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Perfect 1-factorisations of complete k-uniform hypergraphs

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Abstract

A 1-factorisation of a graph is called perfect if it satisfies each of the following equivalent conditions: the union of each pair of 1-factors is isomorphic to the same connected subgraph, the union of each pair of 1-factors is connected, and the union of each pair of 1-factors is a Hamilton cycle. A 1-factorisation of a graph is called uniform if the union of each pair of 1-factors is isomorphic to the same subgraph.

In this paper, we generalise the concept of uniform 1-factorisations from graphs to hypergraphs in the natural way, and, based on the three conditions above, we define four generalisations of perfect 1-factorisations of graphs to the context of hypergraphs (called connected-uniform, connected, Hamilton ℓ, and Hamilton Berge 1-factorisations). We then ask, for which values of k and n does the complete k-uniform hypergraph K_n^k admit such 1-factorisations. We show that, for $k \geq 3$, uniform and uniform-connected 1-factorisations of complete k-uniform hypergraphs can exist only when $k = 3$, and when they exist they can be used to construct biplanes. We also show that, for $k \geq 2$, all 1-factorisations of K_{2k}^k and K_{3k}^k are connected 1-factorisations, and prove the existence of non-connected 1-factorisations of K_{mk}^k for every $m \geq 4$. We prove that Hamilton ℓ 1-factorisations of complete k-uniform hypergraphs do not exist for $k \geq 3$. We then prove that, for $k \geq 2$, all 1-factorisations of K_{2k}^k are Hamilton Berge 1-factorisations, and demonstrate a strong connection between Hamilton Berge 1-factorisations of K_{3k}^k and Häggkvist's conjecture on Hamilton cycles in 2-connected k-regular bipartite graphs, leading us to conjecture that all 1-factorisations of K_{3k}^k are Hamilton Berge 1-factorisations.

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1 Introduction

A 1-regular spanning subgraph of a graph is known as a *1-factor*. A partition of the edge set of a graph G into α 1-factors is called a *1-factorisation* of G (often denoted by $\mathcal{F} = \{F_1, \ldots, F_\alpha\}$). A natural question is: under what conditions does a 1-factorisation of the complete graph on n vertices, K_n, exist? Clearly n must be even, and one of the earliest proofs that this condition is sufficient is Kirkman's 1847 construction of 1-factorisations of K_n for all even integers $n \geq 2$ [21].

Given a 1-factorisation of a graph G, a well-studied problem is to ask if the 1-factorisation has the property that the union of each pair of 1-factors is isomorphic to the same subgraph H of G. Such a 1-factorisation is called a *uniform 1-factorisation* (U1F) of G and the subgraph H is called the *common graph*. Furthermore, a uniform 1-factorisation in which the common graph is a Hamilton cycle is called a *perfect 1-factorisation* (P1F). The following famous conjecture is due to Kotzig [23].

Conjecture 1.1. [23] *For any $n \geq 2$, K_{2n} admits a perfect 1-factorisation.*

Kotzig [22] provided an infinite family of 1-factorisations of the complete graph K_{2n} that are perfect when $2n-1$ is an odd prime. Bryant, Maenhaut, and Wanless [7] constructed another infinite family of P1Fs of K_{2n} where $2n-1$ is an odd prime, which is not isomorphic to the family given by Kotzig. Anderson [1] gave an infinite family of 1-factorisations of K_{2n} that are perfect when n is an odd prime. Besides these infinite families there are a number of sporadic values of n such that K_{2n} has been shown to admit a P1F. Most recently a P1F of K_{56} was found by Pike [27], which leaves K_{64} as the smallest complete graph for which the existence of a P1F is unknown; for more information on the orders of complete graphs with known P1Fs, a paper on the number of non-isomorphic P1Fs of K_{16} by Gill and Wanless [15] is recommended.

For uniform 1-factorisations that are not perfect, the common graph will be a collection of two or more disjoint cycles of even lengths. We say that a U1F has *type* $(c_1, c_2, \ldots, c_t)$ if the common graph of the U1F is a collection of t cycles of lengths $c_1, c_2, \ldots, c_t$. For complete graphs K_{2n} with $2n \leq 16$, all types of U1Fs up to isomorphism are known due to a result by Meszka and Rosa [25].

Theorem 1.2. [25] *If $\mathcal{F}$ is a U1F of K_{2n}, where $2n \leq 16$, then $\mathcal{F}$ is one of the following:*

- *(a) a P1F;*
- *(b) a U1F of K_8 of type $(4,4)$;*
- *(c) a U1F of K_{10} of type $(4,6)$;*
- *(d) a U1F of K_{12} of type $(6,6)$;*
- *(e) a U1F of K_{16} of type $(4,4,4,4)$.*

Further, the U1Fs from cases (b), (c), (d), (e) are unique up to isomorphism.

Besides the above U1Fs there are several known infinite families of U1Fs; for further information on these families the survey paper on P1Fs by Rosa [28] is recommended.

The goal of this paper is to generalise the concepts of uniform and perfect 1-factorisations from graphs to hypergraphs. A *hypergraph* $\mathcal{H}$ consists of a non-empty vertex set $V(\mathcal{H})$ and an edge set $E(\mathcal{H})$ where each element of $E(\mathcal{H})$ is a non-empty subset of the vertex set $V(\mathcal{H})$. The complete k-uniform hypergraph of order n, denoted K_n^k, is the hypergraph on n vertices, where the edges are precisely all the k-sets of the vertex set. In this paper, to avoid the case of graphs we will consider only k-uniform hypergraphs for $k \geq 3$.

A *path* between two vertices, x and y, of a hypergraph is an alternating sequence of vertices and edges of the hypergraph:

$$[x = v_1, e_1, v_2, e_2, \ldots, v_s, e_s, v_{s+1} = y]$$

such that $v_1, v_2, \ldots, v_{s+1}$ are distinct vertices, and $e_1, e_2, \ldots e_s$ are distinct edges such that $v_i \in e_i$ for $1 \leq i \leq s$ and $v_j \in e_{j-1}$ for $2 \leq j \leq s+1$. If every two vertices of a hypergraph $\mathcal{H}$ have a path between them we say that $\mathcal{H}$ is *connected.* Generalising the concept of 1-factors and 1-factorisations from graphs to hypergraphs is relatively straightforward. A 1-factor of a hypergraph is a spanning 1-regular sub-hypergraph, and a decomposition of a hypergraph into edge-disjoint 1-factors is a *1-factorisation.* An obvious necessary condition for the existence of a 1-factorisation of the complete k-uniform hypergaph on n vertices is that $k|n$. Baranyai [3] showed that for $k \geq 3$, this condition is also sufficient.

Theorem 1.3. [3] *Suppose $k \geq 3$ is an integer. A 1-factorisation of the complete k-uniform hypergraph on n vertices, K_n^k, exists if and only if k divides n.*

In Sections 2, 3, and 4, we propose four generalisations of the concept of a perfect 1-factorisation of a graph to the context of hypergraphs. With each generalisation we will give some existence results, and also provide some interesting connections to other combinatorial objects and conjectures. These four generalisations come from three equivalent definitions of a perfect 1-factorisation of a graph:

1. The union of each pair of 1-factors in the 1-factorisation is isomorphic to the same connected subgraph.

2. The union of each pair of 1-factors in the 1-factorisation is connected.

3. The union of each pair of 1-factors in the 1-factorisation is a Hamilton cycle.

In Section 5 we will discuss some known 1-factorisations of complete k-uniform hypergraphs and which, if any, of the four generalisations they satisfy. Finally, in Section 6 we pose several open problems for the four generalisations.

2 Uniform and Uniform-connected 1-Factorisations

The concept of a uniform 1-factorisation of a graph generalises naturally to hypergraphs.

Definition 2.1. A 1-factorisation of a hypergraph $\mathcal{H}$ is a *uniform 1-factorisation* (U1F) if the union of each pair of 1-factors of the 1-factorisation is isomorphic to the same hypergraph, called the *common hypergraph.*

We can then define uniform-connected 1-factorisations to be uniform 1-factorisations for which the common hypergraph is connected.

Definition 2.2. A 1-factorisation of a hypergraph $\mathcal{H}$ is a *uniform-connected 1-factorisation* (UC1F) if the union of each pair of 1-factors of the 1-factorisation is isomorphic to the same connected hypergraph.

We begin our investigation of U1Fs of complete k-uniform hypergraphs by introducing some terminology, notation and lemmas. For two distinct 1-factors F_1 and F_2 of a hypergraph, we say that a set of vertices, $B = \{v_1, v_2, \ldots, v_w\}$, is *repeated* in the pair F_1 and F_2 if $B \subseteq e$ for some edge $e \in F_1$ and $B \subseteq e'$ for some edge $e' \in F_2$. By counting repeated $(k-1)$-sets in pairs of 1-factors of a 1-factorisation of K_n^k, we determine a necessary condition for the existence of a U1F of K_n^k.

Lemma 2.1. *For $k \geq 3$, if K_n^k admits a U1F then*

$$\frac{\binom{n}{k-1}\binom{n-k+1}{2}}{\binom{\binom{n-1}{k-1}}{2}}$$

is a positive integer.

Proof. Suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_\alpha\}$ is a U1F of K_n^k. First we count the number of repeated $(k-1)$-sets occurring over all pairs of 1-factors of $\mathcal{F}$. Consider a $(k-1)$-set, $\{a_1, a_2, \ldots, a_{k-1}\}$ of vertices of K_n^k. This set occurs in an edge of $n-(k-1)$ factors of $\mathcal{F}$, so this $(k-1)$-set will be repeated in $\binom{n-(k-1)}{2}$ pairs of 1-factors. Thus, there are $\binom{n}{k-1}\binom{n-k+1}{2}$ repeated $(k-1)$-sets over all vertices and all pairs of 1-factors. Next, we count the number of pairs of 1-factors of $\mathcal{F}$. A 1-factorisation of K_n^k has $\binom{n-1}{k-1}$ 1-factors, so there are $\binom{\binom{n-1}{k-1}}{2}$ pairs of 1-factors.

Now, since the union of each pair of 1-factors of $\mathcal{F}$ is isomorphic to the same hypergraph, it follows that each pair of 1-factors must have exactly

$$\frac{\binom{n}{k-1}\binom{n-k+1}{2}}{\binom{\binom{n-1}{k-1}}{2}}$$

repeated $(k-1)$-sets of vertices. Thus, this expression must be a positive integer. $\square$

The necessary condition in Lemma 2.1 quickly rules out the existence of U1Fs of K_n^k for some values of n and k. Note that a 1-factorisation of K_k^k is trivially a U1F, so we only consider 1-factorisations of K_n^k with $n \geq 2k$.

Lemma 2.2. *For $k \geq 3$ and $n \geq 2k$, if a U1F of K_n^k exists, then $k = 3$.*

Proof. To rule out the existence of a U1F of K_n^k with $k \geq 4$ and $n \geq 2k$ we first consider the case $k = 4$ with $n = 8$; this case is quickly ruled out by Lemma 2.1. We then rule out the existence of a U1F of K_n^k for $n \geq 9$ and $4 \leq k \leq \frac{n}{2}$ by showing that

$$\binom{n}{k-1}\binom{n-k+1}{2} < \binom{\binom{n-1}{k-1}}{2} \text{ and hence } \frac{\binom{n}{k-1}\binom{n-k+1}{2}}{\binom{\binom{n-1}{k-1}}{2}} < 1. \tag{1}$$

Using the identity $\binom{a}{b}\binom{a-b}{c} = \binom{a}{c}\binom{a-c}{b}$ we can see that

$$\binom{n}{k-1}\binom{n-(k-1)}{2} < \binom{\binom{n-1}{k-1}}{2}$$

if and only if

$$n(n-1)\binom{n-2}{k-1} < \binom{n-1}{k-1}\left(\binom{n-1}{k-1} - 1\right).$$

This clearly holds for $n = 9$, $k = 4$ and for $n = 10$, $k = 4$ or 5. For larger values of n it suffices to show that $n(n-1) < \binom{n-1}{k-1} - 1$. We note that for $n \geq 11$ and $4 \leq k \leq \frac{n}{2}$,

$$n(n-1) < \binom{n-1}{3} - 1 \leq \binom{n-1}{k-1} - 1.$$

Hence (1) holds. Thus, for $k \geq 3$ and $n \geq 2k$, a U1F of K_n^k can exist only when $k = 3$. ☐

We say that a pair of 1-factors in a 1-factorisation of a hypergraph has *pair overlap number d* if the number of repeated 2-sets in that pair of 1-factors is d. A 1-factorisation of a hypergraph is said to have *pair overlap number d* if the pair overlap number is d for every pair of 1-factors in the 1-factorisation. Note that a uniform 1-factorisation must have pair overlap number d for some positive integer d, but that a 1-factorisation with pair overlap number d might not be uniform. For $k = 3$, Lemma 2.1 gives the pair overlap number of any U1F of K_n^3, and it is 2. We will now consider 1-factorisations of the 3-uniform complete hypergraph on n vertices that have a pair overlap number 2.

Generalising a construction that was used by Husain in [18] to construct a biplane of order 4, we can construct symmetric balanced incomplete block designs of index 2 from 1-factorisations of K_n^3 with pair overlap number 2.

Lemma 2.3. *If there exists a 1-factorisation of K_n^3 with pair overlap number 2, then there exists a symmetric $(\binom{n}{2} + 1, n, 2)$-design.*

Proof. Let $\mathcal{F} = \{F_1, F_2, \ldots, F_{\binom{n-1}{2}}\}$ be a 1-factorisation of K_n^3 with pair overlap number 2. Consider the design $\mathcal{D} = (\mathcal{V}, \mathcal{B})$ constructed in the following way:

- Let $\mathcal{V} = V(K_n^3) \cup \{F_1, F_2, \ldots, F_{\binom{n-1}{2}}\}$.
- Let $B_0 = V(K_n^3)$.
- For every unordered pair of distinct vertices $x, y \in V(K_n^3)$, let

$$B_{xy} = \{x, y\} \cup \{F_i \mid \text{there exists } \; e \in F_i \; \text{ such that } \; \{x, y\} \subset e\}.$$

- Let $\mathcal{B} = B_0 \cup \bigcup_{xy} B_{xy}$.

We claim that this design is a symmetric $(\binom{n}{2} + 1, n, 2)$-design. First, we can see that $\mathcal{D}$ has $n + \binom{n-1}{2} = \binom{n}{2} + 1$ points. Also, from the definition it is clear we have $\binom{n}{2} + 1$ blocks. Second, we note that each block contains n points. For B_0 this is obvious and for B_{xy} this follows because each unordered pair of distinct vertices appears together in an edge in $n-2$ of the 1-factors of $\mathcal{F}$. Furthermore, each point occurs in n blocks. For the points corresponding to the vertices of K_n^3, this is obvious. If u' is a point corresponding to a 1-factor F_i, u' will occur in block B_{xy} for each pair of vertices x, y that come from the same edge of F_i; as any 1-factor has $\frac{n}{3}$ edges, and each edge contains 3 pairs of vertices, it follows that u' occurs in n blocks.

Finally we show that every pair of points occurs in exactly two blocks. Any pair of points, u, v, each corresponding to a vertex of K_n^3 will appear in B_0 and B_{uv}. Any pair of points u', v', each corresponding to a 1-factor of $\mathcal{F}$ will occur in the blocks B_{ab} and B_{cd}, where a, b and c, d are the repeated 2-sets in the pair of 1-factors corresponding to u' and v'. Consider the pair of points of the design u, v', where u corresponds to a vertex of K_n^3, and v' corresponds to a 1-factor, F_i, of $\mathcal{F}$. Then the pair of points u, v' will appear in the blocks B_{ua} and B_{ub} where a, b are the vertices that appear with u in an edge of F_i. Thus it follows that any pair of points in the design appear in precisely two blocks. Therefore $\mathcal{D}$ is a symmetric $(\binom{n}{2} + 1, n, 2)$-design. □

With this connection, we can use the Bruck-Ryser-Chowla theorem [10] to rule out the existence of a U1F of K_n^3 for some values of n. Applying the Bruck-Ryser-Chowla theorem and Lemma 2.3 for values of $n \leq 30$ we see that if there exists a U1F of K_n^3 then $n \in \{3, 6, 9, 18, 21, 27\}$. Below we give examples of U1Fs (which are in fact UC1Fs) of K_3^3, K_6^3 and K_9^3. Thus $n = 18$ is the smallest value for which the existence of a U1F of K_n^3 is unknown.

We note that K_3^3 is the hypergraph that consists of 3 vertices and 1 edge and the 1-factorisation of K_3^3 is trivially uniform (uniform-connected). Also the unique 1-factorisation of K_6^3 is uniform, and in fact uniform-connected.

The 103 000 non-isomorphic 1-factorisations of K_9^3 were enumerated by Khatirinejad and Östergård [20]. Any U1F of K_9^3 will be a UC1F, and by exhaustive computer search we were able to determine that, up to isomorphism, there is a unique

{123, 456}	{135, 246}	{124, 356}	{136, 245}	{125, 346}
{145, 236}	{126, 345}	{146, 235}	{134, 256}	{156, 234}

Table 1: The unique 1-factorisation of K_6^3.

U1F (UC1F) of K_9^3, see Table 2. This 1-factorisation is the 1-factorisation with automorphism group of size 1512 found by Khatirinejad and Östergård [20]. This 1-factorisation is isomorphic to the 1-factorisation of K_9^3 in the infinite family provided by Chen and Lu [9].

{123, 456, 789}	{134, 268, 579}	{146, 259, 378}	{159, 236, 478}
{124, 369, 578}	{135, 249, 678}	{147, 289, 356}	{167, 234, 589}
{125, 347, 689}	{136, 257, 489}	{148, 235, 679}	{168, 239, 457}
{126, 358, 479}	{137, 269, 458}	{149, 237, 568}	{169, 278, 345}
{127, 359, 468}	{138, 247, 569}	{156, 248, 379}	{178, 256, 349}
{128, 367, 459}	{139, 258, 467}	{157, 238, 469}	{179, 245, 368}
{129, 348, 567}	{145, 267, 389}	{158, 279, 346}	{189, 246, 357}

Table 2: The unique UC1F of K_9^3.

3 Connected 1-Factorisations

Another natural way of generalising the concept of perfect 1-factorisations to hypergraphs is by asking that the union of every pair of 1-factors be a connected hypergraph. We note that in the case of graphs, the definition below is equivalent to the standard definition of a perfect 1-factorisation.

Definition 3.1. A 1-factorisation of a hypergraph is a *connected 1-factorisation* (C1F) if the union of each pair of 1-factors of the 1-factorisation is connected.

Note that the 1-factorisation of K_k^k is trivially a C1F, and it is easy to see that the 1-factorisation of K_{2k}^k is also a C1F. We will now show that that every 1-factorisation of the complete k-uniform hypergraph on $3k$ vertices is a C1F.

Theorem 3.1. *For $k \geq 3$, every 1-factorisation of K_{3k}^k is a C1F.*

Proof. Let $V = V(K_{3k}^k)$ and consider a 1-factorisation $\mathcal{F}$ of K_{3k}^k. Let $\mathcal{H}$ be the hypergraph formed by the union of two 1-factors $F_x = \{e_1^x, e_2^x, e_3^x\}$ and $F_y = \{e_1^y, e_2^y, e_3^y\}$ of $\mathcal{F}$. To show that $\mathcal{H}$ is connected, we consider an arbitrary vertex $v \in e_1^x$ and show that there exists a path in $\mathcal{H}$ from v to every other vertex in V. Note that since the factors are edge-disjoint, there exist $\alpha, \beta \in \{1, 2, 3\}$ with $\alpha \neq \beta$ such that $e_1^x \cap e_\alpha^y \neq \emptyset$ and $e_1^x \cap e_\beta^y \neq \emptyset$. Thus there exists a path from v to each of the $2k$ vertices in $e_\alpha^y \cup e_\beta^y$. Furthermore $(e_\alpha^y \cup e_\beta^y) \cap e_i^x \neq \emptyset$ for each $i = 1, 2, 3$ so there exists a path from v to every vertex in V. □

To show the existence of non-connected 1-factorisations of K^k_{mk} for $m \geq 4$ we rely on a result from Häggkvist and Hellgren [17] that shows that a 1-factorisation of K^k_m can be embedded in a 1-factorisation of K^k_n if and only if $n \geq 2m$ and k divides both m and n.

Lemma 3.2. *For $m \geq 4$, there exists a 1-factorisation of K^k_{mk} that is not a C1F.*

Proof. Let $\mathcal{F}$ be the unique 1-factorisation of K^k_{2k} with vertex set V. Using the result of Häggkvist and Hellgren, we can embed this 1-factorisation $\mathcal{F}$ of K^k_{2k} into a 1-factorisation $\mathcal{F}'$ of K^k_{mk} with vertex set $V \cup S$ for any $m \geq 4$. To show that $\mathcal{F}'$ is not a C1F, take two 1-factors F_1, F_2 from $\mathcal{F}$ and consider the two corresponding 1-factors from $\mathcal{F}'$, say F'_1, F'_2, that contain F_1 and F_2 respectively. Clearly the union of these two 1-factors is disconnected since there is no edge $e \in F'_1 \cup F'_2$ that has both a vertex from V and a vertex from S. Thus $\mathcal{F}'$ is not a C1F. □

It is natural to ask for which values of n and k a C1F of K^k_n can exist. In Section 5 we identify some C1Fs of K^k_{mk} for values of $m \geq 4$.

4 Hamilton ℓ 1-factorisations and Hamilton Berge 1-Factorisations

To generalise the idea of P1Fs to hypergraphs by requiring the union of each pair of 1-factors to be a Hamilton cycle, we must first decide which definition of a Hamilton cycle in a hypergraph to use. We will consider two common definitions, Hamilton ℓ-cycles and Hamilton Berge cycles, and propose a generalisation of a P1F of a graph for each.

Given an integer ℓ with $1 \leq \ell < k$, a k-uniform hypergraph C is an *ℓ-cycle* if there exists a cyclic ordering of the vertices of C such that every edge of C consists of k consecutive vertices in the ordering and such that every two consecutive edges, in the natural ordering of the edges, intersect in precisely ℓ vertices. An ℓ-cycle C is a *Hamilton ℓ-cycle* of a k-uniform hypergraph $\mathcal{H}$ if $V(C) = V(\mathcal{H})$ and $E(C) \subseteq E(\mathcal{H})$. Note that a Hamilton ℓ-cycle in a hypergraph $\mathcal{H}$ has $|V(\mathcal{H})|/(k-\ell)$ edges. If $\ell = k-1$ the ℓ-cycle is called a *tight* cycle, and if $\ell = 1$, it is called a *loose* cycle. Decomposition of complete k-uniform hypergraphs into tight Hamilton cycles has been studied by various authors, see for example [2] and [26].

Using this definition of a Hamilton ℓ-cycle, a natural approach to generalise P1Fs to hypergraphs is to ask for the union of each pair of 1-factors of a 1-factorisation to be a Hamilton ℓ-cycle. Since a Hamilton ℓ-cycle in a k-uniform hypergraph of order n has $\frac{n}{k-\ell}$ edges and a 1-factor of a k-uniform hypergraph of order n has $\frac{n}{k}$ edges, this generalisation requires $\ell = \frac{k}{2}$. We propose the following definition of a Hamilton ℓ 1-factorisation of a k-uniform hypergraph, as a generalisation of a P1F of a graph. Note that for $k = 2$, this agrees with the standard definition of a P1F.

Definition 4.1. A 1-factorisation of a k-uniform hypergraph is called a *Hamilton ℓ 1-Factorisation* (HL1F) if the union of every pair of 1-factors of the 1-factorisation is a Hamilton $\left(\frac{k}{2}\right)$-cycle.

It is quick to see that an obvious necessary condition will quickly rule out the existence of such 1-factorisations for complete k-uniform hypergraphs.

Lemma 4.1. *Let $\mathcal{H}$ be a k-uniform hypergraph. If $\mathcal{H}$ admits an HL1F, then each edge of $\mathcal{H}$ intersects all other edges in either 0 or $\frac{k}{2}$ vertices.*

Proof. Let $\mathcal{H}$ be a k-uniform hypergraph and suppose for a contradiction that $\mathcal{F}$ is an HL1F of $\mathcal{H}$ and that $\mathcal{H}$ contains two edges, e_1, e_2 such that $|e_1 \cap e_2| \notin \{0, \frac{k}{2}\}$. Note that this implies $k \geq 3$. As $\mathcal{F}$ is a 1-factorisation, e_1 and e_2 must belong to separate 1-factors; call these F_1 and F_2 respectively. Now consider the union of F_1 and F_2. Clearly this union does not form a Hamilton $\left(\frac{k}{2}\right)$-cycle, and thus $\mathcal{F}$ cannot be an HL1F. □

Corollary 4.2. *K_n^k does not admit an HL1F for any $k \geq 3$ and $n \geq 3$.*

Although complete k-uniform graphs do not admit HL1Fs, it is possible to find other k-uniform hypergraphs that do. For example, consider the 4-uniform hypergraph on 8 vertices whose vertices and edge set are the points and blocks of a Steiner quadruple system on 8 vertices constructed from a Steiner triple system on 7 vertices, see Table 3. It is easy to confirm that it is an HL1F where $\ell = 2$.

{1248, 3567}	{2358, 1467}	{3468, 1257}	{4578, 1236}
{1568, 2347}	{2678, 1345}	{1378, 2456}	

Table 3: An HL1F of a 4-uniform hypergraph on 8 vertices with seven 1-factors.

We leave it as an open question to explore what other families of k-uniform hypergraphs admit HL1Fs.

A *Berge cycle* in a hypergraph $\mathcal{H} = (V, E)$ is an alternating sequence

$$(v_1, e_1, v_2, e_2, \ldots, v_m, e_m)$$

of distinct vertices $v_i \in V$ and distinct edges $e_i \in E$, where e_i contains v_i and v_{i+1} for each $i \in \{1, 2, \ldots, m-1\}$ and e_m contains v_m and v_1. Note that each edge e_i may contain vertices other than v_i and v_{i+1} including vertices outside of $\{v_1, \ldots, v_m\}$. A *Hamilton Berge cycle* in a hypergraph $\mathcal{H}$ is a Berge cycle in $\mathcal{H}$ for which $\{v_1, \ldots, v_m\}$ is the vertex set of $\mathcal{H}$.

It is well-known that for $n \geq 2k$ and $k \geq 2$, K_n^k has a Hamilton Berge cycle and many authors have investigated decompositions of K_n^k into Hamilton Berge cycles, see for example [4], [5], [24], and [29].

Similar to before, when equipped with the definition of a Hamilton Berge cycle, a natural approach to generalise P1Fs to hypergraphs is to ask for the union of each pair of 1-factors of a 1-factorisation to have a Hamilton Berge cycle. However, a Hamilton Berge cycle in K_n^k has n edges and a 1-factor in K_n^k has $\frac{n}{k}$ edges, so this generalisation would require $k = 2$. Hence we propose the following definition of a Hamilton Berge 1-factorisation of a k-uniform hypergraph, as a generalisation of a P1F of a graph.

Definition 4.2. A 1-factorisation of a k-uniform hypergraph is called a *Hamilton Berge 1-factorisation* (HB1F) if the union of each k-set of 1-factors of the 1-factorisation has a Hamilton Berge cycle.

The *incidence graph* of a hypergraph $\mathcal{H} = (V, E)$ is a bipartite graph, denoted $IG(\mathcal{H})$, with vertex set $S = V \cup E$, and where $v \in V$ and $e \in E$ are adjacent if and only if $v \in e$. The incidence graph of the union of k edge-disjoint 1-factors of K_n^k will be a k-regular bipartite graph with n vertices in each part. Let $\mathcal{H}$ be the union of k edge-disjoint 1-factors of K_n^k. We make the observation that finding a Hamilton Berge cycle in $\mathcal{H}$ is equivalent to finding a Hamilton cycle in the incidence graph $IG(\mathcal{H})$. Consider a Hamilton Berge cycle of $\mathcal{H}$, denoted $(v_1, e_1, v_2, e_2, \ldots, v_n, e_n)$; we see that this is also a Hamilton cycle of $IG(\mathcal{H})$, noting that the e_i are now vertices of $IG(\mathcal{H})$. Similarly, a Hamilton cycle of $IG(\mathcal{H})$ corresponds to a Hamilton Berge cycle of $\mathcal{H}$.

Note that the 1-factorisation of K_k^k is trivially an HB1F, and we now show that the 1-factorisation of K_{2k}^k is an HB1F. The following result from [11] (a proof of which can be found in [8]) can be applied to the incidence graph of the union of k edge-disjoint 1-factors of K_{2k}^k, thereby showing that the 1-factorisation of K_{2k}^k is an HB1F.

Proposition 4.3. [11] *If G is a connected spanning $\frac{m}{2}$-regular subgraph of $K_{m,m}$, then G is Hamiltonian.*

Corollary 4.4. *The 1-factorisation of the hypergraph K_{2k}^k is an HB1F.*

We now consider complete uniform hypergraphs where each edge consists of exactly one third of the vertices. For $n = 9$ and $k = 3$ we ran an exhaustive computer search and found a Hamilton Berge cycle in every set of three edge-disjoint 1-factors of K_9^3. This implies that every 1-factorisation of K_9^3 is an HB1F.

The incidence graph of the union of k edge-disjoint 1-factors of K_{3k}^k is a k-regular bipartite graph on $6k$ vertices, and is 2-connected. There is a well-known conjecture from Häggkvist [16] that, if true, would prove that every 1-factorisation of K_{3k}^k is an HB1F.

Conjecture 4.5. [16] *Every 2-connected k-regular bipartite graph on at most $6k$ vertices is Hamiltonian.*

The result that comes closest to proving Häggkvist's conjecture is a result of Jackson and Li [19] that shows that every 2-connected, k-regular bipartite graph on $6k - 38$ vertices is Hamiltonian. This leads us to explicitly conjecture the following.

Conjecture 4.6. *Every 1-factorisation of K^k_{3k} is an HB1F.*

One can also show the existence of 1-factorisations of $K^k_{\ell k}$ that are not HB1Fs for $\ell \geq 4$ using a similar argument to Lemma 3.2. In Section 5 we identify some HB1Fs of $K^k_{\ell k}$ for values of $\ell \geq 4$.

5 Considering known 1-Factorisations of K^k_n

1-Factorisations of complete uniform hypergraphs have been studied under several guises. A 1-factorisation of K^k_n can also be viewed as a proper edge colouring of K^k_n, where each colour class is a 1-factor. A 1-factorisation of K^t_n is also equivalent to a resolvable complete t-design, where the resolution classes are 1-factors. In this section, we investigate some known 1-factorisations of K^k_n with respect to the properties we have introduced. We call a 1-factorisation of K^k_n *uninteresting* if it is not a U1F, C1F, or HB1F.

Deo and Micikevicius [14] construct 1-factorisations of $K^3_9, K^3_{15}, K^3_{21}$. They conjecture that their construction technique can be used to build 1-factorisations of K^3_n where $n \equiv 3 \pmod 6$. The 1-factorisation of K^3_9 given by this construction is not isomorphic to the unique U1F of K^3_9, we also know that it is a C1F and HB1F from our earlier results. Their 1-factorisations of K^3_{15} and K^3_{21} were both found to be uninteresting.

In [6], Beth gives a construction of a 1-factorisation of K^3_n whenever $n - 1$ is a prime. Using this construction we built 1-factorisations of K^3_{12}, K^3_{18}, K^3_{24}, and K^3_{30}. The 1-factorisations of K^3_{12}, K^3_{18}, and K^3_{24} are all HB1Fs but not U1Fs or C1Fs, while the 1-factorisation of K^3_{30} is uninteresting.

Chen and Lu [9] completely classify all non-homogeneous symmetric 1-factorisations of complete uniform hypergraphs. They found two infinite families and 21 sporadic examples; the first infinite family is the unique 1-factorisation of K^k_{2k} (which has been classified in earlier sections), and the second is a 1-factorisation of K^3_{q+1} where $q \equiv 2 \pmod 3$ is a prime power. In this second infinite family we have found some 1-factorisations that are C1Fs but they are not all C1Fs. Of the 21 sporadic examples, only one was found to be interesting; the third 1-factorisation of K^3_{12} from Example 5.3 in [9] is both a C1F and an HB1F.

A large set of Kirkman triple systems (LKTS) of order n is equivalent to a 1-factorisation of K^3_n. The 1-factorisation of K^3_{15} corresponding to the LKTS(15) given by Denniston as a solution to Sylvester's schoolgirl problem [13] was found to be uninteresting. Denniston's construction [12] of an LKTS(3^m) for any $m > 1$, can be quickly shown to be uninteresting for $m > 2$, and when $m = 2$, gives a 1-factorisation of K^3_9 that is a C1F and HB1F (which we know from Sections 3 and 4) but is not a U1F. Similarly, the 1-factorisations of K^3_n corresponding to the LKTS(n)s from Zhang and Zhu's product construction [30] are uninteresting for $n > 9$.

Table 4 presents a summary of these constructions of 1-factorisations K^k_n that were found to be interesting. We have not included 1-factorisations of K^k_{2k} or of K^3_9, as both are fully understood (in terms of interesting 1-factorisations).

n	k	Construction	U1F	UC1F	C1F	HB1F
12	3	Chen & Lu Symmetric Infinite [9]	N	N	Y	Y
12	3	Chen & Lu Symmetric Sporadic 3 [9]	N	N	Y	Y
12	3	Beth [6]	N	N	N	Y
18	3	Beth [6]	N	N	N	Y
24	3	Beth [6]	N	N	N	Y
33	3	Chen & Lu Symmetric Infinite [9]	N	N	Y	Y
129	3	Chen & Lu Symmetric Infinite [9]	N	N	Y	?

Table 4: Some known constructions of 1-factorisations of K_n^k.

6 Open Problems

For future directions, we pose the following existence questions.

Open Problem 6.1. Does K_n^3 admit a U1F for all admissible values of n?

Open Problem 6.2. Does K_n^3 admit a UC1F for all admissible values of n?

Open Problem 6.3. Does K_{mk}^k admit a C1F for all $m \geq 2$, $k \geq 3$?

Open Problem 6.4. Does K_{mk}^k admit an HB1F for all $m \geq 2$, $k \geq 3$?

Beyond questions on existence, we also pose questions about the relationship between the different generalisations. Clearly every UC1F is a C1F. All the known examples of C1Fs are also HB1Fs, but we have examples of 1-factorisations of K_n^k that are HB1Fs but not C1Fs and examples of 1-factorisations of K_n^k that are C1Fs but not UC1Fs. To understand the relationship between these types of 1-factorisations, we pose the following question.

Open Problem 6.5. If $\mathcal{F}$ is a C1F of K_n^k, is $\mathcal{F}$ also an HB1F?

Finally, since we have only three examples of U1Fs of complete 3-uniform hypergraphs, each of which is also a UC1F, we ask if it is possible to find a U1F that is not also a UC1F.

Open Problem 6.6. Does there exist a U1F of K_n^3 that is not a UC1F?

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