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Switching for small strongly regular graphs

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Abstract

We provide an abundance of strongly regular graphs (SRGs) for certain parameters (n, k, λ, μ) with $n < 100$. For this we use Godsil-McKay (GM) switching with a partition of type $4, n-4$ and Wang-Qiu-Hu (WQH) switching with a partition of type $3, 3, n-6$ or $4, 4, n-8$. In most cases, we start with a highly symmetric graph which belongs to a finite geometry. Many of the graphs obtained are new; for instance, we find 16,565,438 strongly regular graphs with parameters $(81, 30, 9, 12)$ while only 15 seem to be described in the literature.

We provide statistics about the size of the occurring automorphism groups. We also find the recently discovered Krčadinac partial geometry, thus finding a third method of constructing it.

1 Introduction

> 'Strongly regular graphs lie on the cusp between highly structured and unstructured. For example, there is a unique strongly regular graph with parameters $(36, 10, 4, 2)$, but there are 32 548 non-isomorphic graphs with parameters $(36, 15, 6, 6)$.' Peter Cameron, "Random Strongly Regular Graphs?"

A strongly regular graph (SRG) is a k-regular graph with n vertices such that any two adjacent vertices have λ common neighbors, while any two non-adjacent vertices have μ common neighbors [15]. We call (n, k, λ, μ) the parameters of an SRG. SRGs are interesting for many reasons. Their existence relates to several combinatorial objects such as Steiner triple systems, quasi-symmetric designs, rank 3 permutation groups, and partial geometries. See [5] for a recent survey. They are also an important class of graphs for isomorphism testing [3, 30] as they are often hard to distinguish which makes it interesting to have many SRGs with the same parameters.

Our main aim is to provide an abundance of small SRGs which can be used to test various conjectures in graph theory. For instance, researchers test conjectures

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by using Spence's collection of SRGs [29]. Sometimes these are later refuted, cf. [13]. A larger selection of easily accessible SRGs as well as an easy method to generate them will hopefully lead to better conjectures. An additional motivation is that [5] cites some of the data of this document and we want to provide a proper reference.

Let us recall special cases of Godsil-McKay (GM) switching [12] and Wang-Qiu-Hu (WQH) switching [33], cf. [16].

Theorem 1.1 (GM Switching) *Let Γ be a graph whose vertex set is partitioned as $C \cup D$. Assume that the induced subgraph on C is regular. Suppose that each $x \in D$ either has 0, $|C|/2$, or $|C|$ neighbours in C. Construct a new graph $\overline{\Gamma}$ by switching adjacency and non-adjacency between $x \in D$ and C when $|\Gamma(x) \cap C| = |C|/2$. Then Γ and $\overline{\Gamma}$ are cospectral.*

Theorem 1.2 (WQH Switching) *Let Γ be a graph whose vertex set is partitioned as $C_1 \cup C_2 \cup D$. Assume that the induced subgraphs on C_1, C_2, and $C_1 \cup C_2$ are regular, and that the induced subgraphs on C_1 and C_2 have the same size and degree. Suppose that each $x \in D$ either has the same number of neighbors in C_1 and C_2, or $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$. Construct a new graph $\overline{\Gamma}$ by switching adjacency and non-adjacency between $x \in D$ and $C_1 \cup C_2$ when $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$. Then Γ and $\overline{\Gamma}$ are cospectral.*

Cospectral SRGs have the same parameters, so WQH switching applied to an SRG yields an SRG with the same parameters. We say that we apply WQH switching with a partition of type $\ell, \ell, n-2\ell$ if $|C_1| = |C_2| = \ell$. The aim of this paper is to provide a large collection of SRGs which can be generated by WQH switching with a partition of type $2, 2, n-4$, a partition of type $3, 3, n-6$, or a partition of type $4, 4, n-8$.

Note that WQH switching with a partition of type $2, 2, n-4$ produces a graph isomorphic to a graph with Godsil-McKay switching if $C = C_1 \cup C_2$. As this is mentioned in both [5] and [16] without proof, let us include one provided personally due to Munemasa [27].

Lemma 1.3 *Theorem 1.1 and Theorem 1.2 produce isomorphic graphs if $C = C_1 \cup C_2$ and $|C| = 4$.*

Proof: Let I and J denote the identity matrix and all-ones matrix, respectively. Let P_1 be the permutation matrix for $(1,2)(3,4)$, P_2 the permutation matrix for $(1,3)(2,4)$, and P_3 the permutation matrix for $(1,4)(2,3)$. Put $Q_i = \frac{1}{2}(J - 2P_i)$. Put

$$P = \begin{pmatrix} \frac{1}{2}J - I & 0 \\ 0 & I \end{pmatrix}, \qquad R = \begin{pmatrix} Q_1 & 0 \\ 0 & I \end{pmatrix}.$$

Let A be the adjacency matrix of Γ. Suppose that $C = C_1 \cup C_2$ corresponds to the first four vertices of A. Then the graph $\overline{\Gamma}_1$ from Theorem 1.1 has adjacency

(n,d,c,a)	#	Type	Seed	$\gg$?
$(57,24,11,9)$	31,490,375	GM(9)	$S(2,3,19)$	no
$(63,30,13,15)$	13,505,292	GM(5)	$Sp(6,2)$	no
$(64,21,8,6)$	76,323	GM(∞)	$Bilin(2,3,2)$	no
$(64,27,10,12)$	8,613,977	GM(5)	$VO^-(6,2)$	yes
$(64,28,12,12)$	11,063,360	GM(5)	$VO^+(6,2)$	maybe
$(70,27,12,9)$	78,900,835	GM(10)	$S(2,3,21)$	no
$(81,24,9,6)$	7,441,608	$\text{WQH}_3(6)$	$VNO_4^+(3)$	maybe
$(81,30,9,12)$	16,565,438	$\text{WQH}_3(\infty)$	$VNO_4^-(3)$	yes
$(81,32,13,12)$	21,392,603	$\text{WQH}_3(6)$	$Bilin(2,2,3)$	maybe
$(85,30,3,5)$	237,787	$\text{WQH}_4(5)$	$Sp(4,4)$	yes
$(96,19,2,4)$	178,040	$\text{WQH}_4(6)$	$Haemers(4)$	maybe
$(96,20,4,4)$	133,005	$\text{WQH}_4(6)$	$GQ(5,3)$	maybe

Table 1: The number of generated graphs.

matrix PAP, and the graph $\overline{\Gamma}_2$ from Theorem 1.2 has adjacency matrix RAR. We have $Q_1Q_2Q_3 = \frac{1}{2}(J-I)$, and $Q_2Q_3 = P_2P_3$ is a permutation. Hence, PAP is a permutation of RAR. Thus, $\overline{\Gamma}_1$ and $\overline{\Gamma}_2$ are isomorphic. □

It was shown in several papers, for instance [1, 16], that GM and WQH switching work well for several families of SRGs. Here we present a more thorough investigation for small parameter sets. Note that WQH switching was almost observed in Definition 3 of [4] by Behbahani, Lam, and Östergård. This led to a similar investigation.

Table 1 summarizes our results. Write GM(m) (respectively, $\text{WQH}_\ell(m)$) if we apply WQH switching up to m times with a partition of type $2, 2, n-4$ (respectively, $\ell, \ell, n-2\ell$) to our seed graph.

Definitions of the graphs are in the corresponding subsections. The last column "$\gg$?" contains a binary statement yes/no to state whether (as far as the author is aware) the number of graphs constructed here is much larger than those found in the literature. References are given in the corresponding subsections. We write "maybe" when there are not many graphs in the literature, but at least one construction, which in general is known to be prolific in some sense, is associated with the given set of parameters. Note that exact counts are out of the scope of this note; for instance, for parameters $(64,27,10,12)$ there are at least 6 different methods of constructing such SRGs, see [5], and it is not clear how many nonisomorphic graphs these yield.

We provide the number of new graphs after each switching step and the automorphism group sizes for all graphs. All graphs can be found on the homepage of the author in Nauty's graph6 format: `http://math.ihringer.org/srgs.php`. There we also provide selected versions of the C program used.

2 Finding Partitions and Other Technicalities

Our investigation itself uses the folklore method of keeping a global record of *canonical representatives* of graphs for isomorphy rejection, see [20, Subsection 4.2.1] for the general technique.

The canonical representative of a graph is given by McKay's and Piperno's nautytraces [26]. A tiny self-written C program applies the switching. We also use nautytraces to calculate the sizes of the automorphism groups. We use cliquer by Östergård [28] to calculate clique numbers in some cases. In two cases we use the default SRG with the corresponding parameters from Sage [31], relying on Cohen's and Pasechnik's implementation of Brouwer's SRG database [9]. Due to hardware constraints, we usually end our search at around 10 million SRGs. A particular emphasis was put on parameters $(70, 27, 12, 9)$ as the existence of a partial geometry $pg(6, 6, 4)$ is open.

We want to calculate all graphs which we can obtain from a seed graph Γ_0 by applying a chosen type of switching up to i times. We describe the general method in the following:

Replace the seed graph Γ_0 by its canonical representative. Note that there are many canonical forms for graphs and one has to use the same method throughout the whole algorithm.

$T \leftarrow \{\Gamma_0\}$, $C \leftarrow \{\Gamma_0\}$, $j \leftarrow 0$

for $j < i$ **do**

  $N \leftarrow \emptyset$

  for $\Gamma \in C$ **do**

    Calculate the set M_Γ of graphs which can be obtained by applying the chosen switching to Γ. See Subsection 2.1 and Subsection 2.2 for details.

    Calculate the set N_Γ of canonical representatives of the graphs in M_Γ. Note that M_Γ might contain distinct, but isomorphic graphs, while N_Γ cannot.

    $N \leftarrow N \cup N_\Gamma$

  end for

  $C \leftarrow N \setminus T$

  $T \leftarrow T \cup C$

  $j \leftarrow j + 1$

end for

Now T is the set of all canonical representatives of graphs which can be obtained from Γ_0 by applying the chosen switching up to i times.

Let us explain T, C, and N: At the beginning of the outer for-loop, T (as in *total*) is the set of graphs after applying the chosen switching j times; C (as in *current*) is the set of graphs in T to which the chosen switching was not yet applied. The inner for-loop applies the chosen switching to all elements in C and collects their canonical representatives in N (as in *new*).

Our partition finding method is very simple and described below. It uses simple pruning techniques. Our vertex set is labeled $V = \{1, \dots, n\}$ and the adjacency matrix of the graph is A.

2.1 Type $2, 2, n - 4$

For WQH switching with a partition of type $2, 2, n-4$, we implemented GM switching with a partition of type $4, n - 4$. The partition $C \cup D$ has to satisfy the following:

(A) The induced subgraph on C is regular.

(B) All $x \in D$ satisfy $|\Gamma(x) \cap C| \in \{0, 2, 4\}$.

(C) There exists an $x \in D$ with $|\Gamma(x) \cap C| = 2$.

The last condition is not stated in Theorem 1.1 above, but otherwise $\Gamma = \overline{\Gamma}$.

Most of our generated graphs have no symmetries,[1] so we naively iterate through all 4-tuples (c_1, c_2, c_3, c_4) with $c_1 < c_2 < c_3 < c_4$ in a nested loop. We only check the conditions in the inner loop. First we check for (A) as it (naively) only involves accessing up to $|C|(|C| - 1) = 12$ entries of A, while (B) and (C) might access up to $|D| \cdot |C| = 4(n - 4)$ entries of A.

2.2 Type $\ell, \ell, n - 2\ell$

For WQH switching with a partition of type $\ell, \ell, n - 2\ell$, the partition $C_1 \cup C_2 \cup D$ has to satisfy the following:

(A) The induced subgraph on C_1 is regular for some degree k_1.

(B) The induced subgraph on C_2 is regular with the same degree k_1.

(C) The bipartite subgraph between C_1 and C_2 (with the edges of Γ) is regular.

(D) All $x \in D$ satisfy $|\Gamma(x) \cap C_1| = |\Gamma(x) \cap C_2|$ or $\Gamma(x) \cap (C_1 \cup C_2) \in \{C_1, C_2\}$.

(E) The second case of (D) occurs.

While Theorem 1.2 asks for the induced subgraph in $C_1 \cup C_2$ to be regular, in light of (A) and (B), testing for (C) suffices and is faster.

Suppose that $C_1 = \{c_1, \dots, c_\ell\}$ and $C_2 = \{c_{\ell+1}, \dots, c_{2\ell}\}$. We pick $c_1, \dots, c_{2\ell}$ in order, where $c_1 < \dots < c_\ell$ and $c_1 < c_{\ell+1} < \dots < c_{2\ell}$. Write $\tilde{C}_m = \{c_1, \dots, c_m\}$.

[1] We have no a priori reason for this.

Let $k_{11}(m)$ (respectively, $k_{22}(m)$) be the minimal degree of the induced subgraph on $\tilde{C}_m$ (respectively, $\tilde{C}_m \setminus \tilde{C}_\ell$) and let $K_{11}(m)$ (respectively, $K_{22}(m)$) be the maximal degree of the induced subgraph on $\tilde{C}_m$ (respectively, $\tilde{C}_m \setminus \tilde{C}_\ell$). We discard $\tilde{C}_m$ if $K_1(m) - k_1(m) > \ell - m$ for $m \leq \ell$ as then (A) is impossible. Similarly, we discard $\tilde{C}_m$ if $K_2(m) - k_2(m) > 2\ell - m$ for $m > \ell$ as then (B) is impossible.

Suppose $\{i, j\} = \{1, 2\}$ and $m > \ell$. Consider the bipartite graph with parts $\tilde{C}_\ell$ and $\tilde{C}_m \setminus \tilde{C}_\ell$ (with the edges as in Γ). Let $k_{12}(m)$ be the minimal degree on $\tilde{C}_\ell$, $k_{21}(m)$ the minimal degree on $\tilde{C}_m \setminus \tilde{C}_\ell$, $K_{12}(m)$ the maximum degree on $\tilde{C}_\ell$, and $K_{12}(m)$ the maximum degree on $\tilde{C}_m \setminus \tilde{C}_\ell$. We discard $\tilde{C}_m$ if $K_{21}(m) > k_{21}(m)$ (we already picked all vertices of $\tilde{C}_\ell$, so all degrees in $\tilde{C}_m \setminus \tilde{C}_\ell$ must be the same by (C)). We also discard $\tilde{C}_m$ if $K_{12}(m) - k_{12}(m) > 2\ell - m$ as otherwise (C) is impossible.

For (D) and (E) we only test in the inner loop after C_1 and C_2 are fully chosen.

3 SRGs

In this section we present the generated SRGs. We apply switchings of type GM and WQH_3 to all the discussed graphs. If any of them does not work, then we try to apply switchings of type WQH_4. We mention precisely the cases for which our technique produces SRGs which are nonisomorphic to the used seed graph.

3.1 Very Small Parameters

SRGs with very small parameters are discussed in [4]. For instance, there are at least 342 SRGs with parameters $(49, 18, 7, 6)$ and one has a GM switching class of size 175.

3.2 SRG(57, 24, 11, 9)

There is a one-to-one correspondence between Steiner triple systems and SRGs derived from a Steiner triple system [19, 30]. Particularly, the complete classification of Steiner triple systems of order 19 [19] yields a large amount of SRGs with parameters $(57, 24, 11, 9)$. All of our graphs might be included in the 11,084,874,829 SRGs of [19]. Similarly to [16, Theorem 4] one can see that certain cycle switches of designs (see [18]) can be interpreted as WQH switchings. Particularly, the so-called Pasch switching corresponds to WQH switching with a partition of type $2, 2, n-4$, that is GM switching with a partition of type $4, n-4$. To our knowledge, this is first observed in [4]. There it is also observed that GM switching can lead to non-geometric SRGs. The authors of [4] only find 338,536 SRGs by GM switching which is small compared to our number.

The number of generated SRGs after applying GM switching up to i times can be found in Table 2.

i	0	1	2	3	4	5	6	7	8	9
New	1	9	102	829	5,408	31,409	171,607	913,192	4,826,290	25,541,528
Total	1	10	112	941	6,349	37,758	209,365	1,122,557	5,948,847	31,490,375

Table 2: SRGs with parameters $(57, 24, 11, 9)$.

The automorphism group sizes can be found in Table 3. The first row denotes the size of the automorphism group, the second row denotes the numbers of SRGs with an automorphism group of that size.

$\|G\|$	1	2	3	6	9
SRGs	31,489,648	468	255	1	3

Table 3: Automorphism group sizes of SRGs with parameters $(57, 24, 11, 9)$.

3.3 SRG(63, 30, 13, 15)

Let us give a short description of the collinearity graph of $Sp(2d, q)$: Vertices are 1-dimensional subspaces of $\mathbb{F}_q^{2d}$. Two 1-dimensional subspaces are adjacent if they are perpendicular with respect to the bilinear form $x_1y_2 - x_2y_1 + \ldots + x_{2d-1}y_{2d} - x_{2d}y_{2d-1}$. For $(d, q) = (3, 2)$, this graph has the desired parameters. WQH switching works for $Sp(2d, q)$, see [1] for $q = 2$ and [16] for the general case.

The graph $Sp(6, 2)$ has an automorphism group of size 1,451,520, clique number 7 and coclique number 7. SRGs with the same parameters as $Sp(6, 2)$ have spectrum $(30, 3^{35}, -5^{27})$, clique number at most 7 and coclique number at most 9. More details on $Sp(6, 2)$ can be found in [5, Section 10.21].

At most 522,079 SRGs are known from intersection-8 graphs of quasi-symmetric 2-$(36, 16, 12)$ designs [24], 4,653 SRGs with these parameters in [23], at least 9 SRGs with these parameters in [2], and one more SRG in [1].

The number of graphs after applying GM switching up to i times can be found in Table 4.

i	0	1	2	3	4	5
New	1	2	52	3,275	254,097	13,247,865
Total	1	3	55	3,330	257,427	13,505,292

Table 4: SRGs with parameters $(63, 30, 13, 15)$.

The automorphism group sizes can be found in Table 5 and Figure 1. The first column denotes the size of the automorphism group, the second column denotes the numbers of SRGs with an automorphism group of that size.

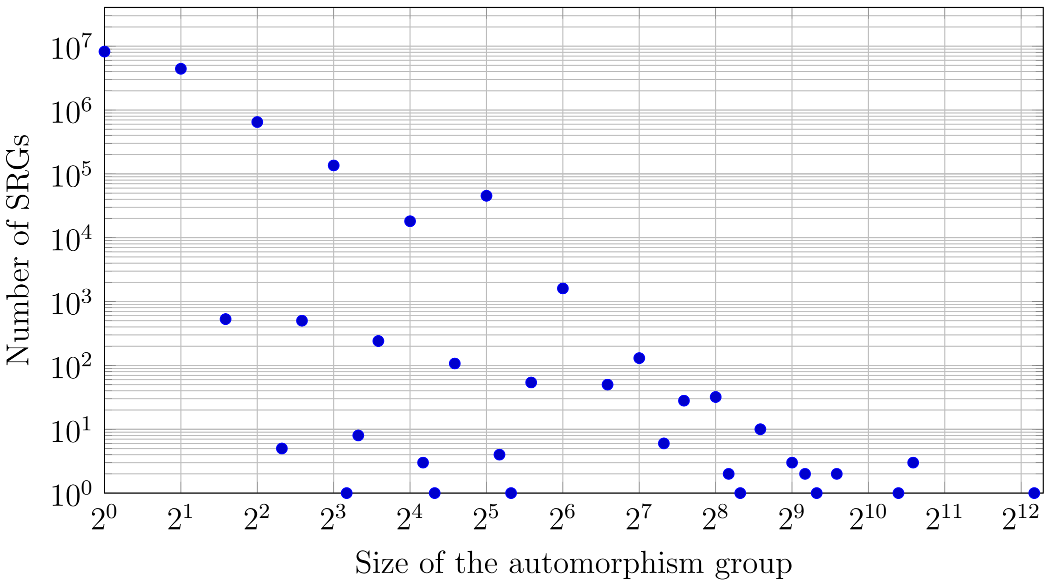

Figure 1: Group sizes for parameters $(63, 30, 13, 15)$.

$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs
1	8,226,588	9	1	32	45,390	160	6	576	2
2	4,428,326	10	8	36	4	192	28	640	1
3	531	12	241	40	1	256	32	768	2
4	648,049	16	18,136	48	54	288	2	1,344	1
5	5	18	3	64	1,605	320	1	1,536	3
6	501	20	1	96	50	384	10	4,608	1
8	135,468	24	107	128	130	512	3	1,451,520	1

Table 5: Automorphism group sizes of SRGs with parameters $(63, 30, 13, 15)$.

The number of cliques of size 7 can be found in Table 6. We also found graphs with no cliques of size 6, but these examples are not reached in five steps. While a partial geometry of type $pg(6, 4, 3)$ is known, none of the 29,017 graphs with at least 45 cliques belongs to a partial geometry.

Cls	SRGs	Cls	SRGs	Cls	SRGs	Cls	SRGs	Cls	SRGs
0	24	14	379,654	28	180,134	42	1,490	56	2
1	54	15	1,025,530	29	400,493	43	32,343	57	29
2	202	16	480,205	30	120,076	44	404	59	227
3	2,837	17	1,087,195	31	420,594	45	6,777	60	3
4	2,574	18	530,185	32	67,720	46	215	61	13
5	15,844	19	1,214,285	33	178,932	47	14,592	62	2
6	15,519	20	497,876	34	40,152	48	72	63	198
7	76,236	21	1,015,163	35	191,629	49	1,560	67	29
8	53,325	22	433,987	36	19,761	50	30	71	69
9	199,053	23	1,052,324	37	77,842	51	2,918	79	7
10	131,289	24	346,865	38	10,441	52	22	87	6
11	436,005	25	705,665	39	100,499	53	181	103	1
12	246,544	26	263,539	40	3,668	54	3	135	1
13	694,608	27	699,321	41	24,189	55	2,060		

Table 6: Cliques of size 7 for parameters $(63, 30, 13, 15)$.

3.4 SRG(64, 21, 8, 6)

See Subsection 3.10 for a description of the graph $Bilin(2, 3, 2)$ which is our seed graph. The search in [4] found more than 500,000 SRGs, while the GM switching class of $Bilin(2, 3, 2)$ has only size 76,323, therefore we omit any further details. We calculated the index chromatic number of a random subset of size 1000, but failed to find a counterexample to the conjecture in [8], namely that the chromatic index of an SRG with n even is always k unless the SRG is the Petersen graph.

3.5 SRG(64, 27, 10, 12)

Let us give a short description of the graph $VO^-(2d, q)$. Let $Q(x) = \alpha x_1^2 + \beta x_1 x_2 + x_2^2 + \ldots + x_{2d}^2$ such that $\alpha x_1^2 + \beta x_1 x_2 + x_2^2$ is irreducible over $\mathbb{F}_q$. For $q = 2$, we can choose $(\alpha, \beta) = (1, 1)$. The vertices of $VO^-(2d, q)$ are the vectors of $\mathbb{F}_q^{2d}$. Two vertices x, y are adjacent if $Q(x-y) = 0$. The graph $VO^-(6, 2)$ has an automorphism group of size 3,317,760. More details on $VO^-(6, 2)$ can be found in [5, Section 10.25].

We find at least 9 SRGs with these parameters in [2].

The number of graphs after applying GM switching up to i times can be found in Table 7.

i	0	1	2	3	4	5
New	1	2	43	2,116	158,036	8,453,779
Total	1	3	46	2,162	160,198	8,613,977

Table 7: SRGs with parameters $(64, 27, 10, 12)$.

The automorphism group sizes are as in Table 8. The first column denotes the size of the automorphism group, the second column denotes the numbers of SRGs with an automorphism group of that size.

$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs
1	4,799,279	12	362	64	2,640	256	68	1,152	2
2	2,962,488	16	16,612	72	3	288	2	1,536	1
3	379	18	1	80	1	320	1	3,072	2
4	681,960	20	6	96	59	384	16	4,096	1
5	4	24	119	128	338	512	16	4,608	2
6	453	32	37,068	144	2	640	1	6,144	1
8	111,971	40	1	160	6	768	6	73,728	1
10	1	48	68	192	31	1,024	4	3,317,760	1

Table 8: Automorphism group sizes of SRGs with parameters $(64, 27, 10, 12)$.

The graph $VO^-(6, 2)$ is known to be $(K_5 - e)$-free and its complement is $(K_7 - e)$-free. Therefore, it is a witness for the Ramsey number $R(K_5 - e, K_7 - e) \geq 65$, see [5, Section 10.25]. In fact, $R(K_5 - e, K_7 - e) = 65$. Among the 8,613,977 in our collection, it is the only graph with that property. Indeed, it includes only 8 K_5-free graphs for which the complement is K_7-free.

3.6 SRG(64, 28, 12, 12)

The graph $VO^+(2d, q)$ can be constructed the same way as $VO^-(2d, q)$ from the preceding section, but with (α, β) chosen such that $\alpha x_1^2 + \beta x_1 x_2 + x_2^2$ is reducible over $\mathbb{F}_q$. For $q = 2$, we can choose $(\alpha, \beta) = (1, 0)$. The graph $VO^+(6, 2)$ has an automorphism group of size 2,580,480. More details on $VO^+(6, 2)$ can be found in [5, Section 10.26].

We find a at least 9 SRGs with these parameters in [2]. We find 15 SRGs with these parameters in [17].

The number of graphs after applying GM switching up to i times can be found in Table 9.

The automorphism group sizes can be found in Table 10. The first column denotes the size of the automorphism group, the second column denotes the numbers of SRGs

i	0	1	2	3	4	5
New	1	1	52	2,680	201,883	10,858,742
Total	1	3	55	2,735	204,618	11,063,360

Table 9: SRGs with parameters $(64, 28, 12, 12)$.

with an automorphism group of that size.

$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs
1	6,419,836	10	1	40	1	168	1	1,536	4
2	3,522,363	12	326	48	74	192	23	3,072	2
3	363	16	18,812	64	3,061	256	75	4,096	2
4	872,266	18	1	72	5	384	14	6,144	1
5	1	20	6	80	1	512	19	24,576	1
6	336	24	123	96	50	576	2	2,580,480	1
8	160,568	32	64,629	128	380	768	4		
9	1	36	1	144	2	1,024	4		

Table 10: Automorphism group sizes of SRGs with parameters $(64, 28, 12, 12)$.

3.7 SRG$(70, 27, 12, 9)$

The classification of Steiner triple systems of order 21 [21, 22] is a rich source for a large number of SRGs with parameters $(70, 27, 12, 9)$. Maybe our list of 78,900,835 SRGs is mostly disjoint to the 13,168,639 SRGs in [22] and the 83,003,869 SRGs in [21] as the extra conditions in [21, 22] appear to be restrictive.

Our seed graph belongs to a Steiner triple system on 21 points and has an automorphism group of size 126.

The number of graphs after applying GM switching up to i times can be found in Table 11.

i	0	1	2	3	4	5	6	7	8	9	10
New	1	1	7	85	775	6,094	43,397	286,285	1,799,283	10,976,064	65,788,843
Total	1	2	9	94	869	6,963	50,360	336,645	2,135,928	13,111,992	78,900,835

Table 11: SRGs with parameters $(70, 27, 12, 9)$.

The automorphism group sizes can be found in Table 12. The first row denotes the size of the automorphism group, the second row denotes the numbers of SRGs with an automorphism group of that size.

$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs
1	78,899,457	3	460	21	1
2	911	6	5	126	1

Table 12: Automorphism group sizes of SRGs with parameters $(70, 27, 12, 9)$.

The complement of an SRG with parameters $(70, 27, 12, 9)$ can be the point graph of a partial geometry of type $pg(6, 6, 4)$. An SRG belonging to such a partial geometry has at least 70 cocliques of size 7 which pairwise meet in at most one vertex. In the following, we list the number of the cocliques of size 7.

CoCls	SRGs	CoCls	SRGs	CoCls	SRGs	CoCls	SRGs
4	9	22	3,441,902	40	600,491	58	108
5	26	23	4,097,976	41	426,302	59	69
6	101	24	4,693,209	42	297,407	60	38
7	416	25	5,175,736	43	203,004	61	13
8	1,165	26	5,509,532	44	136,577	62	10
9	3,248	27	5,668,239	45	88,848	63	4
10	8,476	28	5,640,194	46	57,586	64	6
11	19,938	29	5,436,783	47	36,770	65	2
12	43,282	30	5,082,314	48	22,903	66	2
13	86,276	31	4,612,777	49	13,991	68	2
14	162,550	32	4,074,374	50	8,570	69	1
15	287,714	33	3,491,432	51	5,205	70	2
16	477,322	34	2,922,707	52	3,170		
17	749,488	35	2,382,507	53	1,790		
18	1,115,548	36	1,893,952	54	1,018		
19	1,585,748	37	1,468,602	55	601		
20	2,146,052	38	1,111,984	56	348		
21	2,778,104	39	826,135	57	179		

Table 13: Cocliques of size 7 for parameters $(70, 27, 12, 9)$.

We found only two graphs with a sufficient amount of cocliques. One has an automorphism group of size 6, one of size 1. Both have at most 16 cocliques which pairwise meet in at most one vertex. Hence, we do not obtain a $pg(6, 6, 4)$.

3.8 SRG$(81, 24, 9, 6)$

A nice geometric graph with the given parameters can be obtained as follows. Let $Q(x) = x_1^2 - x_2^2 + x_3^2 + x_4^2$. The vertices are the vectors of $\mathbb{F}_3^4$. Two vertices x and y

are adjacent if $Q(x-y)=1$. This graph is also known as $VNO^{+}(4,3)$ and has an automorphism group of size 93,312.

We find 13 graphs with these parameters in [4].

The number of graphs after applying WQH switching with a partition of type $3,3,n-6$ up to i times can be found in Table 14.

i	0	1	2	3	4	5	6
New	1	2	31	596	15,183	377,270	7,048,525
Total	1	3	34	630	15,813	393,083	7,441,608

Table 14: SRGs with parameters $(81,24,9,6)$.

The automorphism group sizes can be found in Table 15. The first column denotes the size of the automorphism group, the second column denotes the numbers of SRGs with an automorphism group of that size.

$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs
1	7,213,765	9	1,820	48	5	162	6	1,944	1
2	62,221	12	311	54	56	216	4	93,312	1
3	154,705	18	664	72	4	324	3		
4	635	24	20	81	5	432	1		
6	7,228	27	27	108	18	486	1		
8	6	36	96	144	1	972	4		

Table 15: Automorphism group sizes of SRGs with parameters $(81,24,9,6)$.

3.9 SRG(81, 30, 9, 12)

Van Lint and Schrijver discovered a partial geometry of type $pg(5,5,2)$ [32], the vL-S partial geometry. The point graph of this partial geometry is an SRG with parameters $(81,30,9,12)$. Recently, a second partial geometry of the same type was discovered by Krčadinac [25] and, almost at the same time, by Crnković, Švob and Tonchev [10]. More details on $VNO^{-}(4,3)$ can be found in [5, Section 10.29].

We can describe the SRG derived from the vL-S geometry as follows. Let $Q(x)=x_1^2+x_2^2+x_3^2+x_4^2$. The vertices are the vectors of $\mathbb{F}_3^4$. Two vertices x and y are adjacent if $Q(x-y)=1$. This graph is also known as $VNO_4^{-}(3)$.

The number of graphs after applying WQH switching with a partition of type $3,3,n-6$ up to i times can be found in Table 16. Further applications of the switching operation do not yield more graphs.

i	0	1	2	3	4	5	6	7
New	1	2	21	144	1,249	12,560	107,665	691,650
Total	1	3	24	168	1,417	13,977	121,642	813,292

i	8	9	10	11	12
New	2,957,467	7,041,075	4,892,852	835,010	25,742
Total	3,770,759	10,811,834	15,704,686	16,539,696	16,565,438

Table 16: SRGs with parameters $(81, 30, 9, 12)$.

The automorphism group sizes can be found in Table 17. The first column denotes the size of the automorphism group, the second column denotes the numbers of SRGs with an automorphism group of that size.

$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs
1	966	16	4	81	12	1,944	1
2	482	18	2,142	108	68	3,888	1
3	16,369,234	24	65	144	3	5,832	1
4	176	27	514	162	12	116,640	1
6	184,747	36	64	216	13		
8	6	48	5	324	6		
9	5,760	54	292	432	2		
12	845	72	8	972	8		

Table 17: Automorphism group sizes of SRGs with parameters $(81, 30, 9, 12)$.

Our seed graph, the point graph of the vL-S partial geometry, has an automorphism group of size 116,640. By comparing automorphism group sizes, we see that our list cannot contain all of the 14 new SRGs described in [10].

A partial geometry $pg(5, 5, 2)$ necessarily has at least 81 cliques of size 6 which pairwise meet in at most one vertex. Our search produced 38 SRGs with sufficiently many cliques of size 6, see Table 18. Only the ones corresponding to the vL-S partial geometry and the Krčadinac partial geometry are point graphs of partial geometries. Hence, we rediscover the Krčadinac partial geometry via a third method. The distance between the vL-S partial geometry and the Krčadinac partial geometry is 6 using WQH switchings with $|C_1| = |C_2| = 3$.

3.10 SRG(81, 32, 13, 12)

The graph $Bilin(2, m-2, q)$, $n \geq 4$, can be described as follows. The vertices are the set of all 2-spaces of $\mathbb{F}_q^m$ which are disjoint to a fixed $(m-2)$-space. Two 2-spaces

Cls	SRGs	Cls	SRGs	Cls	SRGs	Cls	SRGs	Cls	SRGs
0	10,807,396	16	38	32	47	50	14	81	10
1	60	17	11	33	421	51	1	84	3
2	146	18	45,898	34	15	52	2	90	14
3	3,788,193	19	2	35	4	54	321	98	2
4	95	20	48	36	2,514	56	17	102	1
5	17	21	16,740	37	1	57	10	108	6
6	1,157,783	22	39	38	24	58	1	126	1
7	19	23	8	39	1,006	59	2	162	1
8	43	24	8,805	40	38	60	18		
9	514,162	25	2	42	630	63	53		
10	60	26	13	43	18	65	2		
11	12	27	6,021	44	18	66	31		
12	169,427	28	35	45	447	70	8		
13	13	29	1	46	2	72	69		
14	54	30	3,120	47	3	76	4		
15	41,094	31	12	48	286	78	6		

Table 18: Cliques of size 6 for parameters $(81, 30, 9, 12)$.

are adjacent if their meet is a 1-space. This yields an SRG. For $n = 4$ and $q = 3$, its parameters are $(81, 32, 13, 12)$ and it has an automorphism group of size 186,624.

The number of graphs after applying WQH switching with a partition of type $3, 3, n - 6$ up to i times can be found in Table 19. The automorphism group sizes can be found in Table 20. The first row denotes the size of the automorphism group, the second row denotes the numbers of SRGs with an automorphism group of that size.

i	0	1	2	3	4	5	6
New	1	2	41	963	29,120	841,699	20,520,777
Total	1	3	44	1,007	30,127	871,826	21,392,603

Table 19: SRGs with parameters $(81, 32, 13, 12)$.

3.11 SRG(85, 20, 3, 5)

See Subsection 3.3 for a description of the graph $Sp(4, 4)$. It has an automorphism group of size 1,958,400.

The number of graphs after applying WQH switching with a partition of type $4, 4, n-8$ up to i times can be found in Table 21. Van Dam and Guo provide 127,433

$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs
1	20,082,770	12	1,225	48	8	144	4	972	6
2	286,231	16	16	54	111	162	3	3,888	1
3	961,829	18	3,623	64	1	216	9	186,624	1
4	6,108	24	74	72	7	288	1		
6	45,007	27	163	81	5	324	5		
8	206	32	3	96	3	432	2		
9	4,971	36	182	108	27	486	1		

Table 20: Automorphism group sizes of SRGs with parameters $(81, 32, 13, 12)$.

graphs with parameters $(85, 20, 3, 5)$ in [11][2]. The list of graphs here shares precisely 3,501 entries with their list.

i	0	1	2	3	4	5
New	1	1	16	442	12,303	225,024
Total	1	2	18	460	12,763	237,787

Table 21: SRGs with parameters $(85, 20, 3, 5)$.

The automorphism group sizes can be found in Table 22. The first column denotes the size of the automorphism group, the second column denotes the numbers of SRGs with an automorphism group of that size.

$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs	$\lvert G\rvert$	SRGs
1	100,064	12	47	64	15	384	3
2	116,790	16	206	72	2	768	1
3	87	20	3	96	8	1,280	1
4	19,268	24	23	128	5	1,536	1
5	2	32	51	192	1	2,304	1
6	29	36	1	240	1	1,958,400	1
8	1,167	48	8	288	1		

Table 22: Automorphism group sizes of SRGs with parameters $(85, 20, 3, 5)$.

3.12 SRG$(96, 19, 2, 4)$

See [6, Section 8.A] for a construction of graphs of type Haemers(q). Note that even for fixed q, this does not uniquely determine the graph. Our seed graph has an

[2] As of 12 April 2012. Their data changed after acceptance of this article.

automorphism group of size 9,216.

In [14] we find 2 graphs with these parameters. Surely, there are many more as several constructions are known and the constructions of type Haemers(4) allow for some freedom.

The number of graphs after applying WQH switching with a partition of type $4, 4, n-8$ up to i times can be found in Table 23.

i	0	1	2	3	4	5	6
New	1	2	17	160	1,680	17,578	158,602
Total	1	3	20	180	1,860	19,438	178,040

Table 23: SRGs with parameters $(96, 19, 2, 4)$.

The automorphism group sizes can be found in Table 24. The first column denotes the size of the automorphism group, the second column denotes the numbers of SRGs with an automorphism group of that size.

$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs
1	122,184	12	78	64	38	288	1
2	43,093	16	510	72	1	384	1
3	122	18	2	96	13	768	3
4	9,605	24	33	128	19	1,024	1
6	41	32	144	144	1	1,536	1
8	2,113	36	1	192	5	9,216	1
9	1	48	25	256	3		

Table 24: Automorphism group sizes of SRGs with parameters $(96, 19, 2, 4)$.

3.13 SRG(96, 20, 4, 4)

Our seed graph is the point graph of the unique generalized quadrangle of order $(5, 3)$ and has a group of size 138,240. In [14] we find 6 graphs with these parameters. Surely, there are many more as plenty constructions are known, but we are unaware of any counts.

The number of graphs after applying WQH switching with a partition of type $4, 4, n-8$ up to i times can be found in Table 25.

The automorphism group sizes can be found in Table 26. The first column denotes the size of the automorphism group, the second column denotes the numbers of SRGs with an automorphism group of that size.

i	0	1	2	3	4	5	6
New	1	2	13	95	949	10,773	121,172
Total	1	3	16	111	1,060	11,833	133,005

Table 25: SRGs with parameters $(96, 20, 4, 4)$.

$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs	$\|G\|$	SRGs
1	18,759	20	1	128	139	1,024	4
2	56,510	24	47	144	2	1,152	1
3	18	32	1,351	192	10	1,536	6
4	38,277	48	53	240	1	3,072	4
6	34	54	1	256	59	7,680	2
8	12,673	64	463	384	8	138,240	1
12	51	72	2	512	17		
16	4,475	80	1	640	1		
18	1	96	24	768	9		

Table 26: Automorphism group sizes of SRGs with parameters $(96, 20, 4, 4)$.

4 Future Work

It might be very fruitful to use switching to optimize SRGs for a certain parameter. For instance, switching embeds naturally in a threshold accepting algorithm. Note that for Steiner triple systems, one can find a similar suggestion in [18].

Our investigation is incomplete in at least two ways. Firstly, we might not have checked all known SRGs with less than 100 vertices for the considered switchings (as there are too many constructions known). Secondly, surely there exist SRGs for some of the parameters which are at the time of writing unknown. Here is a list of all sets of parameters for which SRGs are known, but we failed at finding a graph for which our switching works. Note that we did not investigate parameter sets which are completely classified.

$(37, 18, 8, 9)$, $(41, 20, 9, 10)$, $(45, 22, 10, 11)$, $(49, 24, 11, 12)$,
$(50, 21, 8, 9)$, $(53, 26, 12, 13)$, $(65, 32, 15, 16)$, $(70, 27, 12, 9)$,
$(73, 36, 17, 18)$, $(81, 40, 19, 20)$, $(82, 36, 15, 16)$, $(89, 44, 21, 22)$
$(97, 48, 23, 24)$, $(99, 48, 22, 24)$.

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A Ambiguous Seed Graphs

Here we list the used seed graphs for the less beautiful seed graphs.

A.1 SRG(57, 24, 11, 9)

```
x`MjkWRlZLZHuJY^J]~NvkT?^_KB[Sl_LAimY_Wxy_WFSGj`M_zopIn|?ZeaYYMo{Ceu
NC\Lap{\_]???^~{LiEiOKMaiaLQISj?taGIsd[cIMQLRoHMSMq[`wg@@PUDl@xpcG[p
@|QlDCSeedCiHOrJ_yOOwdzLASGw`zhrE_OjhCwlACKySW?Q@|[ouOBOBtuPrHaFGHuO
eb?sYB[ob`KOe_u@rbE@jGHoMY[@p{_deCTgBbe_qqCVgBb_]E@rK{?????~~~~
```

A.2 SRG(70, 27, 12, 9)

```
~?@EQd_pJPwdUi{chWU`w]W^hm`Xt?}pX^HYwRu\G}WF~~sLX?Sdp?kd_HmLGAtS`BFT
WGdYLGqAy[HaIpe]?NADaiCLQcXOc}pO_`Z{aaCI^eGbhlGXCRTqcF_@}?N{RWiY[QPf
Eal`_???F~~{TDQGdcAIbKCXaAdOa`[cBUgGSMe?ss[_EIOakhi?cW`SqET?`ESzpHe@
?havWEX_oEWWXURAHe_ChDpg_rW?YNoo?pwK?oxkKQeABD[PqQLHADAyakGXOCHoI}V@
Dg?`i@^\BgJQ_GbqIIV@RD@@UooE?@hQ_|h_tH?ASl?zUOy?R@PhOJ_FeoBGITgCw@xr
?eAa?^JUORW@cDG@{ueAMW?F`tgC\KCMg?F`ySAiqAVS?Bow@{c?{o~???????~~~~w
```

A.3 SRG(96, 19, 2, 4)

```
~?@_????CA?[_K?C?G?S_Ao???q@cACo?W?AA??OO??oEHC_WQCCGOGGCO_?Ew?CQ?d?P
_G[?S?d?@O?s?ScY??D`J??@EGE@OPG`GDE?@o??@OD@_`A_GaP@PC??a_aG_?OSH`IA?
Q@@GSGAOC?WBD?S?GWCg_g?Q?CgOPJ?OAEGAKoCCSAH?OgGGgHCAD?cO_CPG@_W_OG_oA
OAGDCO@C?`C?oa?__Pg@?O?oCC`OG@?C__`?`AG`?AC?x?cD@?CA@WB?wKe?K?Ka@__aK
g?SaGQGAaQ_DAEE?M??_^_??GOAQC`I@O?C_@cAQDA_??EECKe?W?WKKA_SI?@G?AQOWD
@_?E?B??N~??????F}???????@}@QpO?OOa?K?GUcW?OOa?H?CoCS?@OaaG_CE@D??DAI
Ga?OAa_PC@@CY?@?II@@GC@Cw?C?C`X?`GCU????CHQOGWCS_???G?????????F~~?BBo
@_?oK?w_AGPGPa?_O@?QDAHCPDG?`?C?SD@_??@~w????Ko?I????AYeT@CB@@O????XU
Y_`@GO@~_?????@xw?@oOIOOB_Gp?QG?OOOh?OF?aP?QG@?IR?E_GQA?OGK?_Ap_CgGO_
_OCH?_DG_hO`?G?gA_@??UGGdAOA?I?a?O?EPPPCG@O`???iG?BDDAO_D?c??AgO????B
_?wW?Fo?B_?K??AHO_P_gs?QOA_??@@aOGcDJ?BC@O???O@KE?]?wB?_?o??????????
?@~~~
```

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