AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **84(2)** (2022), Pages 314–324

Decompositions of complete 3-uniform hypergraphs into cycles of some special even lengths

R. Lakshmi* T. Poovaragavan

Department of Mathematics, Annamalai University
Annamalainagar-608 002, India
mathlakshmi@gmail.com poovamath@gmail.com

Abstract

The complete 3-uniform hypergraph $K_n^{(3)}$ is a simple 3-uniform hypergraph with vertex set V having order $|V| = n$, and the set of all 3-subsets of V as its edge set. A t-cycle in this hypergraph is $v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$ where $v_1, v_2, \ldots, v_t$ are distinct vertices and $e_1, e_2, \ldots, e_t$ are distinct edges such that $v_i, v_{i+1} \in e_i$ for $i \in \{1, 2, \ldots, t-1\}$ and $v_t, v_1 \in e_t$. A decomposition of a hypergraph is a partition of its edge set into edge-disjoint subsets. In this paper, we prove the existence of a t-cycle decomposition of $K_n^{(3)}$ for values of $t \equiv 2$ or $4 \pmod 6$ that satisfy the divisibility condition $t|(n-2)$ or $t|n$ or $2t|(n-1)$. Using this, we characterize the existence of a decomposition of $K_n^{(3)}$ into 2^ℓ-cycles, where $\ell \geq 2$ is a positive integer. Consequently, the main result of the paper by Jordan and Newkirk [*Australas. J. Combin.* 71(2) (2018), 312–323] is a corollary.

1 Introduction

A *hypergraph* $\mathcal{H}$ consists of a finite nonempty set V of *vertices* and a set $\mathcal{E} = \{e_1, e_2, \ldots, e_m\}$ of *edges* where each $e_i \subseteq V$ with $|e_i| > 0$ for $i \in \{1, 2, \ldots, m\}$. If $|e_i| = h$, then we call e_i an *h-edge.* If every edge of $\mathcal{H}$ is an h-edge for some h, then we say that $\mathcal{H}$ is *h-uniform.* The *complete h-uniform hypergraph* $K_n^{(h)}$ is the hypergraph with vertex set V, where $|V| = n$, in which every h-subset of V determines an h-edge. It then follows that $K_n^{(h)}$ has $\binom{n}{h}$ edges. When $h = 2$, $K_n^{(2)} = K_n$, the complete graph on n vertices.

A *decomposition* of a hypergraph $\mathcal{H}$ is a set $\mathcal{F} = \{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k\}$ of *subhypergraphs* of $\mathcal{H}$ such that $\mathcal{E}(\mathcal{F}_1) \cup \mathcal{E}(\mathcal{F}_2) \cup \cdots \cup \mathcal{E}(\mathcal{F}_k) = \mathcal{E}(\mathcal{H})$ and $\mathcal{E}(\mathcal{F}_i) \cap \mathcal{E}(\mathcal{F}_j) = \emptyset$ for all $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. We denote this by $\mathcal{H} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k$.

* Also at: Department of Mathematics, Dharumapuram Gnanambigai Government Arts College for Women, Mayiladuthurai-609 001, India.

ISSN: 2202-3518

If $\mathcal{H} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k$ is a decomposition such that $\mathcal{F}_1 \cong \mathcal{F}_2 \cong \cdots \cong \mathcal{F}_k \cong \mathcal{G}$, where $\mathcal{G}$ is a fixed hypergraph, then $\mathcal{F}$ is called a *$\mathcal{G}$-decomposition of $\mathcal{H}$*.

A *cycle* of length t in a hypergraph $\mathcal{H}$ is a sequence of the form $v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$, where $v_1, v_2, \ldots, v_t$ are distinct vertices and $e_1, e_2, \ldots, e_t$ are distinct edges satisfying $v_i, v_{i+1} \in e_i$ for $i \in \{1, 2, \ldots, t-1\}$ and $v_t, v_1 \in e_t$.

Decompositions of $K_n^{(3)}$ into Hamilton cycles were considered in [1, 2] and the proof of their existence was given in [10]. Decompositions of $K_n^{(h)}$ into Hamilton cycles were considered in [5, 8], a complete solution for $h \geq 4$ and $n \geq 30$ was given in [5], and cyclic decompositions were considered in [8]. In [3], necessary and sufficient conditions were given for a $\mathcal{G}$-decomposition of $K_n^{(3)}$, where $\mathcal{G}$ is any 3-uniform hypergraph with at most three edges and at most six vertices. In [4], decompositions of $K_n^{(3)}$ into 4-cycles were considered and their existence was established. In [7], decompositions of $K_n^{(3)}$ into 6-cycles were considered and their existence was given. In [6], decompositions of $K_n^{(3)}$ into p-cycles were considered and their existence was given, whenever p is prime.

In this paper, we are interested in the following problem.

Problem 1.1. *Given a positive integer $n \geq 3$, find all positive integers $\ell \geq 2$, such that there exists a 2^ℓ-cycle decomposition of $K_n^{(3)}$.*

For any positive integer $t \geq 3$, a necessary condition for the existence of a t-cycle decomposition of $K_n^{(3)}$ is: t divides the number of edges in $K_n^{(3)}$, that is, $t | \binom{n}{3}$.

In this paper, we consider values of $t \equiv 2$ or $4 \pmod 6$ and we prove the existence of a t-cycle decomposition in the following three special cases: (i) $t|(n-2)$; (ii) $t|n$; (iii) $2t|(n-1)$.

By the assumption on t, we have: $t \not\equiv 0 \pmod 3$, both n and $n-2$ are even in cases (i) and (ii), and $t|\frac{n-1}{2}$ in case (iii). Thus, we have $t|\binom{n}{3}$.

The main result of this paper is stated below and is proved in Section 3.

Theorem 1.1. *If $t \geq 4$ is an integer with $t \equiv 2$ or $4 \pmod 6$ and n is congruent to $0 \pmod t$, $2 \pmod t$ or $1 \pmod{2t}$, then $K_n^{(3)}$ has a t-cycle decomposition.*

2 Tools

In this section, we prove some results which are required to prove Theorem 1.1.

We will assume the vertex set of $K_n^{(3)}$ is $\{v_i : i \in \mathbb{Z}_n\}$, where $\mathbb{Z}_n$ is the set of integers modulo n. For non-negative integers i and j with $i < j$, we denote the set $\{v_i, v_{i+1}, \ldots, v_j\}$ by $[v_i, v_j]$, and the set $\{i, i+1, \ldots, j\}$ by $[i, j]$.

For convenience, we will often write the edge $\{v_a, v_b, v_c\}$ as v_a-v_b-v_c and the t-cycle $v_1, e_1, v_2, e_2, \ldots, v_t, e_t, v_1$ as $(v_1\text{-}y_1\text{-}v_2, v_2\text{-}y_2\text{-}v_3, \ldots, v_t\text{-}y_t\text{-}v_1)$, where $e_i = v_i\text{-}y_i\text{-}v_{i+1}$ for $i \in \{1, 2, \ldots, t-1\}$ and $e_t = v_t\text{-}y_t\text{-}v_1$.

2.1 The hypergraph $Z^{(3)}_{p,q,r}$

Define the 3-uniform hypergraph $Z^{(3)}_{p,q,r}$ of order $p+q+r$ as follows: $V(Z^{(3)}_{p,q,r}) = \{v_i : i \in \mathbb{Z}_{p+q+r}\}$ grouped as $G_0 = [v_0, v_{p-1}]$, $G_1 = [v_p, v_{p+q-1}]$ and $G_2 = [v_{p+q}, v_{p+q+r-1}]$ and let $\mathcal{E}(Z^{(3)}_{p,q,r})$ be the set of all 3-edges v_a-v_b-v_c such that $v_a \in G_0$, $v_b \in G_1$ and $v_c \in G_2$. Note that $|\mathcal{E}(Z^{(3)}_{p,q,r})| = pqr$. A necessary condition for the existence of a t-cycle decomposition of $Z^{(3)}_{p,q,r}$ is that $t|pqr$.

Lemma 2.1. *If $t \geq 4$ is an even integer, then $Z^{(3)}_{1,t,t}$ admits a t-cycle decomposition.*

To prove this lemma, we need the following theorem.

Theorem 2.1. [9]. *Let P_{k+1} be the path of length k, and let $k, m, n \in \mathbb{N}$ with m, n even and $m \geq n$. The complete bipartite graph $K_{m,n}$ has a P_{k+1}-decomposition if and only if $m \geq \lceil \frac{k+1}{2} \rceil$, $n \geq \lceil \frac{k}{2} \rceil$ and $mn \equiv 0 \pmod{k}$.*

. *Proof of Lemma 2.1.* Consider the complete bipartite graph $K_{t,t}$ with bipartition $([v_1, v_t], [v_{t+1}, v_{2t}])$. By Theorem 2.1, we have a decomposition, say $\mathscr{F}$, of $K_{t,t}$ into paths of length t. For each path $(x_1, x_2, \ldots, x_t, x_{t+1})$ of length t in $\mathscr{F}$, consider the corresponding t-cycle of the form

$$(v_0\text{-}x_1\text{-}x_2,\ x_2\text{-}v_0\text{-}x_3,\ x_3\text{-}v_0\text{-}x_4,\ x_4\text{-}v_0 - x_5,\ \ldots,\ x_{t-1}\text{-}v_0\text{-}x_t,\ x_t\text{-}x_{t+1}\text{-}v_0)$$

in $Z^{(3)}_{1,t,t}$. This collection of t-cycles yields a decomposition of $Z^{(3)}_{1,t,t}$ into t-cycles. □

Corollary 2.1. *If $t \geq 4$ is an even integer and if $p \geq 1$ is an integer, then $Z^{(3)}_{p,t,t}$ decomposes into t-cycles. In particular, if $t \geq 4$ is an even integer, then $Z^{(3)}_{t,t,t}$ decomposes into t-cycles.*

Proof. We may think of $Z^{(3)}_{p,t,t}$ as an edge-disjoint union of p copies of $Z^{(3)}_{1,t,t}$. Apply Lemma 2.1 to each one of these p copies. □

Lemma 2.2. *If $t \geq 4$ is an even integer, then $Z^{(3)}_{1,2t,2t}$ decomposes into t-cycles.*

Proof. To see this, we write $Z^{(3)}_{1,2t,2t}$ as an edge-disjoint union of four copies of $Z^{(3)}_{1,t,t}$ with vertex set grouped into

$$\begin{aligned}
(G_0, G_1, G_2) &= (\{v_0\}, [v_1, v_t], [v_{2t+1}, v_{3t}]),\\
(G_0, G_1, G_2) &= (\{v_0\}, [v_1, v_t], [v_{3t+1}, v_{4t}]),\\
(G_0, G_1, G_2) &= (\{v_0\}, [v_{t+1}, v_{2t}], [v_{2t+1}, v_{3t}]),\\
(G_0, G_1, G_2) &= (\{v_0\}, [v_{t+1}, v_{2t}], [v_{3t+1}, v_{4t}]),
\end{aligned}$$

for the first, second, third and fourth copy respectively. Now apply Lemma 2.1 to each copy of $Z^{(3)}_{1,t,t}$. □

Lemma 2.3. *If $t \geq 4$ is an even integer and if $p \geq 1$ is an integer, then $Z^{(3)}_{p,2t,2t}$ decomposes into t-cycles. In particular, if $t \geq 4$ is an even integer, then $Z^{(3)}_{2t,2t,2t}$ decomposes into t-cycles.*

Proof. We may think of $Z^{(3)}_{p,2t,2t}$ as an edge-disjoint union of p copies of $Z^{(3)}_{1,2t,2t}$. Apply Lemma 2.1 to each one of these p copies. □

2.2 The hypergraph $K_{m,n}^{(3)}$

Define the 3-uniform hypergraph $K_{m,n}^{(3)}$ of order $m+n$ as follows. Let $V(K_{m,n}^{(3)}) = \{v_i : i \in \mathbb{Z}_{m+n}\}$, grouped as $G_0 = [v_0, v_{m-1}]$ and $G_1 = [v_m, v_{m+n-1}]$. Let $\mathcal{E}(K_{m,n}^{(3)})$ be the set of all 3-edges v_a-v_b-v_c such that v_a, v_b and v_c are not all from the same set G_i, $i \in \{0,1\}$; that is, for each $i \in \{0,1\}$, $\{v_a, v_b, v_c\} \cap G_i \neq \emptyset$. Note that $\left|\mathcal{E}(K_{m,n}^{(3)})\right| = \frac{mn(m+n-2)}{2}$. A necessary condition for the existence of a t-cycle decomposition of $K_{m,n}^{(3)}$ is that $2t|mn(m+n-2)$.

Lemma 2.4. *If $t \geq 4$ is an even integer, then $K_{1,2t}^{(3)}$ decomposes into t-cycles.*

Proof. Consider the hypergraph $K_{1,2t}^{(3)}$ (respectively, its spanning subhypergraph $Z_{1,t,t}^{(3)}$) with vertex set $[v_0, v_{2t}]$ which is grouped into $\{v_0\}$ and $[v_1, v_{2t}]$ (respectively, $\{v_0\}$, $[v_1, v_t]$ and $[v_{t+1}, v_{2t}]$). For convenience, relabel the vertices v_i and v_{t+i}, $i \in [1,t]$ by x_{i-1} and y_{i-1}, respectively. The subscripts of the vertices x and y are taken modulo t.

Consider the complete bipartite graph $K_{t,t}$ with bipartition $([x_0, x_{t-1}], [y_0, y_{t-1}])$. Let

$$P^0 = (x_0, y_{t-1}, x_1, y_{t-2}, x_2, y_{t-3}, \ldots, y_{\frac{t}{2}+2}, x_{\frac{t}{2}-2}, y_{\frac{t}{2}+1}, x_{\frac{t}{2}-1}, y_{\frac{t}{2}}, x_{\frac{t}{2}}).$$

(Observe that, if we denote the length of the edge x_iy_j as $(j-i) \pmod{t}$, then the lengths of the edges of P^0 in order are: $t-1, t-2, t-3, \ldots, 1, 0$.) Let $\mathscr{F} = \{P^0, P^1, P^2, \ldots, P^{t-1}\}$, where for each $i \in [1, t-1]$, P^i is obtained from P^{i-1} by adding 1 and reducing the subscripts modulo t in each of the vertices of P^{i-1}. By the above observation, $\mathscr{F}$ is a decomposition of $K_{t,t}$ into paths of length t.

Using P^0, obtain a t-cycle C^0 in $Z_{1,t,t}^{(3)}$ as shown below:

$$\begin{aligned} C^0 &= (v_0\text{-}x_0\text{-}y_{t-1},\ y_{t-1}\text{-}v_0\text{-}x_1,\ x_1\text{-}v_0\text{-}y_{t-2},\ y_{t-2}\text{-}v_0\text{-}x_2,\ x_2\text{-}v_0\text{-}y_{t-3},\ \ldots, \\ &\qquad y_{\frac{t}{2}+2}\text{-}v_0\text{-}x_{\frac{t}{2}-2},\ x_{\frac{t}{2}-2}\text{-}v_0\text{-}y_{\frac{t}{2}+1},\ y_{\frac{t}{2}+1}\text{-}v_0\text{-}x_{\frac{t}{2}-1},\ x_{\frac{t}{2}-1}\text{-}v_0\text{-}y_{\frac{t}{2}},\ y_{\frac{t}{2}}\text{-}x_{\frac{t}{2}}\text{-}v_0). \end{aligned}$$

Similarly, for each $i \in [1, t-1]$, using P^i, obtain a t-cycle C^i in $Z_{1,t,t}^{(3)}$. As $\mathscr{F}$ is a decomposition of $K_{t,t}$ into paths of length t, $\{C^0, C^1, C^2, \ldots, C^{t-1}\}$ is a t-cycle decomposition of $Z_{1,t,t}^{(3)}$.

As $Z_{1,t,t}^{(3)}$ is a spanning subhypergraph of $K_{1,2t}^{(3)}$, to find a t-cycle decomposition of $K_{1,2t}^{(3)}$, it is enough to find a t-cycle decomposition of $K_{1,2t}^{(3)} \setminus \mathcal{E}(\bigcup_{i=1}^{t-1} C^i)$.

Consider two disjoint copies of K_t, say $K_t^{'}$ and $K_t^{''}$, with vertex sets $[x_0, x_{t-1}]$ and $[y_0, y_{t-1}]$, respectively. As t is even, each of $K_t^{'}$ and $K_t^{''}$ is Hamilton path decomposable. For $i \in [1, \frac{t}{2}]$, let

$$\begin{aligned} P_i^1 &= (x_{i-1}, x_i, x_{i-2}, x_{i+1}, x_{i-3}, x_{i+2}, \ldots, x_{\frac{t}{2}+1+i}, x_{\frac{t}{2}-2+i}, x_{\frac{t}{2}+i}, x_{\frac{t}{2}-1+i}) \\ \text{and } P_i^2 &= (y_{i-1}, y_i, y_{i-2}, y_{i+1}, y_{i-3}, y_{i+2}, \ldots, y_{\frac{t}{2}+1+i}, y_{\frac{t}{2}-2+i}, y_{\frac{t}{2}+i}, y_{\frac{t}{2}-1+i}). \end{aligned}$$

Then, clearly, $\{P_i^1 : i \in [1, \frac{t}{2}]\}$ and $\{P_i^2 : i \in [1, \frac{t}{2}]\}$ are, respectively, decompositions of $K_t^{'}$ and $K_t^{''}$ into paths of length $t-1$.

For each $i \in [1, \frac{t}{2}]$, let

$$\begin{aligned} P_i^{'} &= (x_{i-1}\text{-}v_0\text{-}x_i,\ x_i\text{-}v_0\text{-}x_{i-2},\ x_{i-2}\text{-}v_0\text{-}x_{i+1},\ x_{i+1}\text{-}v_0\text{-}x_{i-3},\ x_{i-3}\text{-}v_0\text{-}x_{i+2}, \\ &\quad \ldots, x_{\frac{t}{2}+1+i}\text{-}v_0\text{-}x_{\frac{t}{2}-2+i}, i\ x_{\frac{t}{2}-2+i}\text{-}v_0\text{-}x_{\frac{t}{2}+i},\ x_{\frac{t}{2}+i}\text{-}v_0\text{-}x_{\frac{t}{2}-1+i}) \end{aligned}$$

and

$$\begin{aligned} P_i^{''} &= (y_{i-1}\text{-}v_0\text{-}y_i,\ y_i\text{-}v_0\text{-}y_{i-2},\ y_{i-2}\text{-}v_0\text{-}y_{i+1},\ y_{i+1}\text{-}v_0\text{-}y_{i-3},\ y_{i-3}\text{-}v_0\text{-}y_{i+2}, \\ &\quad \ldots, y_{\frac{t}{2}+1+i}\text{-}v_0\text{-}y_{\frac{t}{2}-2+i},\ y_{\frac{t}{2}-2+i}\text{-}v_0\text{-}y_{\frac{t}{2}+i},\ y_{\frac{t}{2}+i}\text{-}v_0\text{-}y_{\frac{t}{2}-1+i}), \end{aligned}$$

where $P_i^{'}$ and $P_i^{''}$ are paths of length $t-1$ in $K_{1,2t}^{(3)}$ obtained from P_i^1 and P_i^2, respectively. This results in a decomposition $\{P_i^{'}, P_i^{''} : i \in [1, \frac{t}{2}]\}$ of $K_{1,2t}^{(3)} \setminus \mathcal{E}(C^0 \cup C^1 \cup \cdots \cup C^{t-1})$ into paths of length $t-1$.

Consider $P_1^{'}$. By rewriting the last edge $x_{\frac{t}{2}+1}\text{-}v_0\text{-}x_{\frac{t}{2}}$ of $P_1^{'}$ as $x_{\frac{t}{2}+1}\text{-}x_{\frac{t}{2}}\text{-}v_0$ and adding, at the end, the first edge $v_0\text{-}x_0\text{-}y_{t-1} = v_0\text{-}y_{t-1}\text{-}x_0$ of C^0, we obtain the t-cycle

$$\begin{aligned} C_1^{'} &= (x_0\text{-}v_0\text{-}x_1,\ x_1\text{-}v_0\text{-}x_{t-1},\ x_{t-1}\text{-}v_0\text{-}x_2,\ x_2\text{-}v_0\text{-}x_{t-2},\ x_{t-2}\text{-}v_0\text{-}x_3, \\ &\quad \ldots, x_{\frac{t}{2}+2}\text{-}v_0\text{-}x_{\frac{t}{2}-1},\ x_{\frac{t}{2}-1}\text{-}v_0\text{-}x_{\frac{t}{2}+1},\ x_{\frac{t}{2}+1}\text{-}x_{\frac{t}{2}}\text{-}v_0,\ v_0\text{-}y_{t-1}\text{-}x_0). \end{aligned}$$

Now consider $P_i^{'}$, for $i \neq 1$. By rewriting the last edge $x_{\frac{t}{2}+i}\text{-}v_0\text{-}x_{\frac{t}{2}-1+i}$ of $P_i^{'}$ as $x_{\frac{t}{2}+i}\text{-}x_{\frac{t}{2}-1+i}\text{-}v_0$ and adding, at the end, the $(2i-1)^{\text{st}}$ edge $x_{i-1}\text{-}v_0\text{-}y_{t-i} = v_0\text{-}y_{t-i}\text{-}x_{i-1}$ of C^0, we obtain the t-cycle

$$\begin{aligned} C_i^{'} &= (x_{i-1}\text{-}v_0\text{-}x_i,\ x_i\text{-}v_0\text{-}x_{i-2},\ x_{i-2}\text{-}v_0\text{-}x_{i+1},\ x_{i+1}\text{-}v_0\text{-}x_{i-3},\ x_{i-3}\text{-}v_0\text{-}x_{i+2}, \\ &\quad \ldots, x_{\frac{t}{2}+1+i}\text{-}v_0\text{-}x_{\frac{t}{2}-2+i},\ x_{\frac{t}{2}-2+i}\text{-}v_0\text{-}x_{\frac{t}{2}+i},\ x_{\frac{t}{2}+i}\text{-}x_{\frac{t}{2}-1+i}\text{-}v_0,\ v_0\text{-}y_{t-i}\text{-}x_{i-1}). \end{aligned}$$

Similarly, by rewriting the first edge $y_0\text{-}v_0\text{-}y_1$ of $P_1^{''}$ as $v_0\text{-}y_0\text{-}y_1$ and adding, at the end, the t^{th} edge $y_{\frac{t}{2}}\text{-}x_{\frac{t}{2}}\text{-}v_0$ of C^0, we obtain the t-cycle

$$\begin{aligned} C_1^{''} &= (v_0\text{-}y_0\text{-}y_1,\ y_1\text{-}v_0\text{-}y_{t-1},\ y_{t-1}\text{-}v_0\text{-}y_2,\ y_2\text{-}v_0\text{-}y_{t-2},\ y_{t-2}\text{-}v_0\text{-}y_3, \\ &\quad \ldots, y_{\frac{t}{2}+2}\text{-}v_0\text{-}y_{\frac{t}{2}-1},\ y_{\frac{t}{2}-1}\text{-}v_0\text{-}y_{\frac{t}{2}+1},\ y_{\frac{t}{2}+1}\text{-}v_0\text{-}y_{\frac{t}{2}},\ y_{\frac{t}{2}}\text{-}x_{\frac{t}{2}}\text{-}v_0); \end{aligned}$$

and, for $i \neq 1$, by rewriting the first edge $y_{i-1}\text{-}v_0\text{-}y_i$ of $P_i^{''}$ as $v_0\text{-}y_{i-1}\text{-}y_i$ and adding, at the end, the $(t-2i+2)^{\text{nd}}$ edge, i.e., the $2(\frac{t}{2}-i+1)^{\text{st}}$ edge, $y_{t-(\frac{t}{2}-1+i)}\text{-}v_0\text{-}x_{\frac{t}{2}-i+1} = y_{\frac{t}{2}-1+i}\text{-}v_0\text{-}x_{\frac{t}{2}-i+1} = y_{\frac{t}{2}-1+i}\text{-}x_{\frac{t}{2}-i+1}\text{-}v_0$ of C^0, we obtain the t-cycle

$$\begin{aligned} C_i^{''} &= (v_0\text{-}y_{i-1}\text{-}y_i,\ y_i\text{-}v_0\text{-}y_{i-2},\ y_{i-2}\text{-}v_0\text{-}y_{i+1},\ y_{i+1}\text{-}v_0\text{-}y_{i-3},\ y_{i-3}\text{-}v_0\text{-}y_{i+2}, \\ &\quad \ldots, y_{\frac{t}{2}+1+i}\text{-}v_0\text{-}y_{\frac{t}{2}-2+i},\ y_{\frac{t}{2}-2+i}\text{-}v_0\text{-}y_{\frac{t}{2}+i},\ y_{\frac{t}{2}+i}\text{-}v_0\text{-}y_{\frac{t}{2}-1+i},\ y_{\frac{t}{2}-1+i}\text{-}x_{\frac{t}{2}-i+1}\text{-}v_0). \end{aligned}$$

The collection of these t-cycles $\{C_i^{'} : i \in [1, \frac{t}{2}]\} \cup \{C_i^{''} : i \in [1, \frac{t}{2}]\}$ yields a decomposition of $K_{1,2t}^{(3)} \setminus \mathcal{E}(C^1 \cup C^2 \cup \cdots \cup C^{t-1})$. □

Lemma 2.5. *If $t \geq 4$ is an even integer, then $K_{2,t}^{(3)}$ decomposes into t-cycles.*

Proof. Consider $K_{2,t}^{(3)}$ where its vertex set is grouped into $\{v_0, v_1\}$ and $[v_2, v_{t+1}]$. For convenience, relabel the vertex v_i, $i \in [2, t+1]$, by u_{i-2}, where the subscripts of u are taken modulo t. The complete graph K_t with vertex set $[u_0, u_{t-1}]$ is Hamilton path decomposable, because t is even. Let $\{P_j : j \in [0, \frac{t}{2}-1]\}$ be the Hamilton path decomposition of K_t, where

$$P_j = (u_j, u_{1+j}, u_{t-1+j}, u_{2+j}, u_{t-2+j}, u_{3+j}, \ldots, u_{\frac{t}{2}+2+j}, u_{\frac{t}{2}-1+j}, u_{\frac{t}{2}+1+j}, u_{\frac{t}{2}+j}).$$

For each $j \in [0, \frac{t}{2}-1]$, using P_j, obtain the paths

$$\begin{aligned} P_j^0 &= (u_j\text{-}v_0\text{-}u_{1+j}, u_{1+j}\text{-}v_0\text{-}u_{t-1+j}, u_{t-1+j}\text{-}v_0\text{-}u_{2+j}, u_{2+j}\text{-}v_0\text{-}u_{t-2+j}, u_{t-2+j}\text{-}v_0\text{-}u_{3+j}, \\ &\quad \ldots, u_{\frac{t}{2}+2+j}\text{-}v_0\text{-}u_{\frac{t}{2}-1+j}, u_{\frac{t}{2}-1+j}\text{-}v_0\text{-}u_{\frac{t}{2}+1+j}, u_{\frac{t}{2}+1+j}\text{-}v_0\text{-}u_{\frac{t}{2}+j}) \end{aligned}$$

and

$$\begin{aligned} P_j^1 &= (u_j\text{-}v_1\text{-}u_{1+j},\ u_{1+j}\text{-}v_1\text{-}u_{t-1+j},\ u_{t-1+j}\text{-}v_1\text{-}u_{2+j},\ u_{2+j}\text{-}v_1\text{-}u_{t-2+j}, \\ &\quad u_{t-2+j}\text{-}v_1\text{-}u_{3+j}, \ldots, u_{\frac{t}{2}+2+j}\text{-}v_1\text{-}u_{\frac{t}{2}-1+j}, u_{\frac{t}{2}-1+j}\text{-}v_1\text{-}u_{\frac{t}{2}+1+j}, u_{\frac{t}{2}+1+j}\text{-}v_1\text{-}u_{\frac{t}{2}+j}) \end{aligned}$$

of length $t-1$ in $K_{2,t}^{(3)}$. This results in the decomposition $\{P_j^0 : j \in [0, \frac{t}{2}-1]\} \cup \{P_j^1 : j \in [0, \frac{t}{2}-1]\}$ of $K_{2,t}^{(3)} \setminus \{v_0\text{-}v_1\text{-}u_j : j \in [0, t-1]\}$ into paths of length $t-1$.

Consider P_j^0. By rewriting the last edge $u_{\frac{t}{2}+1+j}\text{-}v_0\text{-}u_{\frac{t}{2}+j}$ of P_j^0 as $u_{\frac{t}{2}+1+j}\text{-}u_{\frac{t}{2}+j}\text{-}v_0$ and adding, at the end, the edge $v_0\text{-}v_1\text{-}u_j$, we obtain the t-cycle

$$\begin{aligned} C_j^0 &= (u_j\text{-}v_0\text{-}u_{1+j}, u_{1+j}\text{-}v_0\text{-}u_{t-1+j}, u_{t-1+j}\text{-}v_0\text{-}u_{2+j}, u_{2+j}\text{-}v_0\text{-}u_{t-2+j}, u_{t-2+j}\text{-}v_0\text{-}u_{3+j}, \ldots, \\ &\quad u_{\frac{t}{2}+2+j}\text{-}v_0\text{-}u_{\frac{t}{2}-1+j}, u_{\frac{t}{2}-1+j}i\text{-}v_0\text{-}u_{\frac{t}{2}+1+j}, u_{\frac{t}{2}+1+j}\text{-}u_{\frac{t}{2}+j}\text{-}v_0, v_0\text{-}v_1\text{-}u_j). \end{aligned}$$

Next consider P_j^1. By rewriting the first edge $u_j\text{-}v_1\text{-}u_{1+j}$ of P_j^1 as $v_1\text{-}u_j\text{-}u_{1+j}$ and adding, at the end, the edge $v_0\text{-}v_1\text{-}u_{\frac{t}{2}+j} = u_{\frac{t}{2}+j}\text{-}v_0\text{-}v_1$, we obtain the t-cycle

$$\begin{aligned} C_j^1 &= (v_1\text{-}u_j\text{-}u_{1+j},\ u_{1+j}\text{-}v_1\text{-}u_{t-1+j},\ u_{t-1+j}\text{-}v_1\text{-}u_{2+j},\ u_{2+j}\text{-}v_1\text{-}u_{t-2+j},\ u_{t-2+j}\text{-}v_1\text{-}u_{3+j}, \\ &\quad \ldots, u_{\frac{t}{2}+2+j}\text{-}v_1\text{-}u_{\frac{t}{2}-1+j},\ u_{\frac{t}{2}-1+j}\text{-}v_1\text{-}u_{\frac{t}{2}+1+j},\ u_{\frac{t}{2}+1+j}\text{-}v_1\text{-}u_{\frac{t}{2}+j},\ u_{\frac{t}{2}+j}\text{-}v_0\text{-}v_1). \end{aligned}$$

The collection of these t-cycles $\{C_j^0 : j \in [0, \frac{t}{2}-1]\} \cup \{C_j^1 : j \in [0, \frac{t}{2}-1]\}$ yields a decomposition of $K_{2,t}^{(3)}$. □

Lemma 2.6. *If $t \geq 4$ is an even integer, then $K_{t,t}^{(3)}$ decomposes into t-cycles.*

Proof. Consider $K_{t,t}^{(3)}$ with its vertex set grouped into $[v_0, v_{t-1}]$ and $[v_t, v_{2t-1}]$. Let K_t' and K_t'' be two disjoint copies of K_t with vertex sets $[v_0, v_{t-1}]$ and $[v_t, v_{2t-1}]$, respectively. As t is even, each of K_t' and K_t'' can be decomposed into i $\frac{t}{2}-1$ Hamilton cycles and a 1-factor. Denote by $H_1' \oplus H_2' \oplus \cdots \oplus H_{\frac{t}{2}-1}' \oplus F'$ and $H_1'' \oplus H_2'' \oplus \cdots \oplus H_{\frac{t}{2}-1}'' \oplus F''$, decompositions of K_t' and K_t'', respectively, where, for each $i \in [1, \frac{t}{2}-1]$, H_i' and H_i'' are Hamilton cycles and F' and F'' are 1-factors.

Consider $H_i^{'}$, $i \in [1, \frac{t}{2}-1]$. If $H_i^{'} = (v_{i_0}, v_{i_1}, \ldots, v_{i_{t-1}}, v_{i_0})$, where $i_0, i_1, \ldots, i_{t-1}$ is a permutation of $0, 1, \ldots, t-1$, then for each $r \in [t, 2t-1]$, obtain the t-cycle $(v_{i_0}\text{-}v_r\text{-}v_{i_1},\ v_{i_1}\text{-}v_r\text{-}v_{i_2},\ v_{i_2}\text{-}v_r\text{-}v_{i_3}, \ldots, v_{i_{t-2}}\text{-}v_r\text{-}v_{i_{t-1}},\ v_{i_{t-1}}\text{-}v_r\text{-}v_{i_0})$ in $K_{t,t}^{(3)}$.

Similarly, for each $j \in [1, \frac{t}{2}-1]$ and $s \in [0, t-1]$, using the Hamilton cycle $H_j^{''}$ and the vertex v_s, obtain a t-cycle in $K_{t,t}^{(3)}$ as follows: if $H_j^{''} = (v_{j_0}, v_{j_1}, \ldots, v_{j_{t-1}}, v_{j_0})$, where $j_0, j_1, \ldots, j_{t-1}$ is a permutation of $t, t+1, \ldots, 2t-1$, then the t-cycle is:

$$(v_{j_0}\text{-}v_s\text{-}v_{j_1},\ v_{j_1}\text{-}v_s\text{-}v_{j_2}, i\ v_{j_2}\text{-}v_s\text{-}v_{j_3}, \ldots, v_{j_{t-2}}\text{-}v_s\text{-}v_{j_{t-1}},\ v_{j_{t-1}}\text{-}v_s\text{-}v_{j_0}).$$

This produces $2(\frac{t}{2}-1)t$ edge disjoint t-cycles in $K_{t,t}^{(3)}$.

If necessary, after relabeling the vertices, let $F^{'} = \{v_0v_1, v_2v_3, \ldots, v_{t-2}v_{t-1}\}$ and $F^{''} = \{v_tv_{t+1}, v_{t+2}v_{t+3}, \ldots, v_{2t-2}v_{2t-1}\}$. For convenience, relabel the vertex v_{t+i}, $i \in [0, t-1]$, by u_i, where subscripts of u are taken modulo t. In this notation, $F^{''} = \{u_0u_1, u_2u_3, \ldots, u_{t-2}u_{t-1}\}$.

To complete the proof, we have to find t edge disjoint t-cycles from the set

$$\begin{aligned}&\{v_0\text{-}v_1\text{-}u_r,\ v_2\text{-}v_3\text{-}u_r, \ldots, v_{t-2}\text{-}v_{t-1}\text{-}u_r : r \in [0, t-1]\}\\ &\cup\{u_0\text{-}u_1\text{-}v_s, u_2\text{-}u_3\text{-}v_s, \ldots, u_{t-2}\text{-}u_{t-1}\text{-}v_s : s \in [0, t-1]\}\end{aligned}$$

of edges.

For each $i \in [0, \frac{t}{2}-1]$, let

$$\begin{aligned}C_i^{'} \;=\; &(v_0\text{-}v_1\text{-}u_{2i},\ u_{2i}\text{-}u_{2i+1}\text{-}v_2,\\ &v_2\text{-}v_3\text{-}u_{2i+2},\ u_{2i+2}\text{-}u_{2i+3}\text{-}v_4,\\ &v_4\text{-}v_5\text{-}u_{2i+4},\ u_{2i+4}\text{-}u_{2i+5}\text{-}v_6,\\ &\ldots,\\ &v_{t-4}\text{-}v_{t-3}\text{-}u_{2i-4},\ u_{2i-4}\text{-}u_{2i-3}\text{-}v_{t-2},\\ &v_{t-2}\text{-}v_{t-1}\text{-}u_{2i-2},\ u_{2i-2}\text{-}u_{2i-1}\text{-}v_0)\end{aligned}$$

and

$$\begin{aligned}C_i^{''} \;=\; &(v_1\text{-}v_0\text{-}u_{2i+1},\ u_{2i+1}\text{-}u_{2i}\text{-}v_{t-1},\\ &v_{t-1}\text{-}v_{t-2}\text{-}u_{2i-1},\ u_{2i-1}\text{-}u_{2i-2}\text{-}v_{t-3},\\ &v_{t-3}\text{-}v_{t-4}\text{-}u_{2i-3},\ u_{2i-3}\text{-}u_{2i-4}\text{-}v_{t-5},\\ &\ldots,\\ &v_5\text{-}v_4\text{-}u_{2i+5},\ u_{2i+5}\text{-}u_{2i+4}\text{-}v_3,\\ &v_3\text{-}v_2\text{-}u_{2i+3},\ u_{2i+3}\text{-}u_{2i+2}\text{-}v_1).\end{aligned}$$

Clearly, $\{C_i^{'},\ C_i^{''} : i \in [0, \frac{t}{2}-1]\}$ is the required collection of t edge-disjoint t-cycles, which completes the proof. □

Lemma 2.7. *If $t \geq 4$ is an even integer, then $K_{2t,2t}^{(3)}$ decomposes into t-cycles.*

Proof. Consider $K^{(3)}_{2t,2t}$ with its vertex set grouped into $[v_0, v_{2t-1}]$ and $[v_{2t}, v_{4t-1}]$. Write $K^{(3)}_{2t,2t}$ as an edge-disjoint union of eight subhypergraphs, out of which four are copies of $K^{(3)}_{t,t}$ with vertex sets grouped into

(i) $[v_0, v_{t-1}]$ and $[v_{2t}, v_{3t-1}]$,
(ii) $[v_0, v_{t-1}]$ and $[v_{3t}, v_{4t-1}]$,
(iii) $[v_t, v_{2t-1}]$ and $[v_{2t}, v_{3t-1}]$, and
(iv) $[v_t, v_{2t-1}]$ and $[v_{3t}, v_{4t-1}]$;

and the remaining four are copies of $Z^{(3)}_{t,t,t}$ with vertex sets grouped into

(i) $[v_0, v_{t-1}]$, $[v_t, v_{2t-1}]$ and $[v_{2t}, v_{3t-1}]$,
(ii) $[v_0, v_{t-1}]$, $[v_t, v_{2t-1}]$ and $[v_{3t}, v_{4t-1}]$,
(iii) $[v_0, v_{t-1}]$, $[v_{2t}, v_{3t-1}]$ and $[v_{3t}, v_{4t-1}]$, and
(iv) $[v_t, v_{2t-1}]$, $[v_{2t}, v_{3t-1}]$ and $[v_{3t}, v_{4t-1}]$.

Since the hypergraphs $K^{(3)}_{t,t}$ and $Z^{(3)}_{t,t,t}$ are decomposable into t-cycles by Lemma 2.6 and Corollary 2.1, respectively, we have the required decomposition. □

2.3 Decompositions of $K^{(3)}_{2t+1}$ and $K^{(3)}_{t+2}$

A *Hamilton* cycle of a hypergraph $\mathcal{H}$ on n vertices is a cycle of length n.

Theorem 2.2. [1, 2, 10] *If $n \equiv 1,\ 2,\ 4$ or $5 \pmod 6$, then $K^{(3)}_n$ decomposes into Hamilton cycles.*

Lemma 2.8. *If $t \geq 4$ and $t \equiv 2$ or $4 \pmod 6$, then $K^{(3)}_{2t}$ decomposes into t-cycles.*

Proof. By Theorem 2.2 and Lemma 2.6, $K^{(3)}_t$ and $K^{(3)}_{t,t}$ are, respectively, t-cycle decomposable, and hence so is $K^{(3)}_{2t} = 2K^{(3)}_t \oplus K^{(3)}_{t,t}$. □

Lemma 2.9. *If $t \geq 4$ and $t \equiv 2$ or $4 \pmod 6$, then $K^{(3)}_{2t+1}$ decomposes into t-cycles.*

Proof. By Lemmas 2.8 and 2.4, $K^{(3)}_{2t}$ and $K^{(3)}_{1,2t}$ are, respectively, t-cycle decomposable, and hence so is $K^{(3)}_{2t+1} = K^{(3)}_{2t} \oplus K^{(3)}_{1,2t}$. □

Lemma 2.10. *If $t \geq 4$ and $t \equiv 2$ or $4 \pmod 6$, then $K^{(3)}_{t+2}$ decomposes into t-cycles.*

Proof. By Theorem 2.2 and Lemma 2.5, $K^{(3)}_t$ and $K^{(3)}_{2,t}$ are, respectively, t-cycle decomposable, and hence so is $K^{(3)}_{t+2} = K^{(3)}_t \oplus K^{(3)}_{2,t}$. □

3 Proof of Theorem 1.1.

To prove Theorem 1.1, we consider three cases:

Case 1. $n \equiv 0 \pmod{t}$.

Then $n = kt$ for some positive integer k. We may think of $K_{kt}^{(3)}$ as an edge-disjoint union of k copies of $K_t^{(3)}$, $\frac{k(k-1)}{2}$ copies of $K_{t,t}^{(3)}$ and $\frac{k(k-1)(k-2)}{6}$ copies of $Z_{t,t,t}^{(3)}$. That is,

$$K_{kt}^{(3)} = \underbrace{K_t^{(3)} \oplus K_t^{(3)} \oplus \cdots \oplus K_t^{(3)}}_{k \text{ times}} \oplus \underbrace{K_{t,t}^{(3)} \oplus K_{t,t}^{(3)} \oplus \cdots \oplus K_{t,t}^{(3)}}_{\frac{k(k-1)}{2} \text{ times}} \oplus \underbrace{Z_{t,t,t}^{(3)} \oplus Z_{t,t,t}^{(3)} \oplus \cdots \oplus Z_{t,t,t}^{(3)}}_{\frac{k(k-1)(k-2)}{6} \text{ times}}.$$

As each of the hypergraphs $K_t^{(3)}$, $K_{t,t}^{(3)}$ and $Z_{t,t,t}^{(3)}$ is decomposable into t-cycles by Theorem 2.2, Lemma 2.6 and Corollary 2.1, respectively, we have the required decomposition.

Case 2. $n \equiv 2 \pmod{t}$.

Then $n = kt + 2$ for some positive integer k. We may think of $K_{kt+2}^{(3)}$ as an edge-disjoint union of k copies of $K_{t+2}^{(3)}$, $\frac{k(k-1)}{2}$ copies of $K_{t,t}^{(3)}$, $\frac{k(k-1)(k-2)}{6}$ copies of $Z_{t,t,t}^{(3)}$ and $k(k-1)$ copies of $Z_{1,t,t}^{(3)}$. That is, $K_{kt+2}^{(3)} = \underbrace{K_{t+2}^{(3)} \oplus K_{t+2}^{(3)} \oplus \cdots \oplus K_{t+2}^{(3)}}_{k \text{ times}} \oplus \underbrace{K_{t,t}^{(3)} \oplus K_{t,t}^{(3)} \oplus \cdots \oplus K_{t,t}^{(3)}}_{\frac{k(k-1)}{2} \text{ times}} \oplus \underbrace{Z_{t,t,t}^{(3)} \oplus Z_{t,t,t}^{(3)} \oplus \cdots \oplus Z_{t,t,t}^{(3)}}_{\frac{k(k-1)(k-2)}{6} \text{ times}} \oplus \underbrace{Z_{1,t,t}^{(3)} \oplus Z_{1,t,t}^{(3)} \oplus \cdots \oplus Z_{1,t,t}^{(3)}}_{\frac{k(k-1)}{2} \text{ times}} \oplus \underbrace{Z_{1,t,t}^{(3)} \oplus Z_{1,t,t}^{(3)} \oplus \cdots \oplus Z_{1,t,t}^{(3)}}_{\frac{k(k-1)}{2} \text{ times}}$. As each of the hypergraphs $K_{t+2}^{(3)}$, $K_{t,t}^{(3)}$, $Z_{t,t,t}^{(3)}$ and $Z_{1,t,t}^{(3)}$ is decomposable into t-cycles by Lemma 2.10, Lemma 2.6, Corollary 2.1 and Lemma 2.1, respectively, we have the required decomposition.

Case 3. $n \equiv 1 \pmod{2t}$.

Then $n = 2kt + 1$ for some positive integer k. We may think of $K_{2kt+1}^{(3)}$ as an edge-disjoint union of k copies of $K_{2t+1}^{(3)}$, $\frac{k(k-1)}{2}$ copies of $K_{2t,2t}^{(3)}$, $\frac{k(k-1)(k-2)}{6}$ copies of $Z_{2t,2t,2t}^{(3)}$ and $\frac{k(k-1)}{2}$ copies of $Z_{1,2t,2t}^{(3)}$. That is,

$$\begin{aligned} K_{2kt+1}^{(3)} &= \underbrace{K_{2t+1}^{(3)} \oplus K_{2t+1}^{(3)} \oplus \cdots \oplus K_{2t+1}^{(3)}}_{k \text{ times}} \oplus \underbrace{K_{2t,2t}^{(3)} \oplus K_{2t,2t}^{(3)} \oplus \cdots \oplus K_{2t,2t}^{(3)}}_{\frac{k(k-1)}{2} \text{ times}} \\ &\oplus \underbrace{Z_{2t,2t,2t}^{(3)} \oplus Z_{2t,2t,2t}^{(3)} \oplus \cdots \oplus Z_{2t,2t,2t}^{(3)}}_{\frac{k(k-1)(k-2)}{6} \text{ times}} \oplus \underbrace{Z_{1,2t,2t}^{(3)} \oplus Z_{1,2t,2t}^{(3)} \oplus \cdots \oplus Z_{1,2t,2t}^{(3)}}_{\frac{k(k-1)}{2} \text{ times}}. \end{aligned}$$

As each of the hypergraphs $K_{2t+1}^{(3)}$, $K_{2t,2t}^{(3)}$, $Z_{2t,2t,2t}^{(3)}$ and $Z_{1,2t,2t}^{(3)}$ is decomposable into t-cycles by Lemmas 2.9, 2.7, 2.3 and 2.2, respectively, we have the required decomposition. □

Among even t, consider those t of the form 2^ℓ, where $\ell \geq 2$. Observe that $2^\ell | \binom{n}{3}$ if and only if $2^{\ell+1}|(n-1)$ or $2^{\ell+1}|n(n-2)$. But $2^{\ell+1}|n(n-2)$ if and only if $2^\ell|n$ or $2^\ell|(n-2)$, and hence $2^\ell|\binom{n}{3}$ if and only if $2^\ell|(n-2)$ or $2^{\ell+1}|(n-1)$ or $2^\ell|n$.

From the above necessary condition and Theorem 1.1 with $t = 2^\ell$, we have the following.

Corollary 3.1. *If $n \geq 2^\ell$ and $\ell \geq 2$, then $K_n^{(3)}$ has a 2^ℓ-cycle decomposition if and only if $n \equiv 0 \pmod{2^\ell}$, $2 \pmod{2^\ell}$ or $1 \pmod{2^{\ell+1}}$.*

Taking $\ell = 2$, we have

Corollary 3.2. *[4] If $n \geq 4$, then $K_n^{(3)}$ has a 4-cycle decomposition if and only if $n \equiv 0 \pmod 4$, $2 \pmod 4$ or $1 \pmod 8$.*

Acknowledgements

We would like to thank the reviewers for their valuable suggestions which improved the presentation of the article.

References

[1] J. C. Bermond, Hamiltonian decompositions of graphs, directed graphs and hypergraphs, *Ann. Discrete Math.* 3 (1978), 21–28.

[2] J. C. Bermond, A. Germa, M. C. Heydemann and D. Sotteau, Hypergraphes hamiltoniens, in Problèmes combinatoires et théorie des graphes, (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), pp. 39–43; *Colloq. Internat.* CNRS, 260, CNRS, Paris, 1978.

[3] D. Bryant, S. Herke, B. Maenhaut and W. Wannasit, Decompositions of complete 3-uniform hypergraphs into small 3-uniform hypergraphs, *Australas. J. Combin.* 60(2) (2014), 227–254.

[4] H. Jordon and G. Newkirk, 4-Cycle decompositions of complete 3-uniform hypergraphs, *Australas. J. Combin.* 71(2) (2018), 312–323.

[5] D. Kühn and D. Osthus, Decompositions of complete uniform hypergraphs into Hamilton Berge cycles, *J. Combin. Theory Ser. A* 126 (2014), 128–135.

[6] R. Lakshmi and T. Poovaragavan, Decompositions of complete 3-uniform hypergraphs into cycles of constant prime length, *Opuscula Math.* 40(4) (2020), 509–516.

[7] R. Lakshmi and T. Poovaragavan, 6-Cycle decompositions of complete 3-uniform hypergraphs, *Australas. J. Combin.* 80(1) (2021), 79–88.

[8] P. Petecki, On cyclic hamiltonian decompositions of complete k-uniform hypergraphs, *Discrete Math.* 325 (2014), 74–76.

[9] M. Truszczyǹski, Note on the decomposition of $\lambda K_{m,n}$ ($\lambda K^*_{m,n}$) into paths, *Discrete Math.* 55 (1985), 89–96.

[10] H. Verrall, Hamilton decompositions of complete 3-uniform hypergraphs, i *Discrete Math.* 132 (1994), 333–348.

(Received 17 Dec 2021; revised 16 Aug 2022)