complete that task. Many of the graph parameters for which throttling has been studied arose from applications. One such parameter is the power domination number, which originated from the problem of optimal placement of Phasor Measurement Units (PMUs) to monitor an electric power network at minimum cost.

The power domination problem was modeled using graphs by Haynes et al. in [9]; Brueni and Heath [3] showed that a simplified version of the propagation rules is equivalent to the original version in [9], and we use their propagation rules. Let G be a graph and let S be a non-empty subset of vertices of G; $N[S]$ denotes the closed neighborhood of S. Define the sequences of sets $P^{(i)}(S)$ and $P^{[i]}(S)$ by the following recursive rules:

(1) $P^{[0]}(S) = P^{(0)}(S) = S$, $P^{[1]}(S) = N[S]$ and $P^{(1)}(S) = N[S] \setminus S$.

(2) For $i \geq 1$,

$$\begin{aligned} P^{(i+1)}(S) &= \left\{w \in V(G) \setminus P^{[i]}(S) : \exists u \in P^{[i]}(S),\, N_G(u) \setminus P^{[i]}(S) = \{w\}\right\}, \\ P^{[i+1]}(S) &= P^{[i]}(S) \cup P^{(i+1)}(S). \end{aligned}$$

For $v \in P^{(k)}(S)$, we say *v is observed in round k*. If for every vertex v there is some round in which v is observed, then S is a *power dominating set* of G. The *power domination number* of G, denoted by $\gamma_P(G)$, is the minimum cardinality of a power dominating set. When S is a power dominating set, the least positive integer t with the property that $P^{[t]}(S) = V(G)$ is the *power propagation time* of S in G, denoted by $\mathrm{pt}_{\mathrm{pd}}(G; S)$; if S is not a power dominating set, then $\mathrm{pt}_{\mathrm{pd}}(G; S) = \infty$. We require t to be positive because we adopt the perspective that step (1) of power domination always occurs, so $\mathrm{pt}_{\mathrm{pd}}(G; S) \geq 1$ for every S, including $S = V(G)$.[1] For $k \in \mathbb{Z}^+$, $\mathrm{pt}_{\mathrm{pd}}(G, k) = \min_{|S|=k} \mathrm{pt}_{\mathrm{pd}}(G; S)$ and the *power propagation time of G* is $\mathrm{pt}_{\mathrm{pd}}(G) = \mathrm{pt}_{\mathrm{pd}}(G, \gamma_P(G))$.

The large scale deployment of wide area measurement systems of PMUs started in 2010 and continues growing [14]. The analysis of available systems has shown that minimizing the number of PMUs alone yields unsatisfactory state estimation, primarily due to the loss of information in the event of transmission failures [15]. Since failures are inevitable, the proposed solution is to add redundancy [13, 14]. While higher levels of redundancy imply larger numbers of PMUs, which result in increased costs, it has been observed that adding even a few redundant PMUs has a number of advantages that offsets the cost increase [13]. As a result, nowadays the PMU placement problem seeks a compromise between the cost of adding redundancy and the improvements in the upgraded system. In terms of power domination, this new approach to the PMU placement problem creates the need to study properties of

[1] In the original definition of power propagation time in [7], $\mathrm{pt}_{\mathrm{pd}}(G; V(G)) = 0$.

the graph propagation process associated with a power dominating set in addition to its cardinality, as minimum power dominating sets might no longer correspond to the best choice of PMU placements. In this work we study a combination of the number of PMUs and the number of rounds in the power domination propagation process, using a parameter that has proven successful in other forms of graph searching.

Throttling sums was studied first and has been studied more widely than throttling products. Brimkov et al. defined the (sum) power domination throttling number in [2]. In this paper we introduce product throttling for power domination, establish bounds, provide conditions sufficient to guarantee the product power throttling number equals the domination number, and show that these parameters are equal for various families of graphs.

Definition 1.1. Let G be a graph. For a set $S \subseteq V(G)$, $\operatorname{th}_{\mathrm{pd}}^{\times}(G; S) = |S| \operatorname{pt}_{\mathrm{pd}}(G; S)$. The *product power throttling number* of G is

$$\operatorname{th}_{\mathrm{pd}}^{\times}(G) = \min_{S \subseteq V(G)} \operatorname{th}_{\mathrm{pd}}^{\times}(G; S) = \min_{S \subseteq V(G)} |S| \operatorname{pt}_{\mathrm{pd}}(G; S).$$

For $k \in \mathbb{Z}^+$, $\operatorname{th}_{\mathrm{pd}}^{\times}(G, k) = \min_{|S|=k} \operatorname{th}_{\mathrm{pd}}^{\times}(G; S)$.

The product power throttling number, $\operatorname{th}_{\mathrm{pd}}^{\times}(G)$, and the (sum) power domination throttling number, $\operatorname{th}_{\mathrm{pd}}(G) := \min_{S \subseteq V(G)} |S| + \operatorname{pt}_{\mathrm{pd}}(G; S)$, are noncomparable. For K_n, one vertex observes all vertices in one round, so $\operatorname{th}_{\mathrm{pd}}^{\times}(K_n) = 1$, whereas $\operatorname{th}_{\mathrm{pd}}(K_n) = 2$. From Proposition 3.2 below, $\operatorname{th}_{\mathrm{pd}}^{\times}(P_n) = \lceil \frac{n}{3} \rceil$, whereas $\operatorname{th}_{\mathrm{pd}}(P_n) = \lceil \sqrt{2n} - \frac{1}{2} \rceil$ [2].

A main theme of this work is that for many graphs $\operatorname{th}_{\mathrm{pd}}^{\times}(G)$ is equal to the domination number (defined below). Graph families for which this is established include paths and cycles (Section 3), unit interval graphs (Section 5), and Cartesian products of complete graphs with complete graphs, and of path or cycles with paths or cycles (Section 6). We also characterize connected graphs of order n having $\operatorname{th}_{\mathrm{pd}}^{\times}(G) = 1, 2$, and $\frac{n}{2}$ in Section 4; Section 2 contains preliminary results.

In the remainder of this introduction we present additional terminology and make some elementary observations. Note that $P^{(k)}(S)$ is the set of vertices that are first observed in round k, and the sets $P^{(0)}(S), P^{(1)}(S), \ldots, P^{(\operatorname{pt}_{\mathrm{pd}}(G;S))}(S)$ partition the vertices of G when S is a power dominating set of G. For each $v \in V(G)$, define the *round function*, $\operatorname{rd}(v)$, to be number of the round in which vertex v is first observed. That is, $\operatorname{rd}(v) = k$ for $v \in P^{(k)}(S)$.

Power domination can be thought of as a domination step (1) followed by a zero forcing process (2). A set $S \subseteq V(G)$ *dominates* a graph G if $V(G) = N[S]$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. *Zero forcing* is a coloring game on a graph, where the goal is to color all the vertices blue (starting with each vertex colored blue or white). White vertices are colored blue by applying the following *color change rule*: A blue vertex u can change the color of a white vertex w to blue if w is the unique white neighbor of u; in this case we say u *forces* w and write $u \to w$. A set B is a *zero forcing set* of G if all the vertices of G can be colored blue by repeated application of the color change rule when starting with the vertices in B blue and the other vertices white. The domination step in power domination takes the set S to $N[S]$, and S is a power dominating set of G if and only if $N[S]$ is a zero forcing set of G. A blue vertex in zero forcing corresponds to an observed vertex in power domination, because $u \in P^{[i]}(S)$ and $N_G(u) \setminus P^{[i]}(S) = \{w\}$ is equivalent to saying that after the i round, w is the only unobserved neighbor of u, so $u \to w$ is possible.

Notice that in power domination we have performed all independently possible observations simultaneously, whereas in zero forcing as just defined, we perform one color change at a time (and choose which vertex forces w if more than one vertex could force w). Both perspectives are useful. For zero forcing, we can start with a set B of blue vertices and in each round we perform all possible forces that can be done independently of each other (this is propagation for zero forcing – see [12]). Sometimes it is necessary to record how the forcing part of the power domination process is carried out. If $i \geq 1$ and there is at least one vertex $u \in P^{[i]}(S)$ such that $N_G(u) \setminus P^{[i]}(S) = \{w\}$, then one such u is chosen as the vertex to force w, denoted by $u \to w$. In the dominating step, for each vertex $w \in N[S] \setminus S$, we choose an $x \in S$ such that $w \in N(x)$ and record $x \to w$ as a force. When it is desired to distinguish these two kinds of forces, a force in step (1) is called a *domination force* and a force in step (2) is called a *zero force*. For a given set S, we construct the set of all observed vertices, recording each force in order. We consider only a *propagating* set of forces, in which $\mathrm{rd}(u) < \mathrm{rd}(v)$ implies u is forced before v in the ordered list of forces. The symbol $\mathcal{F}$ is used to denote the *set of forces*. Given a power dominating set S and set of forces $\mathcal{F}$, a *forcing chain* is a sequence $v_0 \to v_1 \to \cdots \to v_a$ such that $v_{i-1} \to v_i \in \mathcal{F}$ for $i = 1, \ldots, a$.

Observation 1.2. *If $v_0 \to v_1 \to \cdots \to v_a$ is a forcing chain for a given set $\mathcal{F}$ of forces of a power dominating set S of G, then $\mathrm{rd}(v_i) \geq i$, because $\mathrm{rd}(v_0) \geq 0$ and $\mathrm{rd}(v_{i+1}) \geq \mathrm{rd}(v_i) + 1$.*

Remark 1.3. Let S be a power dominating set of G. It is well known that the number of vertices forced in each round of zero forcing cannot exceed the number of initial blue vertices. After the first round, power domination uses the zero forcing process, so the number of observed vertices that have an unobserved neighbor is at most $|P^{(1)}(S)|$. Thus $|P^{(i+1)}(S)| \leq |P^{(1)}(S)|$, for all $i \geq 0$.

Since electrical power networks are modeled by connected simple finite undirected graphs, in this work we assume every graph G has these properties (although 'connected' is listed as a hypothesis in results since power domination has been studied in graphs that need not be connected).

2 Preliminary results

In this section we present bounds on the product power throttling number in terms of other graph parameters.

Observation 2.1. *For a connected graph G, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \geq 1$; moreover, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 1$ if and only if $\gamma(G) = 1$. This implies that $\mathrm{th}_{\mathrm{pd}}^{\times}(K_n) = 1$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(K_{1,n-1}) = 1$.*

Observation 2.2. *For every connected graph G, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \geq \gamma_P(G)$ because $|S| \geq \gamma_P(G)$ in order to have finite propagation time and $\mathrm{pt}_{\mathrm{pd}}(G; S) \geq 1$.*

Observation 2.3. *For every connected graph G:*

(1) $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \leq \gamma(G)$, *since a minimum dominating set is a power dominating set with power propagation time* 1.

(2) $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \leq \gamma_P(G)\,\mathrm{pt}_{\mathrm{pd}}(G)$, *realized by a minimum power dominating set S such that* $\mathrm{pt}_{\mathrm{pd}}(G; S) = \mathrm{pt}_{\mathrm{pd}}(G)$.

The domination number upper bound in the previous observation is explored further throughout the rest of the paper. Next we give an example showing that the product power throttling number need not be the minimum of the two upper bounds $\gamma(G)$ and $\gamma_P(G)\,\mathrm{pt}_{\mathrm{pd}}(G)$. The *spider* $S(\ell_1, \ldots, \ell_k)$ has one vertex of degree k and k pendent paths on $\ell_1, \ldots, \ell_k$ vertices, respectively. Figure 2.1 shows $S(7, 2, 2, 2, 2, 2)$.

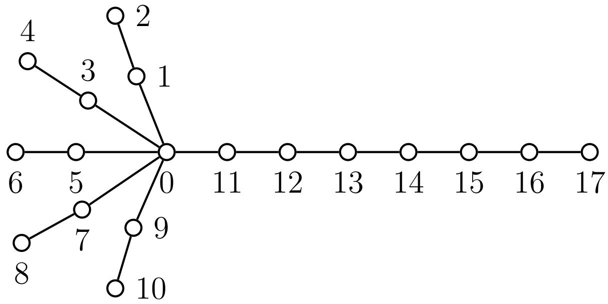

Figure 2.1: The spider $S(7, 2, 2, 2, 2, 2)$.

Example 2.4. Let $G = S(7, 2, 2, 2, 2, 2)$ with the vertices numbered as in Figure 2.1. The product power throttling number of G is 4 using the power dominating set $\{0, 15\}$. This cannot be realized by either a minimum dominating set (since $\gamma(G) = 8$) or a minimum power dominating set (since $\gamma_P(G) = 1$, $\mathrm{pt}_{\mathrm{pd}}(G) = 7$, and $\mathrm{th}_{\mathrm{pd}}^{\times}(G, 1) = 7$).

From Observation 2.3, only subsets $S \subseteq V(G)$ such that $\gamma_P(G) \leq |S| \leq \gamma(G)$ need be considered to determine $\mathrm{th}_{\mathrm{pd}}^{\times}(G)$.

Next we turn our attention to lower bounds. The maximum degree of a graph G is denoted by $\Delta(G)$.

Theorem 2.5. [7] *In a connected graph G,*

$$\gamma_P(G) \geq \frac{|V(G)|}{\mathrm{pt}_{\mathrm{pd}}(G)\Delta(G) + 1}.$$

The argument used to establish Theorem 2.5 in [7] consists of showing that for any power dominating set S of G,

$$|S| \geq \frac{|V(G)|}{\mathrm{pt}_{\mathrm{pd}}(G;S)\Delta(G)+1}. \tag{1}$$

Notice that in the particular case when S is a minimum power dominating set of minimum power propagation time, $|S| = \gamma_P(G)$, $\mathrm{pt}_{\mathrm{pd}}(G;S) = \mathrm{pt}_{\mathrm{pd}}(G)$ and inequality (1) gives the bound in Theorem 2.5. As we show next, in the study of throttling, inequality (1) has additional consequences. The next result is immediate since $\frac{|V(G)|}{\mathrm{pt}_{\mathrm{pd}}(G;S)\Delta(G)+1} \geq \frac{|V(G)|}{\mathrm{pt}_{\mathrm{pd}}(G;S)(\Delta(G)+1)}$.

Corollary 2.6. *In a connected graph G,*

$$\mathrm{th}_{\mathrm{pd}}^{\times}(G) \geq \left\lceil \frac{|V(G)|}{\Delta(G)+1} \right\rceil.$$

3 Conditions resulting in $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$

In this section we present conditions on a graph G that ensure that the product power throttling number is achieved by starting with a dominating set, that is, conditions that guarantee $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$. The next result follows from Corollary 2.6.

Observation 3.1. *Let G be a connected graph of order n with $\gamma(G) = \left\lceil \frac{n}{\Delta(G)+1} \right\rceil$. Then $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \left\lceil \frac{n}{\Delta(G)+1} \right\rceil$.*

Observation 3.2. *Since $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$ and $\Delta(P_n) = 2$, $\mathrm{th}_{\mathrm{pd}}^{\times}(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Similarly, since $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$ and $\Delta(C_n) = 2$, $\mathrm{th}_{\mathrm{pd}}^{\times}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.*

A *d-star cover* of a graph G is a set of subgraphs $G_i = K_{1,p_i}, i = 1, \ldots, d$ such that $\cup_{i=1}^{d} V(G_i) = V(G)$. A star cover is *disjoint* if the vertex sets of the stars are disjoint. For any graph G, any dominating set gives a star cover (which can be chosen disjoint), and $\gamma(G)$ is the minimum d such that G has a d-star cover.

Observation 3.3. [10, p. 50] *A graph G of order n has $\gamma(G) = \frac{n}{\Delta(G)+1}$ if and only if G has a $\frac{n}{\Delta(G)+1}$-star cover that is disjoint and in which each star has order $\Delta(G)+1$.*

Observation 3.4. *If a connected graph G has a star cover consisting of d disjoint copies of $K_{1,d}$ and $\Delta(G) = d$, then $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = d = \gamma(G)$.*

One can construct a graph G of order $d(d+1)$ with $\Delta(G) = d = \gamma(G) = \frac{d(d+1)}{d+1}$ as described in the next example.

Example 3.5. Define the graph G_d to be the graph obtained from d disjoint copies of $K_{1,d}$ by adding all necessary edges so that each leaf of a $K_{1,d}$ is adjacent to the corresponding leaves of the other $d-1$ copies of $K_{1,d}$. Then G_d is a d-regular graph of order $d(d+1)$ with $\gamma(G) = d$; G_3 is shown in Figure 3.1.

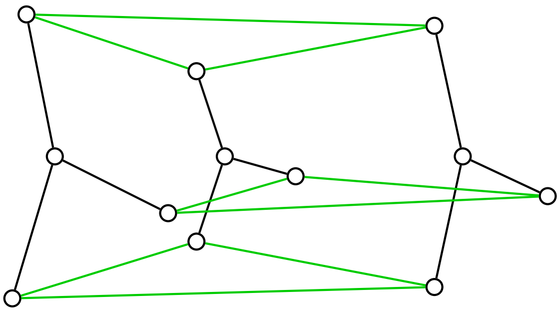

Figure 3.1: The graph G_3, with the edges added to $3K_{1,3}$ shown in green.

The construction in Example 3.5 can be relaxed by adding edges between the degree one vertices of the stars such that no such vertex is incident with more than $d-1$ additional edges and at least $d-1$ additional edges are added to connect the graph. When $\frac{n}{\Delta(G)+1}$ is not an integer and $\gamma(G) = \left\lceil \frac{n}{\Delta(G)+1} \right\rceil$, by Observation 3.1 $\text{th}_{\text{pd}}^{\times}(G) = \gamma(G)$. Although G will have a $\gamma(G)$-star cover, it is possible that none of the stars will have order $\Delta(G)+1$, as Example 3.6 shows.

Example 3.6. Let H be the graph shown in Figure 3.2. Then $\Delta(H) = 4$, $\{x, y, z\}$ is the unique minimum dominating set and H has only one 3-star cover, in which each star is $K_{1,3} = K_{1,\Delta(H)-1}$.

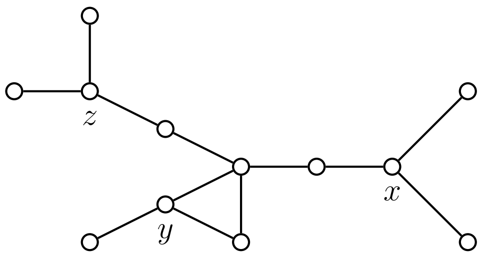

Figure 3.2: The graph H in Example 3.6.

Proposition 3.7. *In any graph G, $\gamma_P(G) = \gamma(G)$ if and only if $\text{pt}_{\text{pd}}(G) = 1$. Moreover, in this case $\text{th}_{\text{pd}}^{\times}(G) = \gamma(G)$.*

Proof. Suppose $\gamma_P(G) = \gamma(G)$. Then $\text{pt}_{\text{pd}}(G) = 1$ because any minimum dominating set S of G is also a minimum power dominating set of G with $\text{pt}_{\text{pd}}(G; S) = 1$. Conversely, if $\text{pt}_{\text{pd}}(G) = 1$, then there exists a minimum power dominating set S of G such that $\text{pt}_{\text{pd}}(G; S) = 1$, which implies S is a dominating set and thus $\gamma_P(G) \le \gamma(G) \le |S| = \gamma_P(G)$. The last statement follows from Observations 2.2 and 2.3. □

Proposition 3.8. *Let G be a connected graph.*

(1) Suppose $\gamma(G) \le b$. *If* $\gamma_P(G) \ge \frac{b}{2}$, *then* $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$. *In particular, if* $\gamma_P(G) \ge \frac{\gamma(G)}{2}$, *then* $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$.

(2) Suppose $\mathrm{th}_{\mathrm{pd}}^{\times}(G; S) = b < \gamma(G)$ *for some* $S \subset V(G)$. *Then* $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \mathrm{th}_{\mathrm{pd}}^{\times}(G, k)$ *for some* k *such that* $\gamma_P(G) \le k \le \lfloor \frac{b}{2} \rfloor$.

Proof. Let S be an arbitrary power dominating set of G. Then, $|S| \ge \gamma_P(G)$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(G; S) = |S| \,\mathrm{pt}_{\mathrm{pd}}(G; S) \ge \gamma_P(G) \,\mathrm{pt}_{\mathrm{pd}}(G; S)$. If S is not a dominating set, then $\mathrm{pt}_{\mathrm{pd}}(G; S) \ge 2$ and this implies $\mathrm{th}_{\mathrm{pd}}^{\times}(G; S) \ge 2|S| \ge 2\gamma_P(G)$.

For (1), if S is not a dominating set, then $\mathrm{th}_{\mathrm{pd}}^{\times}(G; S) \ge 2\gamma_P(G) \ge b \ge \gamma(G)$ by hypothesis. Therefore, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$.

For (2), $\mathrm{th}_{\mathrm{pd}}^{\times}(G, |S|) \le b$ and $|S| \le \frac{b}{2}$ since $b < \gamma(G)$ implies $\mathrm{pt}_{\mathrm{pd}}(G; S) \ge 2$. For any graph, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \mathrm{th}_{\mathrm{pd}}^{\times}(G, k)$ for some k with $\gamma_P(G) \le k \le \gamma(G)$. If $\lfloor \frac{b}{2} \rfloor < k < \gamma(G)$, then $\mathrm{pt}_{\mathrm{pd}}(G, k) \ge 2$, so $\mathrm{th}_{\mathrm{pd}}^{\times}(G, k) \ge 2k > b$. □

Now we apply the first part of the previous result together with some known upper bounds for the power domination number of a graph in terms of its order and minimum degree to derive additional sufficient conditions for $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$.

Theorem 3.9. [10, Theorem 2.1] *Let* G *be a graph of order* n *having no isolated vertices. Then* $\gamma(G) \le \lfloor \frac{n}{2} \rfloor$.

Notice that the condition of G not having isolated vertices can also be stated as $\delta(G) \ge 1$, which is immediate for a connected graph of order at least two.

Corollary 3.10. *Let* G *be a connected graph of order* $n \ge 2$. *If* $\gamma_P(G) \ge \frac{n}{4}$, *then* $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$.

Theorem 3.11. [10, Theorem 2.3] *Let* G *be a connected graph of order* n *with minimum degree* $\delta(G) \ge 2$. *If* $G \notin \mathcal{A}$ *where* $\mathcal{A}$ *is the set of graphs in Figure 3.3, then* $\gamma(G) \le \frac{2n}{5}$.

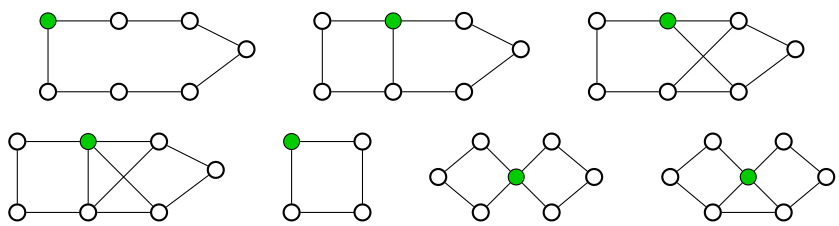

Figure 3.3: The graphs in the family $\mathcal{A}$. Each graph has a vertex (colored green) that is a minimum power dominating set.

Corollary 3.12. *Let* G *be a connected graph of order* n *with minimum degree* $\delta(G) \ge 2$. *If* $\gamma_P(G) \ge \frac{n}{5}$, *then* $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$.

Proof. The graph C_4 has $\mathrm{th}_{\mathrm{pd}}^{\times}(C_4) = 2 = \gamma(C_4)$ and thus satisfies the conclusion. The remaining graphs $H \in \mathcal{A}$ do not satisfy the hypothesis since each has order $n = 7$ and $\gamma_P(H) = 1 < \frac{n}{5}$. When $\gamma_P(G) \geq \frac{n}{5}$ (and $G \neq C_4$) we have $2\gamma_P(G) \geq \frac{2n}{5} \geq \gamma(G)$ by Theorem 3.11 and then the conclusion follows from Proposition 3.8(1). □

Theorem 3.13. [10, Theorem 2.7] *Let G be a graph of order n with minimum degree $\delta(G) \geq 3$. Then, $\gamma(G) \leq \frac{3n}{8}$.*

Corollary 3.14. *Let G be a connected graph of order n with minimum degree $\delta(G) \geq 3$. If $\gamma_P(G) \geq \frac{3n}{16}$, then $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$.*

It should be noted that while the conditions above are sufficient, they are not necessary. Graphs with very low power domination number may still realize $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$. For example, $\gamma_P(P_n) = 1$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$ (Observation 3.2).

There are also many other families of graphs that have $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$, including unit interval graphs (described in Section 5) and some Cartesian products (described in Section 6).

4 Extreme product power throttling numbers

In this section we characterize graphs with extremely low and high product power throttling numbers: The product power throttling number is one if and only if the domination number is one. In Theorem 4.1, we show that the product power throttling number is two if and only if its domination number is two, or the power domination number is one and the power propagation time is two. In Theorem 4.7, we show that the product power throttling number of a connected graph G is half its order if and only if G is K_2, C_4, $C_4 \circ K_1$, or has the form $(H \circ K_1) \circ K_1$ for some connected graph H.

4.1 Low product power throttling numbers

By Observation 2.1, any graph G has $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \geq 1$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 1$ if and only if $\gamma(G) = 1$. The following result characterizes graphs G for which $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 2$.

Theorem 4.1. *A connected graph G has $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 2$ if and only if G satisfies one or both of the following conditions:*

(a) $\gamma(G) = 2$.

(b) $\gamma_P(G) = 1$ *and* $\mathrm{pt}_{\mathrm{pd}}(G) = 2$.

Proof. Suppose G is a graph satisfying at least one of the conditions. Then $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \leq 2$ by Observation 2.3. If $\gamma(G) = 2$ or $\mathrm{pt}_{\mathrm{pd}}(G) = 2$, then $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \geq 2$.

Conversely, assume G is a graph with $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 2$, and let $S \subset V(G)$ such that $\mathrm{th}_{\mathrm{pd}}^{\times}(G; S) = \mathrm{th}_{\mathrm{pd}}^{\times}(G)$. There are only two possibilities: $|S| = 1$ and $\mathrm{pt}_{\mathrm{pd}}(G; S) = 2$, or $|S| = 2$ and $\mathrm{pt}_{\mathrm{pd}}(G; S) = 1$. Suppose first that $|S| = 2$ and $\mathrm{pt}_{\mathrm{pd}}(G; S) = 1$. Then

S is a dominating set of G because $\mathrm{pt}_{\mathrm{pd}}(G; S) = 1$, and we conclude $\gamma(G) \leq |S| = 2$. Furthermore, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 2$ implies $\gamma(G) \geq 2$, so $\gamma(G) = 2$. Finally, consider the case $|S| = 1$ and $\mathrm{pt}_{\mathrm{pd}}(G; S) = 2$. Since $|S| = 1$, S must be a minimum power dominating set of G and $\gamma_P(G) = 1$. Furthermore, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 2$ implies $\mathrm{pt}_{\mathrm{pd}}(G) \geq 2$. □

Remark 4.2. Any connected graph satisfying Theorem 4.1(b) can be constructed as follows: Start with any graph H of order at least two such that $\gamma(H) = 1$. Suppose vertex u is adjacent to every other vertex and let the remaining vertices of H be denoted by $v_1, \dots, v_k$. Add $1 \leq \ell \leq k$ additional vertices $\{w_1, \dots, w_\ell\}$ with v_i adjacent to w_i and to none of the other w_j. Add any subset (possibly empty) of the edges $\{v_s w_j : s = \ell+1, \dots, k, j = 1, \dots, \ell\}$ and any subset (possibly empty) of edges of the form $w_i w_j$.

Observe that conditions (a) and (b) can hold simultaneously. For example, if $G = C_5$ or $G = P_5$, then $\gamma_P(G) = 1$ and $\mathrm{pt}_{\mathrm{pd}}(G) = 2$, and $\gamma(G) = 2$.

4.2 High product power throttling numbers

We know $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \leq \gamma(G) \leq \frac{n}{2}$ for any connected graph G of order $n \geq 2$ (Theorem 3.9). In this section we characterize graphs having $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \frac{n}{2}$.

Remark 4.3. It is known that if a connected graph has a high-degree (≥ 3) vertex, then there is a minimum power dominating set in which each vertex has degree at least three. It is not true that for every graph with a high-degree vertex there is an optimal set for product power throttling that has all high-degree vertices. For example, the spider $S(4, 1, 1)$ has $\mathrm{th}_{\mathrm{pd}}^{\times}(S(4, 1, 1)) = \gamma(S(4, 1, 1)) = 2$ but the power propagation time of the only high-degree vertex is 4. It is true that for any graph that has at least one vertex of degree two or more, there is an optimal set for product power throttling in which all vertices have degree at least two (no leaves). This can be seen by replacing each leaf in an optimal set for product power throttling by its neighbor (no redundancies can be created or the set would not have been optimal).

For a graph H, the *corona* of H with K_1, denoted by $H \circ K_1$, is the graph obtained from H by appending a leaf to each vertex of H.

Theorem 4.4. *If H is a connected graph of order at least two and $G = H \circ K_1$, then $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 2\gamma(H)$. Furthermore, any power dominating set for G that is a subset of $V(H)$ must be a dominating set for H.*

Proof. First we show $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \leq 2\gamma(H)$. Let S be a dominating set of H with $|S| = \gamma(H)$. After the first round, all vertices of H are observed, each vertex of H has at most one unobserved neighbor, and each unobserved vertex has an observed neighbor. Thus, all vertices of G are observed after the second round.

Next we prove that any power dominating set for G that is a subset of $V(H)$ must be a dominating set for H. Let $S \subseteq V(H)$ be a power dominating set of G. Suppose that there exists a vertex $w \in V(H)$ that remains unobserved after the first round, and thus none of w's neighbors are in S. Every $u \in V(H)$ adjacent to w is also adjacent to at least one additional unobserved vertex (its leaf neighbor). Thus w will never be observed by one of its neighbors and this contradicts the assumption that S is a power dominating set.

Finally, we show $\mathrm{th}_{\mathrm{pd}}^{\times}(G) \geq 2\gamma(H)$. First consider the case that $\mathrm{th}_{\mathrm{pd}}^{\times}(G)$ is realized by a power dominating set S with power propagation time at least two. Without loss of generality, we may assume $S \subseteq V(H)$ (cf. Remark 4.3). Then $S \geq \gamma(H)$ since we proved above that S is a dominating set of H. Thus $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = |S| \,\mathrm{pt}_{\mathrm{pd}}(G; S) \geq 2\gamma(H)$. Now consider the case in which $\mathrm{th}_{\mathrm{pd}}^{\times}(G)$ is realized by a dominating set S of G, so $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$. Observe that $\gamma(G) = |V(H)|$ since G has $|V(H)|$ leaves and each leaf must be dominated by a different vertex of G. Thus $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G) = |V(H)| \geq 2\gamma(H)$ by Theorem 3.9. □

Since $\gamma(H' \circ K_1) = n'$ for any connected graph H' of order n', the next result is immediate.

Corollary 4.5. *If H is a connected graph of order n and $G = (H \circ K_1) \circ K_1$, then $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 2n = \frac{1}{2}|V(G)|$.*

We use the next characterization of graphs G of order n having $\gamma(G) = \frac{n}{2}$ to characterize graphs having $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \frac{n}{2}$.

Theorem 4.6. [10, Theorem 2.2] *A connected graph G of order $n \geq 2$ has $\gamma(G) = \frac{n}{2}$ if and only if $G = G' \circ K_1$ for some connected graph G' or $G = C_4$.*

Theorem 4.7. *A connected graph G of order $n \geq 2$ has $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \frac{n}{2}$ if and only if $G = (H \circ K_1) \circ K_1$ for some connected graph H, $G = C_4 \circ K_1$, $G = C_4$, or $G = K_2$.*

Proof. First we can see that each of the graphs G that has one of the specified forms satisfies $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \frac{n}{2}$: Corollary 4.5 implies $\mathrm{th}_{\mathrm{pd}}^{\times}((H \circ K_1) \circ K_1) = \frac{1}{2}|V((H \circ K_1) \circ K_1)|$. Theorem 4.4 implies $\mathrm{th}_{\mathrm{pd}}^{\times}(C_4 \circ K_1) = 4$. Since $\gamma(C_4) = 2$ and $\gamma(K_2) = 1$, $\mathrm{th}_{\mathrm{pd}}^{\times}(C_4) = 2$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(K_2) = 1$ (cf. Section 4.1).

Assume G is a connected graph of order $n \geq 2$ such that $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \frac{n}{2}$, which implies n is even since $\mathrm{th}_{\mathrm{pd}}^{\times}(G)$ is an integer. Then, by Observation 2.3 and Theorem 3.9, $\gamma(G) = \frac{n}{2}$. Thus $G = G' \circ K_1$ for some connected graph G' or $G = C_4$ by Theorem 4.6. If $n = 2$, then $G = K_1 \circ K_1 = K_2$.

It remains to show that G has the specified form when $n \geq 4$ and $G = G' \circ K_1$. Then $\frac{n}{2} = \mathrm{th}_{\mathrm{pd}}^{\times}(G) = 2\gamma(G')$ by Theorem 4.4. Thus $\gamma(G') = \frac{n}{4} = \frac{|V(G')|}{2}$, so $G' = H \circ K_1$ for some connected H or $G' = C_4$. □

A graph $G = (H \circ K_1) \circ K_1$ can also be constructed from a connected graph H of order at least one by appending to each vertex u of H a path of length two (with vertices x_u and y_u) and a path of length one (with vertex z_u) as shown in Figure 4.1.

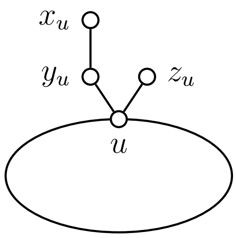

Figure 4.1: Constructing a graph with product power throttling number equal to half its order.

5 Unit interval graphs

A graph G is an *interval graph* if each vertex $v \in V(G)$ can be assigned a closed real interval $I(v)$ so that vertices are adjacent precisely when their assigned intervals intersect. In symbols, for $x, y \in V(G)$ we have $xy \in E(G)$ if and only if $I(x) \cap I(y) \neq \emptyset$. A graph G is a *unit interval graph* if it has such a representation in which each interval has length one. A path is an example of a unit interval graph, a star $K_{1,r}$ with $r \geq 3$ is an interval graph that is not a unit interval graph, and a cycle of order at least four is not an interval graph. Any unit interval graph has a unit interval representation in which all the interval endpoints are distinct, and we assume all our representations have this property. See [8] for additional background. It is convenient to write $I(v) = [\ell(v), r(v)]$ where $r(v) - \ell(v) = 1$. If G is a unit interval graph with a fixed representation, we refer to $\ell(v)$ as the *left endpoint* of the vertex v (as well as the left endpoint of the interval $I(v)$), and analogously for $r(v)$.

In Theorem 5.6, we show that the product power throttling number of a unit interval graph is its domination number. The proof of Theorem 5.6 will depend on several lemmas. We begin with some additional notation. Let G be a unit interval graph and fix a unit interval representation $I(v) = [\ell(v), r(v)]$. Then the order of the left endpoints provides an order on the vertices, called the *induced order*. That is, $v < u$ if and only if $\ell(v) < \ell(u)$.

Observation 5.1. *Let G be a connected unit interval graph with a fixed unit representation. For each vertex v, the closed neighborhood $N[v]$ is a consecutive set of vertices in the induced order.*

The next lemma shows that in a unit interval graph with a fixed representation, the order of the vertices in a forcing chain $v_0 \to v_1 \to \cdots \to v_i$ either follows the induced order $<$ on the vertices, or follows the reverse order (i.e., $v_{j-1} > v_j$ for $j = 1, \ldots, i$). Recall that $\mathrm{rd}(v)$ is the number of the round in which vertex v is first observed.

Lemma 5.2. *Let G be a connected unit interval graph with a fixed unit representation and induced vertex order $<$. Furthermore, let S be a power dominating set of G, $\mathcal{F}$ be a set of forces corresponding to S, and $v_0 \to v_1 \to \cdots \to v_i$ be a forcing chain. Then $v_0 < v_1$ implies $v_1 < v_2 < \cdots < v_i$, and implies $\mathrm{rd}(u) \leq \mathrm{rd}(v) - 1$ when $i \geq 2$ and $v_0 \leq u < v_i$. Analogous statements are true when $v_0 > v_1$.*

Proof. Let $k = \mathrm{rd}(v_i)$ and note that $k \geq i$ (see Observation 1.2). Both statements are proved together by induction assuming $v_0 < v_1$. If $i = 1$, then there is nothing to prove. Now suppose $i \geq 2$ (so $k \geq 2$) and the statement is true for $i-1$. That is, $v_1 < v_2 < \cdots < v_{i-1}$ and $u \in P^{[k-1]}(S)$ for all u such that $v_0 \leq u < v_{i-1}$. Suppose to the contrary that $v_i < v_{i-1}$ ($v_i = v_{i-1}$ is impossible). If $v_0 \leq v_i$, then $v_0 \leq v_i < v_{i-1}$ implies $v_i \in P^{[k-1]}(S)$, contradicting $\mathrm{rd}(v_i) = k$. Suppose $v_i < v_0$. Then $v_{i-1} \in N[v_i]$ and $v_i < v_0 < v_{i-1}$ imply $v_0 \in N[v_i]$ by Observation 5.1, or equivalently, $v_i \in N[v_0]$. This implies $\mathrm{rd}(v_i) = 1$, contradicting $\mathrm{rd}(v_i) = k \geq 2$. Thus $v_i > v_{i-1}$. Since $v_{i-1} \to v_i$ in round k, every other neighbor of v_{i-1} is in $P^{[k-1]}(S)$, i.e., $v_{i-1} \leq u < v_i$ implies $u \in P^{[k-1]}(S)$. □

Lemma 5.3. *Let G be a connected unit interval graph with initial power dominating set S. Then $|P^{(k)}(S)| \leq 2|S|$ for every $k \geq 2$.*

Proof. Fix a unit interval representation of G with induced order $<$. Let $S = \{s_1, s_2, \ldots, s_p\}$ where $s_1 < s_2 < \cdots < s_p$. Suppose that $\mathrm{rd}(v) = k$ for some $k \geq 2$ and $s_j < v < s_{j+1}$ for some j with $1 \leq j \leq p-1$. There exists a forcing chain $v_0 \to v_1 \to \cdots \to v_{i-1} \to v_i = v$ with $i \leq k$ and $v_0 \in S$. If $v_0 < v$, then $\mathrm{rd}(u) \leq k-1$ for all u such that $v_0 \leq u < v$ by Lemma 5.2, and if $v_0 > v$, then then $\mathrm{rd}(u) \leq k-1$ for all u such that $v_0 \geq u > v$. Since $v_0 \leq s_j < v$ (or $v < s_{j+1} \leq v_0$) there are at most two vertices in $P^{(k)}(S)$ between s_j and s_{j+1} (in the induced order). Similarly, there is at most one vertex in $P^{(k)}(S)$ before s_1 and at most one vertex in $P^{(k)}(S)$ after s_p (in the induced order). Thus $|P^{(k)}(S)| \leq 2|S|$ for every $k \geq 2$. □

In power domination (and zero forcing), when a vertex x is first observed in round k (i.e., $\mathrm{rd}(x) = k$), it is not always the case that x has a neighbor y with $\mathrm{rd}(y) = k-1$. However, the next lemma shows that this must happen in a unit interval graph.

Lemma 5.4. *If G is a connected unit interval graph with power dominating set S, then for each $k \geq 1$, every vertex in $P^{(k)}(S)$ is adjacent to a vertex in $P^{(k-1)}(S)$.*

Proof. Fix a unit interval representation of G with induced order $<$, and let S be a power dominating set. The result is clearly true (for any graph) for $k = 1$ and $k = 2$, so we assume $k \geq 3$, and let $\mathrm{rd}(x) = k$. Assume to the contrary that $\mathrm{rd}(y) \neq k-1$ for all $y \in N(x)$. Let z be a neighbor of x such that $z \to x$, so $1 \leq \mathrm{rd}(z) \leq k-2$. Since z did not observe x in round $k-1$, z must have another neighbor w such that $\mathrm{rd}(w) = k-1$. Since x has no neighbors in $P^{(k-1)}(S)$, we know $xw \notin E(G)$. Thus z is adjacent to both x and w, which are not adjacent to each other. Thus $I(z)$ intersects both $I(x)$ and $I(w)$, but $I(x) \cap I(w) = \emptyset$. Therefore, either $w < z < x$ or $x < z < w$. In either case, any vertex v such that $v \to z$ in round $\mathrm{rd}(z) \leq k-2$ would also have been adjacent to a second unobserved vertex (x or w) at that time. This is a contradiction to the rules of power domination if $k \geq 4$. If $k = 3$ then $v \in S$ and consequently either x or w is in $P^{(1)}(S)$, a contradiction because $x \in P^{(k)}(S) = P^{(3)}(S)$ and $w \in P^{(k-1)}(S) = P^{(2)}(S)$. □

For a unit interval graph G, fix a unit representation of G with induced order $<$ and a power dominating set $S = \{s_1, s_2, s_3, \ldots, s_p\}$ where $s_1 < s_2 < \cdots < s_p$. Let u_i be the least neighbor of s_i in the order (if such a neighbor exists) and similarly, let v_i be the greatest neighbor of s_i (if such a neighbor exists). This is illustrated in Figure 5.1. Define $T(S) = \{u_1, u_2, \ldots, u_p\} \cup \{v_1, v_2, \ldots, v_p\}$. By construction, each vertex of S contributes at most 2 vertices to $T(S)$, so $|T(S)| \leq 2|S|$. The next lemma shows that $T(S)$ dominates the vertices in $S \cup P^{(1)}(S) \cup P^{(2)}(S) = P^{[2]}(S)$.

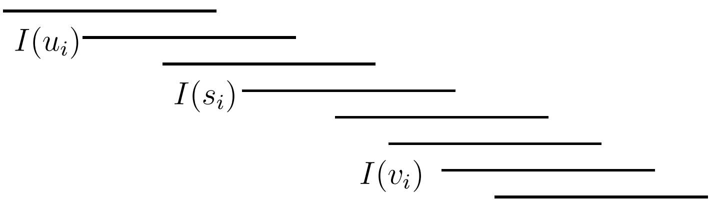

Figure 5.1: Illustration of the values of u_i and v_i for a given s_i.

Lemma 5.5. *Let G be a connected unit interval graph of order at least two with a fixed unit representation and induced order, and let S be a power dominating set of G. If $T(S)$ is the subset of $P^{(1)}(S)$ defined above, then every vertex in $S \cup P^{(1)}(S) \cup P^{(2)}(S)$ is dominated by a vertex in $T(S)$.*

Proof. Since we are considering only graphs that are connected and nontrivial, $T(S)$ dominates S by construction. Next we show $T(S)$ dominates $P^{(1)}(S)$. By definition, any vertex $z \in P^{(1)}(S)$ is adjacent to some $s_i \in S$, so for that value of i we have $I(z) \cap I(s_i) \neq \emptyset$. If $I(z)$ contains the left endpoint $\ell(s_i)$, then $z = u_i$ or $zu_i \in E(G)$, so z is either an element of $T(S)$ or dominated by an element of $T(S)$. The case in which $I(z)$ contains the right endpoint $r(s_i)$ is similar. Thus $T(S)$ dominates $P^{(1)}(S)$

Finally we show $T(S)$ dominates the vertices in $P^{(2)}(S)$. Consider $w \in P^{(2)}(S)$. Suppose $s_i \to z \to w$ with $s_i > z > w$, so $\mathrm{rd}(z) = 1$. By construction, $\ell(z) \geq \ell(u_i)$, so $I(u_i)$ also intersects $I(w)$, and thus w is adjacent to a vertex in $T(S)$. The case in which $s_i < z < w$ is similar. This completes the proof. $\square$

We are now ready to prove the main result of this section.

Theorem 5.6. *If G is a connected unit interval graph, then $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$.*

Proof. Let G be a connected unit interval graph of order at least two with a fixed unit representation and induced order $<$. Let $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \mathrm{th}_{\mathrm{pd}}^{\times}(G; S) = |S|t$ where $t = \mathrm{pt}_{\mathrm{pd}}(G; S)$. We consider three cases:

(i) $t = 1$,

(ii) t is an even integer greater than 1, and

(iii) t is an odd integer greater than 1.

It suffices to show $\gamma(G) \leq \mathrm{th}_{\mathrm{pd}}^{\times}(G)$ by Observation 2.3.

(i): Since $t = 1$, S is a dominating set and $\gamma(G) \leq |S| = |S|t = \mathrm{th}_{\mathrm{pd}}^{\times}(G)$.

Otherwise, we may assume $t \geq 2$. Let $T(S)$ be the set defined just before Lemma 5.5.

(ii): Assume t is even. Let $\hat{S} = T(S) \cup P^{(3)}(S) \cup P^{(5)}(S) \cup \cdots \cup P^{(t-1)}(S)$. By Lemma 5.5, $T(S)$ dominates $S \cup P^{(1)}(S) \cup P^{(2)}(S)$, and by Lemma 5.4, the vertices in $P^{(2j)}(S)$ are dominated by the set $P^{(2j-1)}(S)$ for $2 \leq j \leq \frac{t}{2}$. Thus $\hat{S}$ is a dominating set for G and $|\hat{S}| = |T(S)| + |P^{(3)}(S)| + |P^{(5)}(S)| + \cdots + |P^{(t-1)}(S)|$. By Lemma 5.3, $|P^{(k)}(S)| \leq 2|S|$ for every $k \geq 2$, and as we noted just before Lemma 5.5, $|T(S)| \leq 2|S|$. Thus

$$\gamma(G) \leq |\hat{S}| = |T(S)| + |P^{(3)}(S)| + |P^{(5)}(S)| + \cdots + |P^{(t-1)}(S)| \leq (2|S|)\frac{t}{2} = |S|t = \mathrm{th}_{\mathrm{pd}}^{\times}(G).$$

(iii): Assume t is odd. Let $\hat{S} = S \cup P^{(2)}(S) \cup P^{(4)}(S) \cup \cdots \cup P^{(t-1)}(S)$. The vertices in $P^{(1)}(S)$ are dominated by S by definition, and the vertices in $P^{(2j+1)}(S)$ are dominated by the set $P^{(2j)}(S)$ for $1 \leq j \leq \frac{t-1}{2}$ by Lemma 5.4. Thus $\hat{S}$ is a dominating set for G and $\gamma(G) \leq \mathrm{th}_{\mathrm{pd}}^{\times}(G)$ as in case (ii). □

We observe that the domination number of a connected unit interval graph can be found from a unit interval representation of G using the following greedy algorithm. Let G be a unit interval graph where $V(G) = \{v_1, v_2, \ldots, v_n\}$, interval $I(v_i)$ is assigned to vertex v_i for each i, and, $\ell(v_1) < \ell(v_2) < \cdots < \ell(v_n)$. Start with $S = \emptyset$ and add v_k to S where k is maximum so that $I(v_1) \cap I(v_k) \neq \emptyset$. Now remove v_k and its neighbors from G and iterate. This produces a dominating set for G. More generally, the dominating number of interval graphs (and several related graph classes) can be computed in polynomial time [6].

Theorem 5.6 need not be true for interval graphs in general, as shown by the next example.

Example 5.7. Let G be the graph shown in Figure 5.2. Then $\gamma(G) = 3$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \mathrm{th}_{\mathrm{pd}}^{\times}(G; \{3\}) = 1 \cdot 2 = 2$. Observe that $I(1) = [0, 3]$, $I(2) = [2, 5]$, $I(3) = [4, 9]$, $I(4) = [8, 11]$, $I(5) = [10, 13]$, $I(6) = [6, 7]$ is an interval representation of G. Furthermore, G is not a unit interval graph since $G[\{2, 3, 4, 6\}]$ is a $K_{1,3}$, which is prohibited for a unit interval graph.

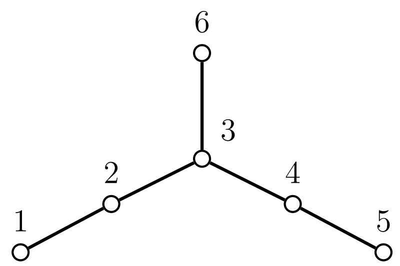

Figure 5.2: An interval graph G with $\mathrm{th}_{\mathrm{pd}}^{\times}(G) < \gamma(G)$.

6 Cartesian products

The *Cartesian product* $G\Box H$ of graphs G and H is the graph whose vertex set is $V(G\Box H) = V(G) \times V(H)$ where two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G\Box H$ if either $x_1 = x_2$ and $y_1y_2 \in E(H)$ or $y_1 = y_2$ and $x_1x_2 \in E(G)$. In this section we provide bounds on the product power throttling number of Cartesian products. We show that the product power throttling number equals the domination number for some families of Cartesian products, including grid graphs (on the plane, cylinder, and torus) and Cartesian products of complete graphs with complete graphs; we also exhibit examples of Cartesian products where the product power throttling number does not equal the domination number.

6.1 Bounds

We begin with upper bounds. We know that $\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box H) \leq \gamma(G\Box H)$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box H) \leq \gamma_P(G\Box H)\,\mathrm{pt}_{\mathrm{pd}}(G\Box H)$ by Observation 2.3. The next result uses the structure of a Cartesian product to obtain additional upper bounds.

Theorem 6.1. *For any connected graphs G and H,*

$$\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box H) \leq \mathrm{th}_{\mathrm{pd}}^{\times}(G)|V(H)| \quad \textit{and} \quad \mathrm{th}_{\mathrm{pd}}^{\times}(G\Box H) \leq \mathrm{th}_{\mathrm{pd}}^{\times}(H)|V(G)|.$$

Proof. Choose a set S such that $\mathrm{th}_{\mathrm{pd}}^{\times}(G; S) = \mathrm{th}_{\mathrm{pd}}^{\times}(G)$, which implies that $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = |S|\,\mathrm{pt}_{\mathrm{pd}}(G; S)$. Let $S' = S \times V(H)$; that is, S' is the set of vertices associated with S in each copy of G. Since S' will power dominate $G\Box H$ using each copy of S simultaneously, S' is a power dominating set of $G\Box H$ and $\mathrm{pt}_{\mathrm{pd}}(G\Box H; S') \leq \mathrm{pt}_{\mathrm{pd}}(G; S)$. Thus,

$$\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box H) \leq |S'|\,\mathrm{pt}_{\mathrm{pd}}(G\Box H; S') = |S||V(H)|\,\mathrm{pt}_{\mathrm{pd}}(G; S) = \mathrm{th}_{\mathrm{pd}}^{\times}(G)|V(H)|.$$

Similarly, $\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box H) \leq \mathrm{th}_{\mathrm{pd}}^{\times}(H)|V(G)|$. □

The Cartesian product in the next example achieves one of the bounds in Theorem 6.1 that is less than $\gamma(G\Box H)$ and $\gamma_P(G\Box H)\,\mathrm{pt}_{\mathrm{pd}}(G\Box H)$.

Example 6.2. Let $G = S(7, 2, 2, 2, 2, 2)$ as shown in Figure 2.1. Consider the graph $G\Box P_2$. We show that $\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box P_2) = \mathrm{th}_{\mathrm{pd}}^{\times}(G)|V(P_2)| = 8 < \gamma(G\Box P_2) = 10 < \gamma_P(G\Box P_2)\,\mathrm{pt}_{\mathrm{pd}}(G\Box P_2) = 14$.

We compute $\gamma(G\Box P_2) = 10$ and $\gamma_P(G\Box P_2) = 2$ [11]. Recall that $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 4$ was established in Example 2.4 and $|V(P_2)| = 2$, so $8 = \mathrm{th}_{\mathrm{pd}}^{\times}(G)|V(P_2)| \geq \mathrm{th}_{\mathrm{pd}}^{\times}(G\Box P_2)$. To show that $\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box P_2) = 8$, by Proposition 3.8(2) we need consider only power dominating sets S such that $2 \leq |S| \leq \frac{8}{2}$, and since $4 < \gamma(G\Box P_2)$, we know $\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box P_2, 4) = 8$. We compute $\mathrm{pt}_{\mathrm{pd}}(G\Box P_2, 2) = 7$ [11], which implies $\gamma_P(G\Box P_2)\,\mathrm{pt}_{\mathrm{pd}}(G\Box P_2) = 14$, and $\mathrm{pt}_{\mathrm{pd}}(G\Box P_2, 3) = 4$ [11], so $\mathrm{th}_{\mathrm{pd}}^{\times}(G\Box P_2, 3) = 12$.

Next we construct an example of a Cartesian product that has power product throttling number less than $\gamma(G\Box H)$, $\gamma_P(G\Box H)\,\mathrm{pt}_{\mathrm{pd}}(G\Box H)$, and the bounds in Theorem 6.1.

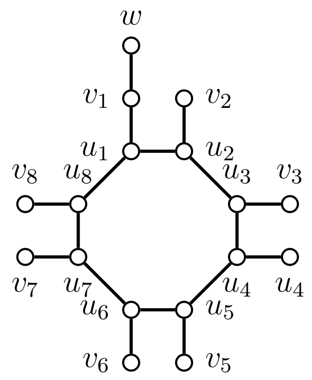

Figure 6.1: The graph G in Example 6.3.

Example 6.3. Let G be the graph in Figure 6.1 with vertex set $\{u_1, u_2, \ldots, u_8\} \cup \{v_1, v_2, \ldots, v_8\} \cup \{w\}$ where the induced subgraph on $\{u_1, u_2, \ldots, u_8\}$ is an 8-cycle, v_i is adjacent to u_i for $1 \le i \le 8$, and w is adjacent to v_1. Thus G is an 8-cycle with a path of length 2 appended to u_1 and a leaf appended to u_j for $2 \le j \le 8$. Note that G has 17 vertices. We denote the vertices of $G \square P_2$ by $\{u_1, \ldots, u_8, v_1, \ldots, v_8, w, u'_1, \ldots, u'_8, v'_1, \ldots, v'_8, w'\}$.

To show that $\mathrm{th}_{\mathrm{pd}}^{\times}(G \square P_2) = 10$, note first that $\mathrm{pt}_{\mathrm{pd}}(G \square P_2, \{u_1, u_5, u'_1, u'_3, u'_7\}) = 2$, so $\mathrm{th}_{\mathrm{pd}}^{\times}(G \square P_2) \le 10$. To show the reverse inequality, we use [11] to compute $\gamma(G \square P_2) = 11$ and $\gamma_P(G \square P_2) = 3$. Since $\mathrm{th}_{\mathrm{pd}}^{\times}(G \square P_2) \le 10 < \gamma(G \square P_2)$, we need consider only power dominating sets S such that $3 \le |S| \le \frac{10}{2}$ by Proposition 3.8(2). We compute $\mathrm{pt}_{\mathrm{pd}}(G \square P_2, 3) = 7$ so $\gamma_P(G \square P_2)\,\mathrm{pt}_{\mathrm{pd}}(G \square P_2) = \mathrm{th}_{\mathrm{pd}}^{\times}(G \square P_2, 3) = 21$, and $\mathrm{pt}_{\mathrm{pd}}(G \square P_2, 4) = 3$ so $\mathrm{th}_{\mathrm{pd}}^{\times}(G \square P_2, 4) = 12$. Since $5 < \gamma(G \square P_2)$, $\mathrm{th}_{\mathrm{pd}}^{\times}(G \square P_2, 5) \ge 10$.

Since $\gamma_P(G) = 3$, $\mathrm{pt}_{\mathrm{pd}}(G) = 2$, and $\gamma(G) = 8$, we have $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = 6$ by Proposition 3.8(2). Thus $\mathrm{th}_{\mathrm{pd}}^{\times}(G)|V(P_2)| = 12$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(P_2)|V(G)| = 17$.

As with upper bounds, the structure of a Cartesian product gives additional lower bounds.

Observation 6.4. *For connected graphs G and H, $\mathrm{th}_{\mathrm{pd}}^{\times}(G \square H) \ge \left\lceil \frac{|V(G)||V(H)|}{\Delta(G)+\Delta(H)+1} \right\rceil$ by Corollary 2.6 and the fact $V(G \square H) = |V(G)||V(H)|$ and $\Delta(G \square H) = \Delta(G) + \Delta(H)$.*

In order to bound the product throttling number of a Cartesian product by the product throttling number of a factor, we need a preliminary result that bounds the power propagation time of a set in a Cartesian product in terms of the power propagation time of a related set in one of the factors. If $G \square H$ is a Cartesian product of graphs G and H and $S \subset V(G \square H)$, define the *projection of S onto G*, denoted by S_G, to be $S_G = \{x : (x, y) \in S \text{ for some } y \in V(H)\}$.

Proposition 6.5. *Let G and H be connected graphs and let S be a power dominating set of $G \square H$. Then S_G is a power dominating set of G. Furthermore,* $\mathrm{pt}_{\mathrm{pd}}(G; S_G) \le \mathrm{pt}_{\mathrm{pd}}(G \square H; S)$.

Proof. Let $S' = S_G \times V(H)$ and note that $S \subseteq S'$. Then S' is a power dominating set of $G \Box H$ since S is a power dominating set of $G \Box H$. For a (propagating) set of forces, all forces for S' in $G \Box H$ have the form $(x_1, y) \to (x_2, y)$, and $\mathrm{rd}(x, y) = \mathrm{rd}(x, z)$ for all $x \in V(G)$ and $y, z \in V(H)$. For $x \in V(G)$ and $y \in V(H)$, note that $\mathrm{rd}(x) = \mathrm{rd}(x, y)$ starting with $S'_G = S_G$ in G and S' in $G \Box H$. Thus, S_G is a power dominating set of G and $\mathrm{pt}_{\mathrm{pd}}(G; S_G) = \mathrm{pt}_{\mathrm{pd}}(G \Box H; S') \leq \mathrm{pt}_{\mathrm{pd}}(G \Box H; S)$. □

Theorem 6.6. *For any connected graphs G and H,*

$$\mathrm{th}^{\times}_{\mathrm{pd}}(G \Box H) \geq \mathrm{th}^{\times}_{\mathrm{pd}}(G) \text{ and } \mathrm{th}^{\times}_{\mathrm{pd}}(G \Box H) \geq \mathrm{th}^{\times}_{\mathrm{pd}}(H).$$

Proof. Choose a set S such that $\mathrm{th}^{\times}_{\mathrm{pd}}(G \Box H; S) = \mathrm{th}^{\times}_{\mathrm{pd}}(G \Box H)$. Then S_G is a power dominating set of G and $\mathrm{pt}_{\mathrm{pd}}(G; S_G) \leq \mathrm{pt}_{\mathrm{pd}}(G \Box H; S)$ by Proposition 6.5. Since $|S_G| \leq |S|$,

$$\mathrm{th}^{\times}_{\mathrm{pd}}(G) \leq |S_G| \, \mathrm{pt}_{\mathrm{pd}}(G; S_G) \leq |S| \, \mathrm{pt}_{\mathrm{pd}}(G \Box H; S) = \mathrm{th}^{\times}_{\mathrm{pd}}(G \Box H).$$

The proof that $\mathrm{th}^{\times}_{\mathrm{pd}}(G \Box H) \geq \mathrm{th}^{\times}_{\mathrm{pd}}(H)$ is similar. □

6.2 Families having $\mathrm{th}^{\times}_{\mathrm{pd}}(G \Box H) = \gamma(G \Box H)$

In this section we show that the product power throttling number equals the domination number for Cartesian products of complete graphs with complete graphs, paths with paths (grid graphs), paths with cycles, and cycles with cycles.

Proposition 6.7. *For $2 \leq n \leq m$, $\gamma_P(K_n \Box K_m) = n - 1$. For $1 \leq n \leq m$, $\mathrm{th}^{\times}_{\mathrm{pd}}(K_n \Box K_m) = \gamma(K_n \Box K_m) = n$.*

Proof. Let $1 \leq n \leq m$. The result $\mathrm{th}^{\times}_{\mathrm{pd}}(K_1 \Box K_m) = 1 = \gamma(K_1 \Box K_m)$ is immediate, so assume $n \geq 2$. Let $V(K_n \Box K_m) = \{(i, j) : 1 \leq i \leq n, 1, \leq j \leq m\}$.

Since $\{(i, 1) : i = 1, \ldots, n - 1\}$ is a power dominating set, $\gamma_P(K_n \Box K_m) \leq n - 1$. We construct a power dominating set S that is not a dominating set and show that S must have at least $n - 1$ vertices in order for step (2) of the power domination process to take place. Without loss of generality, $(1, 1) \in S$ and a neighbor of $(1, 1)$ performs the first zero force (the first force after the domination step). Observe that $N((1, 1)) = \{(i, 1) : i = 2, 3, \ldots, n\} \cup \{(1, j) : j = 2, 3, \ldots, m\}$. Neighbors of the form $(1, j)$ all behave similarly, so suppose first $(1, 2)$ performs the first zero force. There is exactly one unobserved neighbor of $(1, 2)$ after the domination step. Since $\{(1, j) : j = 1, 3, \ldots, m\} \subset N[S]$, without loss of generality the first zero force is $(1, 2) \to (2, 2)$. This implies $\{(i, 2) : i = 1, 3, \ldots, n\} \subset N[S]$ and $(2, 2) \notin N[S]$. Thus $(i, 2) \notin S$ for $i = 1, \ldots, n$, which implies there exist $(i, j_i) \in S$ for $i = 3, \ldots, n$. Thus $|S| \geq n - 1$. If a neighbor of the form $(i, 1)$ performs the first zero force, then $|S| \geq m - 1$. Thus $\gamma_P(K_n \Box K_m) = n - 1$.

Since $\gamma(K_n \Box K_m) = n$ and $\gamma_P(K_n \Box K_m) = n - 1 \geq \frac{n}{2}$, we have $\mathrm{th}^{\times}_{\mathrm{pd}}(K_n \Box K_m) = \gamma(K_n \Box K_m) = n$ by Proposition 3.8(1). □

Proposition 6.8. *Let H be a connected graph of order n and let $G = H \Box K_m$ with $m \geq \Delta(H)(n-1)+1$. Then $\operatorname{th}_{\mathrm{pd}}^{\times}(G) = n = \gamma(G)$.*

Proof. Since $V(H) \times \{y\}$ is a dominating set of G for any $y \in V(K_m)$, we know $\gamma(G) \leq n$, so $\operatorname{th}_{\mathrm{pd}}^{\times}(G) \leq \gamma(G) \leq n$. It remains to show that $\operatorname{th}_{\mathrm{pd}}^{\times}(G) \geq n$. By Observation 6.4, we also know that $\operatorname{th}_{\mathrm{pd}}^{\times}(G) \geq \left\lceil \frac{nm}{m+\Delta(H)} \right\rceil$ since $\Delta(K_m) = m-1$. Note that $\left\lceil \frac{nm}{m+\Delta(H)} \right\rceil \geq n$ since $m \geq \Delta(H)(n-1)+1$. Thus $\operatorname{th}_{\mathrm{pd}}^{\times}(G) \geq n$. □

Since $\Delta(C_n) = 2$ and $\Delta(P_n) = 2$, the next result follows immediately from Proposition 6.8.

Corollary 6.9. *If $G = H \Box K_m$ with $H = C_n$ or P_n and $m \geq 2n-1$, then $\operatorname{th}_{\mathrm{pd}}^{\times}(G) = n$.*

Next we show that $\operatorname{th}_{\mathrm{pd}}^{\times}(P_n \Box P_m) = \gamma(P_n \Box P_m)$, $\operatorname{th}_{\mathrm{pd}}^{\times}(P_n \Box C_m) = \gamma(P_n \Box C_m)$, $\operatorname{th}_{\mathrm{pd}}^{\times}(C_n \Box P_m) = \gamma(C_n \Box P_m)$, and $\operatorname{th}_{\mathrm{pd}}^{\times}(C_n \Box C_m) = \gamma(C_n \Box C_m)$ for all $n \leq m$. The power domination number of a grid graph is known [5]: For $m \geq n \geq 1$,

$$\gamma_P(P_n \Box P_m) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil & \text{if } n \not\equiv 4 \mod 8 \\ \left\lceil \frac{n+1}{4} \right\rceil & \text{if } n \equiv 4 \mod 8 \end{cases}. \tag{2}$$

The domination number is known exactly for only certain values of n; a summary of results appears in [1] and are detailed later as used. Let $J_n = P_n$ or C_n for $n \geq 3$ and $J_n = P_n$ for $n = 1, 2$. Note that $P_n \Box P_m$ is a spanning subgraph of $J_n \Box J_m$, so $\gamma(J_n \Box J_m) \leq \gamma(P_n \Box P_m)$.

We orient $J_n \Box J_m$ near a given vertex x as a grid with n rows and m columns, and refer to the directions from x as north, east, south, and west. When $J_n = C_n$, there is an additional edge between the nothernmost vertex and southernmost vertex of each column, and when $J_m = C_m$, there is an additional edge between the easternmost vertex and westernmost vertex of each row.

Let S be a power dominating set of $J_n \Box J_m$ and let $\mathcal{F}$ be a set of forces of S. For each vertex w in $P^{(2)}(S)$, $\mathcal{F}$ defines a forcing chain $v_0 \to v_1 \to w$. Define the functions $f_1 : P^{(2)}(S) \to P^{(1)}(S)$ and $f_0 : P^{(2)}(S) \to S$ by $f_1(w) = v_1$, and by $f_0(w) = v_0$. By the definition of power domination, f_1 is an injective function (but f_0 need not be injective). For $u \in S$, define $Q_u = \{w \in P^{(2)}(S) : f_0(w) = u\}$. Limiting the size of $P^{(2)}(S)$ is a key idea for the proofs that $\operatorname{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m) = \gamma(J_n \Box J_m)$.

Proposition 6.10. *Let $n, m \geq 4$ and let S be a power dominating set of $J_n \Box J_m$. There is a set of forces $\mathcal{F}$ of S such that $|Q_x| \leq 3$ for each $x \in S$.*

Proof. For any $x \in S$, $|Q_x| \leq 4$ since $\deg(x) \leq 4$. Suppose that $|Q_x| = 4$. We claim that the forces $x \to y \to w$ must occur in the same direction on the grid, e.g., if y is the north neighbor of x, then w is the north neighbor of y. Suppose not, e.g., y is the north neighbor of x and w is the west neighbor of y. Then w is a neighbor of the west neighbor of x, so the west neighbor of x cannot perform a force in round 2, contradicting $|Q_x| = 4$. The other directions are similar. Hence the vertices in

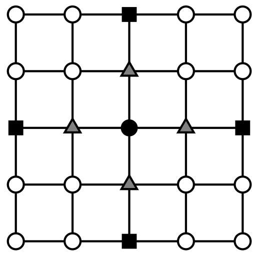

Figure 6.2: The black circle vertex is $x \in S$, the triangle vertices are its neighbors and the square vertices are in Q_x.

Q_x must be the four vertices that are distance 2 from x in the four directions (the square vertices in Figure 6.2); this applies only because $|Q_x| = 4$.

Let x_N be the north neighbor of x, let x_W be the west neighbor of x, and let x_{NN} be the north neighbor of x_N. In order to have $x_N = f_1(x_{NN})$, i.e., $x_N \to x_{NN}$ in round 2, the east and west neighbors of x_N, called x_{NE} and x_{NW}, must be observed in round 0 or round 1. Suppose x_{NW} is observed in round 1. The south neighbor of x_{NW} is x_W and x_W cannot observe x_{NW} in round 1, nor can x_N observe x_{NW} in round 1. Thus either the west neighbor x_{NWW} or the north neighbor x_{NWN} of x_{NW} observes x_{NW} in round 1. If $x_{NWW} \in S$, then the west neighbor of x_W is observed in round 1, so x_W cannot observe it in round 2, and similarly if $x_{NWN} \in S$ then x_{NN} is observed in round 1. So x_{NW} cannot be observed in round 1 and thus $x_{NW} \in S$. Then we can reassign the forcing chain $x \to x_N \to x_{NN}$ to $x_{NW} \to x_N \to x_{NN}$, obtaining $Q'_{x_{NW}} = Q_{x_{NW}} \cup \{x_{NN}\}$ and $Q'_x = Q_x \setminus \{x_{NN}\}$. Since the south neighbor of x_{NW} is x_W, which is still forced by x, with the new assignment $|Q'_{x_{NW}}| \leq 3$, and $|Q'_x| \leq 3$. The other directions are similar. □

Theorem 6.11. *Let $m \geq n \geq 4$. If S is a power dominating set of $J_n \Box J_m$ that is not a dominating set, then $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m; S) \geq \lceil \frac{nm}{4} \rceil$. Furthermore, $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m) = \gamma(J_n \Box J_m)$.*

Proof. Suppose $\mathrm{pt}_{\mathrm{pd}}(J_n \Box J_m; S) \geq 2$ and let $t = \mathrm{pt}_{\mathrm{pd}}(J_n \Box J_m; S)$. Then $|P^{(i+1)}(S)| \leq |P^{(1)}(S)|$ for all $i \geq 0$ by Remark 1.3. Since the maximum degree in $J_n \Box J_m$ is 4, $|P^{(1)}(S)| \leq 4|S|$. By Proposition 6.10, there is an assignment of forcing chains so that for each vertex $x \in S$, $|\{w \in P^{(2)}(S) : f_0(w) = x\}| \leq 3$. Therefore, $|P^{(2)}(S)| \leq 3|S|$, and thus

$$nm = |V(J_n \Box J_m)| = |S| + \sum_{i=1}^{t} |P^{(i)}(S)| \leq |S|(1 + 4 + 3 + 4(t-2)) = |S|(4t).$$

This implies $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m; S) = t|S| \geq \lceil \frac{nm}{4} \rceil$. This lower bound applies whenever $\mathrm{pt}_{\mathrm{pd}}(J_n \Box J_m; S) \geq 2$, i.e., whenever S is not a dominating set.

Since $J_n \Box J_m$ contains $P_n \Box P_m$ as a spanning subgraph, $\gamma(P_n \Box P_m) \geq \gamma(J_n \Box J_m)$. To show that $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m) = \gamma(J_n \Box J_m)$, we combine the bound just obtained, i.e., $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m; S) \geq \lceil \frac{nm}{4} \rceil$ when S is not a dominating set, with known results for the domination number of $P_n \Box P_m$. The cases $n \geq 8$, $n = 7$, $n = 4$, $n = 5$, and

$n = 6$ are analyzed separately, with $n = 4, 5, 6$ and small values of m being done computationally.

It can be verified algebraically that $\frac{nm}{4} \geq \frac{(n+2)(m+2)}{5} - 4$ for $n, m \geq 8$, and Chang showed in [4] that $\left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4 \geq \gamma(P_n \Box P_m)$ for $n, m \geq 8$ (see [1]). For $n = 7$, it is known that $\gamma(P_n \Box P_m) = \left\lfloor \frac{5m+3}{3} \right\rfloor$ [1]. It can be verified algebraically that $\frac{7m}{4} \geq \frac{5m+3}{3}$ for $m \geq 12$, and numerically that $\left\lceil \frac{7m}{4} \right\rceil \geq \left\lfloor \frac{5m+3}{3} \right\rfloor$ for $m = 7, \ldots, 11$. Thus $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m) = \gamma(J_n \Box J_m)$ for $n, m \geq 7$.

For $n = 4$, it is known that $\gamma(P_4 \Box P_m) = m$ if $m \neq 5, 6, 9$ and $\gamma(P_4 \Box P_m) = m + 1$ if $m = 5, 6, 9$ [1]. Since $\frac{4m}{4} = m$, $\mathrm{th}_{\mathrm{pd}}^{\times}(J_4 \Box J_m) = \gamma(J_4 \Box J_m)$ for $m \neq 5, 6, 9$. For $G = C_4 \Box J_m$ with $m = 5, 6, 9$ or $G = P_4 \Box C_m$ with $m = 6$, $\gamma(G) = m$, so $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$. For the cases $G = P_4 \Box P_m$ with $m = 5, 6, 9$ and $G = P_4 \Box C_m$ with $m = 5, 9$, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$ has been verified computationally [11] and these values are listed in Table 1.

For $n = 5$, it is known that $\gamma(P_5 \Box P_m) = \left\lfloor \frac{6m+8}{5} \right\rfloor$ if $m \neq 7$ and $\gamma(P_5 \Box P_7) = 9$ [1]. It can be verified algebraically that $\frac{5m}{4} \geq \frac{6m+8}{5}$ for $m \geq 32$. Straightforward computations show that $\left\lceil \frac{5m}{4} \right\rceil \geq \gamma(P_5 \Box P_m)$ for $5 \leq m \leq 31$ except $m = 8$ and $m = 12$. For $G = P_5 \Box C_m$ or $G = C_5 \Box J_m$ with $m = 8, 12$, $\left\lceil \frac{5m}{4} \right\rceil \geq \gamma(G)$, so $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$. For the case $G = P_5 \Box P_8$, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$ has been verified [11] and this value is listed in Table 1, leaving only $P_5 \Box P_{12}$ (this case is discussed at the end of the proof).

For $n = 6$, it is known that $\gamma(P_6 \Box P_m) = \left\lfloor \frac{10m+12}{7} \right\rfloor$ if $m \not\equiv 1 \mod 7$ and $\gamma(P_6 \Box P_m) = \left\lfloor \frac{10m+10}{7} \right\rfloor$ if $m \equiv 1 \mod 7$ [1]. It is easily verified algebraically that $\frac{6m}{4} \geq \frac{10m+12}{7}$ for $m \geq 24$. Straightforward computations show that $\left\lceil \frac{6m}{4} \right\rceil \geq \gamma(P_6 \Box P_m)$ for $6 \leq m \leq 23$ except $m = 6$ and $m = 10$. For $G = P_6 \Box C_m$ or $G = C_6 \Box J_m$ with $m = 6, 10$, $\left\lceil \frac{6m}{4} \right\rceil \geq \gamma(G)$, so $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$. For the case $G = P_6 \Box P_6$, $\mathrm{th}_{\mathrm{pd}}^{\times}(G) = \gamma(G)$ has been verified [11] and this value is listed in Table 1, leaving only $P_6 \Box P_{10}$.

Table 1: Table of values of $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m)$ and $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m)$ for selected n and m.

$J_n \Box J_m$	$\mathrm{th}_{\mathrm{pd}}^{\times}$	γ		$J_n \Box J_m$	$\mathrm{th}_{\mathrm{pd}}^{\times}$	γ		$J_n \Box J_m$	$\mathrm{th}_{\mathrm{pd}}^{\times}$	γ
$P_4 \Box P_5$	6	6		$P_4 \Box P_6$	7	7		$P_4 \Box P_9$	10	10
$P_4 \Box C_5$	6	6		$P_4 \Box C_9$	10	10				
$P_5 \Box P_8$	11	11		$P_6 \Box P_6$	10	10				

It remains to check $P_5 \Box P_{12}$ and $P_6 \Box P_{10}$, both of which have order 60, domination number 16 [1], and power domination number 2 (see Equation (2)). Let G denote one of these graphs. Suppose S is a power dominating set of G that is not a dominating set of G, so $\mathrm{pt}_{\mathrm{pd}}(G; S) \geq 2$. Then $|S| \, \mathrm{pt}_{\mathrm{pd}}(G; S) \geq \frac{mn}{4} = 15$. Since power propagation time is an integer, this implies $\mathrm{pt}_{\mathrm{pd}}(G; S) \geq \left\lceil \frac{15}{|S|} \right\rceil$. Thus $\mathrm{pt}_{\mathrm{pd}}(G, k) \geq \left\lceil \frac{15}{k} \right\rceil$ by using $k = |S|$. Observe that $k \, \mathrm{pt}_{\mathrm{pd}}(G, k) \geq 16$ for $k \geq 8$, so consider $k \left\lceil \frac{15}{k} \right\rceil$ for $k = 2, \ldots, 7$. It is immediate that $k \left\lceil \frac{15}{k} \right\rceil \geq 16$ unless $k = 3$ or $k = 5$. We use [11] to compute $\mathrm{pt}_{\mathrm{pd}}(P_5 \Box P_{12}, 3) = 6$, $\mathrm{pt}_{\mathrm{pd}}(P_5 \Box P_{12}, 5) = 4$, $\mathrm{pt}_{\mathrm{pd}}(P_6 \Box P_{10}, 3) = 9$, and $\mathrm{pt}_{\mathrm{pd}}(P_6 \Box P_{10}, 5) = 5$. This completes the proof. $\Box$

The only remaining cases are $P_2\Box J_m$ and $J_3\Box J_m$, which are handled in the next theorem.

Theorem 6.12. *For $m \geq 2$, $\mathrm{th}_{\mathrm{pd}}^{\times}(P_2\Box J_m) = \gamma(P_2\Box J_m)$, and for $m \geq 3$, $\mathrm{th}_{\mathrm{pd}}^{\times}(J_3\Box J_m) = \gamma(J_3\Box J_m)$.*

Proof. It is known that $\gamma(P_2\Box P_m) = \left\lfloor \frac{m+2}{2} \right\rfloor$ [1]. Since $P_2\Box C_m$ contains $P_2\Box P_m$ as a spanning subgraph, $\left\lfloor \frac{m+2}{2} \right\rfloor \geq \gamma(P_2\Box C_m)$. We show that if S is a power dominating set of $P_2\Box J_m$ that is not a dominating set, then $\mathrm{th}_{\mathrm{pd}}^{\times}(P_2\Box J_m; S) \geq \left\lceil \frac{2m}{3} \right\rceil$. It is straightforward to check that $\left\lceil \frac{2m}{3} \right\rceil \geq \left\lfloor \frac{m+2}{2} \right\rfloor$ for $m \geq 2$, which then implies that $\mathrm{th}_{\mathrm{pd}}^{\times}(P_2\Box J_m) = \gamma(P_2\Box J_m)$. For $x \in S$, denote the north, east, south, and west neighbors of x by x_N, x_E, x_S and x_W.

Suppose $\mathrm{pt}_{\mathrm{pd}}(P_2\Box J_m; S) \geq 2$ and let $t = \mathrm{pt}_{\mathrm{pd}}(P_2\Box J_m; S)$. Then $|P^{(i+1)}(S)| \leq |P^{(1)}(S)|$ for all $i \geq 0$ by Remark 1.3. Choose a set of forces $\mathcal{F}$ of S, and for $x \in S$ recall that $Q_x = \{w \in P^{(2)}(S) : f_0(w) = x\}$. Since the maximum degree in $P_2\Box J_m$ is 3, $|P^{(1)}(S)| \leq 3|S|$ and $|Q_x| \leq 3$ for $x \in S$. Suppose $x \in S$ is on the bottom row of $P_2\Box J_m$. If x_N forces to the east or west in round 2, then the neighbor of x in the same direction cannot force in round 2. Therefore, $|Q_x| \leq 2$, $|P^{(2)}(S)| \leq 2|S|$, and thus

$$2m = |V(P_2\Box J_m)| = |S| + \sum_{i=1}^{t} |P^{(i)}(S)| \leq |S|(1 + 3 + 2 + 3(t-2)) = |S|(3t).$$

This implies $\mathrm{th}_{\mathrm{pd}}^{\times}(P_2\Box J_m; S) = t|S| \geq \left\lceil \frac{2m}{3} \right\rceil$.

It is known that $\gamma(P_3\Box P_m) = \left\lfloor \frac{3m+4}{4} \right\rfloor$ [1], and thus $\gamma(J_3\Box J_m) \leq \left\lfloor \frac{3m+4}{4} \right\rfloor$. We show that if S is a power dominating set of $J_3\Box J_m$ that is not a dominating set, then $\mathrm{th}_{\mathrm{pd}}^{\times}(J_3\Box J_m; S) \geq \left\lfloor \frac{3m+4}{4} \right\rfloor$ and therefore $\mathrm{th}_{\mathrm{pd}}^{\times}(J_3\Box J_m) = \gamma(J_3\Box J_m)$. Suppose $\mathrm{pt}_{\mathrm{pd}}(J_3\Box J_m; S) \geq 2$ and let $t = \mathrm{pt}_{\mathrm{pd}}(J_3\Box J_m; S)$. Since the maximum degree in $J_3\Box J_m$ is 4, $|P^{(1)}(S)| \leq 4|S|$.

Choose a set of forces $\mathcal{F}$ of S such that for each $y \in P^{(1)}(S)$, if y is adjacent to a vertex in S along a row edge, i.e., y is a row-neighbor of a vertex in S, then y is forced by one of its row-neighbors in S. For $x \in S$, if $\deg(x) = 3$, then $|Q_x| \leq 3$. Let $x \in S$ with $\deg(x) = 4$; we show this implies $|Q_x| \leq 2$. If for both x_N and x_S, this vertex is not forced by x or does not force in round 2, then $|Q_x| \leq 2$ is immediate.

So suppose that $x \to x_N$ and x_N forces in round 2. Then x_N cannot force to the north, because if $J_3 = P_3$, then there is no north neighbor of x_N, and if $J_3 = C_3$, then the north neighbor of x_N is x_S. Without loss of generality, suppose x_N forces its west neighbor x_{NW} in round 2. This implies x_W cannot force in round 2. In order for $x_N \to x_{NW}$ in round 2, the east neighbor x_{NE} of x_N must have $\mathrm{rd}(x_{NE}) = 0$ or $\mathrm{rd}(x_{NE}) = 1$. If $\mathrm{rd}(x_{NE}) = 0$, then $x_{NE} \in S$ and x_N is a row-neighbor of x_{NE}, so x_N would not be forced by x.

Thus $\mathrm{rd}(x_{NE}) = 1$, so x_{NE} is adjacent to a vertex u in S. We show this implies another neighbor of x cannot contribute to Q_x, and thus $|Q_x| \leq 2$. If u is the south neighbor of x_{NE}, then $u = x_E$, so x_E does not contribute to Q_x. If $J_3 = C_3$ and u is the north neighbor of x_{NE}, then u is the east neighbor of x_S, so x_S cannot be forced by x (because it is a row-neighbor of $u \in S$); thus x_S does not contribute to Q_x. If u is the east neighbor of x_{NE}, then x_E cannot force east in round 2, because u is adjacent to the east neighbor of x_E. If x_E forces south in round 2, then x_S cannot force in round 2. Hence $|Q_x| \le 2$.

Since $3 + 3 = 4 + 2 = 6$, we have $|P^{(1)}(S)| + |P^{(2)}(S)| \le 6|S|$, and thus

$$3m = |V(J_3 \Box J_m)| = |S| + \sum_{i=1}^{t} |P^{(i)}(S)| \le |S|(1 + 6 + 4(t-2)) = |S|(4t-1).$$

This implies $\mathrm{th}_{\mathrm{pd}}^{\times}(J_3 \Box J_m; S) = t|S| \ge t \left\lceil \frac{3m}{4t-1} \right\rceil$. Then $t \left\lceil \frac{3m}{4t-1} \right\rceil \ge t \frac{3m}{4t-1} > t \frac{3m}{4t} \ge \left\lfloor \frac{3m}{4} \right\rfloor$. Since the first and last terms are integers, $t \left\lceil \frac{3m}{4t-1} \right\rceil \ge \left\lfloor \frac{3m}{4} \right\rfloor + 1 = \left\lfloor \frac{3m+4}{4} \right\rfloor$. $\square$

We conclude with a corollary that summarizes the situation for Cartesian products of connected graphs in which each factor graph has degree at most 2.

Corollary 6.13. *For all $n, m \ge 1$, $\mathrm{th}_{\mathrm{pd}}^{\times}(J_n \Box J_m) = \gamma(J_n \Box J_m)$, where $J_k = P_k$ or $J_k = C_k$ for $k \ge 3$ and $J_k = P_k$ for $k = 1, 2$.*

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