AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **83(3)** (2022), Pages 312–336

# Chorded pancyclic properties in claw-free graphs

Kathryn Beck

*University of Delaware*
*Newark, DE, U.S.A.*
kbeck@udel.edu

Lisa Cenek

*Amherst College*
*Amherst, MA, U.S.A.*
lcenek21@amherst.edu

Megan Cream

*Lehigh University*
*Bethlehem, PA, U.S.A.*
macd19@lehigh.edu

Brittany Gelb

*Rutgers University*
*Piscataway, NJ, U.S.A.*
brittany.gelb@rutgers.edu

**Abstract**

A graph $G$ is (doubly) chorded pancyclic if $G$ contains a (doubly) chorded cycle of every possible length $m$ for $4 \leq m \leq |V(G)|$. In 2018, Cream, Gould, and Larsen completely characterized the pairs of forbidden subgraphs that guarantee chorded pancyclicity in 2-connected graphs. In this paper, we show that the same pairs also imply doubly chorded pancyclicity. We further characterize conditions for the stronger property of doubly chorded $(k, m)$-pancyclicity where, for $k \leq m \leq |V(G)|$, every set of $k$ vertices in $G$ is contained in a doubly chorded $i$-cycle for all $m \leq i \leq |V(G)|$. In particular, we examine forbidden pairs and degree sum conditions that guarantee this recently defined cycle property.

ISSN: 2202-3518 

## 1 Introduction

Various cycle properties of claw-free graphs have been well-studied throughout history (see [10]). One important cycle property is pancyclicity. A graph is *pancyclic* if it contains at least one cycle of every possible length from three to the order of the graph. Many variations of pancyclicity have been of recent interest in the field. In this paper, we define and consider extensions of two variations of pancyclicity: chorded pancyclity, which was recently defined by Cream, Gould, and Larsen [9]; and $(k, m)$-pancyclicity, a notion introduced by Faudree, Gould, Jacobson, and Lesniak in 2004 [11]. A *chorded cycle* is a cycle containing at least one edge between vertices that are non-adjacent on the cycle. We call such an edge a *chord*. A graph $G$ of order $n$ is *chorded pancyclic* if it contains a chorded cycle of every possible length $i$, for $4 \leq i \leq n$, and $G$ is $(k, m)$-pancyclic if, for $k \leq m \leq n$, every set of $k$ vertices in $G$ is contained in a cycle of every possible length $i$, for $m \leq i \leq n$.

Our property extensions add chords to the aforementioned cycle properties. In particular, we define the properties of doubly chorded pancyclicity and (doubly) chorded $(k, m)$-pancyclicity as follows. A graph $G$ of order $n$ is *doubly chorded pancyclic* if it contains a doubly chorded cycle of every length $i$ for $4 \leq i \leq n$. Further, $G$ is *(doubly) chorded $(k, m)$-pancyclic* if, for $k \leq m \leq n$, every set of $k$ vertices in $G$ is contained in a (doubly) chorded cycle of every length $i$ for $m \leq i \leq n$. Our results explore forbidden subgraphs and degree conditions that guarantee these new properties in graphs.

In 1971, Bondy [1] published his famous metaconjecture that states, "Almost any non-trivial condition that implies a graph is Hamiltonian, also implies that the graph is pancyclic." Nearly 50 years later, this metaconjecture was extended when results of Cream, Gould, and Hirohata suggested that conditions implying Hamilitonicity also imply the stronger property of chorded pancyclicity [8]. The work of Chen, Gould, Gu, and Saito [2] further supports extending Bondy's metaconjecture to chorded pancyclicity. Our results suggest that this metaconjecture may be further extended to the newly defined variations of pancyclicity.

In this article, we consider only finite simple graphs. Further, we let $G$ be a graph of order $n$ with vertex set $V(G)$ and edge set $E(G)$. $P_t$ is a path containing $t$ vertices, and we call a cycle with $m$ vertices an $m$-cycle, denoted $C_m$. Let $C$ be a cycle with a given orientation and $x \in V(C)$. Then $x^+$ denotes the first successor of $x$ on $C$ and $x^-$ denotes the first predecessor of $x$ on $C$. If $x, y \in V(C)$, then $C[x, y]$ denotes the path of $C$ from $x$ to $y$ (including $x$ and $y$) in the given direction on the cycle $C$. We denote the neighborhood of a vertex $v$ in $V(G)$ by $N_G(v)$, that is $N_G(v) = \{x \in V(G) | xv \in E(G)\}$. The degree of $v$ in $G$ is $|N_G(v)|$ and is denoted $\deg_G(v)$. Let $W$ be a subset of the vertices in $V(G)$. Then $N_G(W)$ is the set of all neighbors of the vertices in $W$ in $V(G) \setminus W$. Further, $N_W(v) = N_G(v) \cap W$ and $\deg_W(v) = |N_W(v)|$. To denote the subgraph induced by the vertices of $W$ we use $\langle W \rangle$. Let $H$ be a subgraph of $G$. For $v \in V(G) - V(H)$ we denote $N_H(v) = N_G(v) \cap V(H)$ and $\deg_H(v) = |N_H(v)|$.

Many of our results focus on forbidden subgraphs. We say $G$ is $H$-free if $G$ does

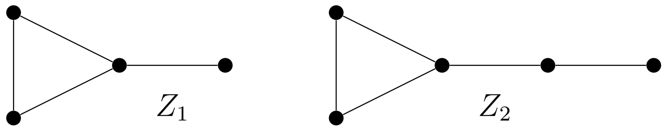


Figure 1: The graphs $Z_1$ and $Z_2$

not contain $H$ as an induced subgraph, and we call $H$ a forbidden subgraph of $G$. In particular, we focus on the claw ($K_{1,3}$), certain paths, and the graphs $Z_1$ and $Z_2$ (shown in Figure 1) as forbidden subgraphs.

We also consider the following degree sum condition of a graph $G$:

$$\sigma_2(G) = \min\{\deg_G(u) + \deg_G(v) \mid u, v \in V(G), uv \notin E(G)\}.$$

By convention we define $\sigma_2(G) = \infty$ when $G$ is the complete graph, $K_n$. For terms not defined here, see [15] for more information.

## 2 Doubly Chorded Pancyclicity

In this section, we examine extensions of chorded pancyclicity to doubly chorded pancyclicity. We begin with two results on chorded pancyclicity by Cream, Gould, and Larsen [9].

**Theorem 2.1.** *(Cream, Gould, and Larsen [9]). Let $G$ be a 2-connected, $\{K_{1,3}, R\}$-free graph of order $n$. Then $G$ is chorded pancyclic if any of the following conditions are satisfied:*
*(i) $R = P_4$ and $n \geq 5$,*
*(ii) $R = P_5$ and $n \geq 8$, or*
*(iii) $R = P_6$ and $n \geq 13$.*

These results are sharp with respect to $n$.

**Theorem 2.2.** *(Cream, Gould, and Larsen [9]). Let $G$ be a 2-connected, $\{K_{1,3}, R\}$-free graph of order $n \geq 10$. Then if $R = Z_1$ or $R = Z_2$, then $G = C_n$ or $G$ is chorded pancyclic.*

Theorem 2.3, 2.4, and 2.5 are extensions of Theorem 2.1, and Theorem 2.6 and 2.7 are extensions of Theorem 2.2. In order to prove these results, we introduce the following lemmas that establish that (chorded) cycles of sufficient length must be doubly chorded with respect to each forbidden pair.

**Lemma 2.1.** *Let $G$ be a 2-connected, $\{K_{1,3}, R\}$-free graph. Then any $m$-cycle must be doubly chorded if any of the following conditions are satisfied:*
*(i) $R = P_4$ and $m \geq 5$,*
*(ii) $R = P_5$ and $m \geq 6$, or*
*(iii) $R = P_6$ and $m \geq 7$.*

*Proof.* To prove part (i), let $C$ be an $m$-cycle in $G$ for any $m \geq 5$. Since $G$ is $P_4$-free, choosing any $P_4$ contained in $C$ implies the existence of one of three possible edges, any of which form a chord on $C$. Then, we can choose a different $P_4$ on $C$ containing exactly one endpoint of the chord. This $P_4$ again must not be induced, which implies the existence of an edge that will add a second chord to $C$. So $C$ is doubly chorded. The proof of parts (ii) and (iii) follow similarly. □

**Lemma 2.2.** *Let $G$ be a 2-connected, $\{K_{1,3}, R\}$-free graph. Then any chorded $m$-cycle $C$ must also be doubly chorded if any of the following conditions are satisfied:*
*(i) $R = Z_1$ and $m \geq 5$, or*
*(ii) $R = Z_2$ and $m \geq 6$.*

*Proof.* To prove part (i), consider a chorded $m$-cycle $C$ in $G$ for $m \geq 5$. If $C$ has exactly one chord, then the chord in $C$ will induce either a claw or a $Z_1$. This implies the existence of a second chord. The proof of part (ii) follows similarly. □

We will use the above lemmas in this section to prove the existence of doubly chorded cycles of sufficiently large lengths; we will see that the more difficult part of the proofs in this paper is showing the existence of relatively small doubly chorded cycles. We begin by addressing the sufficiently large doubly chorded cycles. A graph $G$ of order $n$ is *doubly chorded $k$-pancyclic* if there exists a doubly chorded cycle of length $i$ for every $k \leq i \leq n$. Thus doubly chorded pancyclicity is doubly chorded 4-pancyclicity.

**Lemma 2.3.** *Let $G$ be a 2-connected, $\{K_{1,3}, R\}$-free graph of order $n$. Then $G$ is doubly chorded $k$-pancyclic if any of the following conditions are satisfied:*
*(i) $R = P_4$, $k = 5$, and $n \geq 5$,*
*(ii) $R = P_5$, $k = 6$, and $n \geq 8$,*
*(iii) $R = P_6$, $k = 7$, and $n \geq 13$,*
*(iv) $R = Z_1$, $k = 5$, and $n \geq 10$ (and $G \neq C_n$), or*
*(v) $R = Z_2$, $k = 6$, and $n \geq 10$ (and $G \neq C_n$).*

*Proof.* Let $G$ be a 2-connected, $\{K_{1,3}, R\}$-free graph of order $n$. By Theorem 2.1 or Theorem 2.2, whichever appropriate, $G$ is chorded pancyclic. So consider a chorded $m$-cycle $C$ in $G$ for $m \geq k$. Applying Lemma 2.1 or Lemma 2.2, whichever appropriate, $C$ must also be doubly chorded. Therefore, $G$ is doubly chorded $k$-pancyclic. □

We now describe Algorithm 1, which we use in the proofs of Theorem 2.3 through Theorem 2.7. In these proofs, we use the conditions placed on the graph to establish the existence of singly chorded cycles of small sizes in claw- and $R$-free graphs where $R$ is any one of $P_4, P_5, P_6, Z_1$, and $Z_2$. We then input these singly chorded cycles into Algorithm 1, which uses the forbidden subgraphs to show the existence of small doubly chorded cycles. See [9] for examples of proofs that employ the techniques used in Algorithm 1, the appendix for an example of the steps of Algorithm 1, and below for pseudocode.

The initial arguments of Algorithm 1 are the representation of an unchorded cycle $C$ and one chord not yet on the cycle. In the first step of the first call of the algorithm, we add the chord to the cycle to construct a chorded cycle.

Three scenarios are checked. First, if the current graph is claw-free, $R$-free, and does not contain a doubly chorded $m$-cycle, false is returned and the iteration ends. If the current graph contains a doubly chorded $m$-cycle, the most recently added edge is removed, true is returned, and the iteration ends.

If none of the above has occurred, the current graph is checked for an induced claw. If an induced claw exists, the edges that would eliminate the induced claw are stored in a list called *edges*. If no induced claw exists in the current graph, the graph is checked for an induced $R$, and the edges that would eliminate the induced $R$ are stored in *edges*. Then the iteration ends.

After each iteration, the algorithm is called again with the arguments being the current graph and the next edge in *edges*. If all of the return values are true, then any version of the initial graph with added edges so that it is claw- and $R$-free will also contain a doubly chorded $m$-cycle. If at least one of the return values is false, then a counterexample is found.

**Algorithm 1** Algorithm for forbidden subgraph subcases

```
add(edge)
if C does not contain doubly chorded m-cycle nor induced claw nor induced R
then
  return false
else if doubly chorded m-cycle exists in C then
  remove(edge)
  return true
else
  edges = []
  if induced claw exists in C then
    edges = findInducedClaw()
  else
    edges = findInducedR()
  end if
  for i in length(edges) do
    algorithm(edges[i])
  end for
  remove(edge)
end if
```

**Theorem 2.3.** *Let $G$ be a 2-connected, $\{K_{1,3}, P_4\}$-free graph of order $n \geq 7$. Then $G$ is doubly chorded pancyclic. This result is sharp.*

*Proof.* Let $G$ be a 2-connected, $\{K_{1,3}, P_4\}$-free graph of order $n \geq 7$. By Lemma 2.3, $G$ is doubly chorded 5-pancyclic. Thus, it remains to show that $G$ contains a doubly

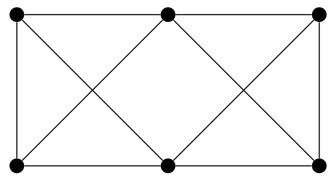

Figure 2: This graph has no doubly chorded 4-cycle, which shows that Theorem 2.3 is sharp.

chorded 4-cycle. Since $G$ is chorded pancyclic, $G$ must contain a chorded 7-cycle $C = v_1v_2 \dots v_7v_1$. There are exactly two possible cases, up to symmetry, for the location of the chord on $C$: $v_1v_3$ or $v_1v_4$. For each case, we run Algorithm 1 on $C$ using the chord as the initial edge input and setting $m = 4$ and $R = P_4$. The algorithm never returns false, verifying that $G$ contains a doubly chorded 4-cycle in every possible case. Therefore, $G$ is doubly chorded pancyclic. □

Figure 2 shows a 6-vertex, 2-connected, claw-free, and $P_4$-free graph that does not contain a doubly chorded 4-cycle. This shows that Theorem 2.3 is sharp with respect to $n$. Also note that these conditions guarantee chorded pancyclicity for $n \geq 5$, but $n \geq 7$ is required to guarantee doubly chorded pancyclicity.

We similarly extend part (ii) of Theorem 2.1 as follows:

**Theorem 2.4.** *Let $G$ be a 2-connected, $\{K_{1,3}, P_5\}$-free graph of order $n \geq 9$. Then $G$ is doubly chorded pancyclic. This result is sharp.*

*Proof.* Suppose $G$ is a 2-connected, $\{K_{1,3}, P_5\}$-free graph of order $n \geq 9$. By Lemma 2.3, $G$ is doubly chorded 6-pancyclic. We now use Algorithm 1 to prove that $G$ contains a doubly chorded 5-cycle and a doubly chorded 4-cycle. Since $G$ is chorded pancyclic, $G$ must contain a chorded 9-cycle $C = v_1v_2 \dots v_9v_1$. It follows by symmetry that there are exactly three cases for the location of the chord in $C$: $v_1v_3$, $v_1v_4$, or $v_1v_5$. For each case, we run Algorithm 1 on $C$ using the chord as the initial edge input and setting $m = 4$ and $R = P_5$. The algorithm never returns false, verifying that $G$ must contain a doubly chorded 4-cycle. We similarly set $m = 5$ and run the algorithm for each case and find that $G$ always contains a doubly chorded 5-cycle. Therefore, $G$ is doubly chorded pancyclic. □

Figure 3 shows an 8-vertex, 2-connected, claw-free, and $P_5$-free graph which does not contain a doubly chorded 4-cycle. This shows that Theorem 2.4 is sharp.

**Theorem 2.5.** *Let $G$ be a 2-connected, $\{K_{1,3}, P_6\}$-free graph of order $n \geq 13$. Then $G$ is doubly chorded pancyclic. This result is sharp.*

*Proof.* Suppose $G$ is a 2-connected, $\{K_{1,3}, P_6\}$-free graph of order $n \geq 13$. By Lemma 2.3, $G$ is doubly chorded 7-pancyclic, and thus it remains to show that $G$ contains a doubly chorded $m$-cycle for each $m \in \{4, 5, 6\}$. We use Algorithm 1 to verify the existence of these cycles. Since $G$ is chorded pancyclic, $G$ must contain a chorded 13-cycle $C = v_1v_2 \dots v_{13}v_1$. It follows by symmetry that there are exactly

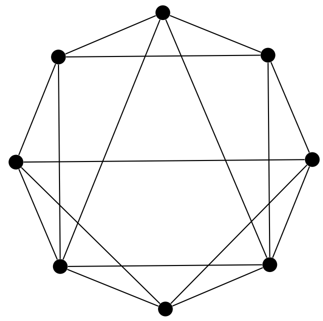

Figure 3: This graph is $\{K_{1,3}, P_5\}$-free but has no doubly chorded 4-cycle, which shows that Theorem 2.4 is sharp with respect to $n$.

five cases for the location of the chord in $C$, which can be any one of the edges $\{v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_1v_7\}$. For each $m \in \{4, 5, 6\}$, and for each possible chord location, we run Algorithm 1 on $C$ using the chord as the initial edge input and setting $R = P_6$. The algorithm never returns false, verifying that $G$ must contain a doubly chorded 4-, 5-, and 6-cycle. So $G$ is doubly chorded pancyclic. □

The sharpness of Theorem 2.5 with respect to $n$ follows directly from the sharpness of part (iii) of Theorem 2.1.

**Theorem 2.6.** *Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 7$. Then $G$ is doubly chorded pancyclic. This result is sharp.*

*Proof.* We first prove the result for $n \geq 10$ and then lower the bound to $n \geq 7$. Suppose $G$ is a 2-connected, $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 10$. We have doubly chorded 5-pancyclicity by Lemma 2.3, so it remains to show that $G$ contains a doubly chorded 4-cycle. Since $G$ is chorded 5-pancyclic, $G$ must contain a chorded 10-cycle $C = v_1v_2 \ldots v_{10}v_1$. By symmetry, there are exactly four cases for the location of the chord in $C$, which can be any one of the edges $\{v_1v_3, v_1v_4, v_1v_5, v_1v_6\}$. Setting $m = 4$ and $R = Z_1$, we run Algorithm 1 on each of the four cases using the chord as the initial edge input. The algorithm never returns false and thus verifies that $G$ must contain a doubly chorded 4-cycle. So $G$ is doubly chorded pancyclic.

Suppose $n \geq 7$. Since $G$ is 2-connected and $\{K_{1,3}, Z_1\}$-free, $G$ is Hamiltonian by a theorem of Goodman and Hedetniemi [13]. Since $G \neq C_n$, $G$ must contain a chorded Hamiltonian cycle. If $n = 7$, then without loss of generality the chord of the Hamiltonian cycle $C_7$ is one of the edges in $\{v_1v_3, v_1v_4\}$. If $n = 8$ or $n = 9$, then without loss of generality the chord of the Hamiltonian cycle $C_8$ or $C_9$ is one of the edges in $\{v_1v_3, v_1v_4, v_1v_5\}$. For $n \in \{7, 8, 9\}$, for $m \in \{4, \ldots, n\}$, and for each possible chord location, we run Algorithm 1 on $C_n$ using the chord as the initial edge input and setting $R = Z_1$. The algorithm never returns false, and thus verifies that $G$ is doubly chorded pancyclic for $n \geq 7$. □

The appendix details an example of the algorithm steps as it is used in the proof

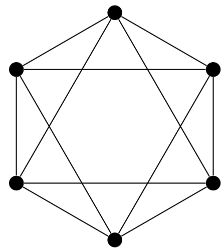

Figure 4: This graph has no doubly chorded 4-cycle, which shows that Theorem 2.6 is sharp.

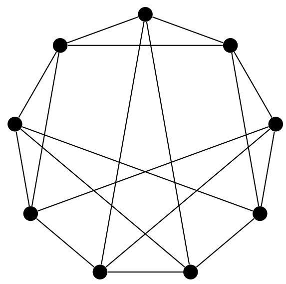

Figure 5: This graph is $\{K_{1,3}, Z_2\}$-free but has no doubly chorded 4-cycle (or doubly chorded 5-cycle), which shows that Theorem 2.7 is sharp with respect to $n$.

above. Figure 4 shows a graph of order six that is $\{Z_1, K_{1,3}\}$-free and does not contain a doubly chorded 4-cycle, which demonstrates that Theorem 2.6 is sharp.

**Theorem 2.7.** *Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n \geq 10$. Then $G$ is doubly chorded pancyclic. This result is sharp.*

*Proof.* Suppose $G$ is a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n \geq 10$. By Lemma 2.3, $G$ is doubly chorded 6-pancyclic. To show that $G$ contains a doubly chorded 4-cycle and a doubly chorded 5-cycle, we use Algorithm 1. Note that $G$ must contain a chorded 10-cycle $C = v_1v_2 \dots v_{10}v_1$, and by symmetry there are exactly four cases for the location of the chord, which can be any one of the edges $\{v_1v_3, v_1v_4, v_1v_5, v_1v_6\}$. Setting $R = Z_2$, we run the algorithm on each of the four cases for each $m \in \{4, 5\}$. All cases return true, verifying the existence of a doubly chorded 4-cycle and a doubly chorded 5-cycle in $G$. □

Figure 5 shows a $\{K_{1,3}, Z_2\}$-free graph on nine vertices with no doubly chorded 4-cycle. So Theorem 2.7 is sharp.

## 3 (Doubly) Chorded $(k, m)$-pancyclicity

Recall that a graph $G$ is doubly chorded $(k, m)$-pancyclic if, for $k \leq m \leq |V(G)|$, every set of $k$ vertices in $G$ is contained in a doubly chorded $i$-cycle for all $m \leq i \leq$

$|V(G)|$. We say that a result is sharp with respect to $m$ if the given conditions do not guarantee doubly chorded $(k, m-1)$-pancyclicity. We say that a result is sharp with respect to $k$ if the bound on $k$ cannot be lowered.

In this section, we extend results on $(k, m)$-pancyclicity to chorded or doubly chorded $(k, m)$-pancyclicity under forbidden subgraph conditions. One critical property for proving these results is cycle extendability. A non-Hamiltonian cycle $C$ in a graph $G$ is *extendable* if there exists a cycle $C'$ in $G$ such that $V(C') = V(C) \cup \{v\}$ for some $v \in V(G) - V(C)$. Suppose $x, y \in V(C)$, then we say that $C$ is *extended* on $x$ and $y$ to $C'$ if $vx, vy \in E(C')$. A graph $G$ is *cycle-extendable* if every non-Hamiltonian cycle is extendable [14].

**Theorem 3.1.** *(Faudree and Gould [14]). Let $R$ and $S$ be connected graphs ($R, S \neq P_3$) and let $G$ be a 2-connected graph of order $n \geq 10$. Then $G$ is $\{R, S\}$-free implies $G$ is cycle extendable if and only if $R = K_{1,3}$ and $S$ is one of the graphs $C_3, P_4, Z_1$, or $Z_2$.*

The next three lemmas relate cycle extendability to chords. We will use these lemmas in the proofs of the doubly chorded $(k, m)$-pancyclicity results.

**Lemma 3.1.** *For any extendable cycle $C$ with $k$ vertices and $j$ chords, where $k \geq 3$ and $j \geq 0$, any extended cycle $C'$ with $k+1$ vertices has at least $j+1$ chords.*

*Proof.* Suppose $G$ contains an extendable cycle $C$ with $k$ vertices and $j$ chords. Let $H$ be the subgraph of $G$ where $V(H) = V(C)$ and $E(H) = E(C) \cup J$, where $J$ is the set of $j$ chords between the vertices of $C$. Then $H$ has exactly $k + j$ edges.

Consider the extended cycle $C'$ where $V(C') = V(C) \cup \{v\}$ for some $v \in V(G) - V(C)$. Let $x_1$ and $x_2$ denote the vertices in $V(C)$ such that $C'$ is extended on $x_1$ and $x_2$.

Let $H'$ be the subgraph of $G$ where $V(H') = V(C) \cup \{v\}$ and $E(H') = E(H) \cup \{vx_1, vx_2\}$ for some $x, y \in V(C)$. Then $H'$ has $j + k + 2$ edges. Since $C'$ has exactly $k + 1$ edges, there are $j + 1$ edges in $E(H')$ that are not on the cycle $C'$. Since $V(C') = V(H')$, any of these edges must be chords on $C'$. Thus $C'$ must contain at least $j + 1$ chords. □

Lemma 3.1 allows us to reduce the problem of finding a doubly chorded $i$-cycle for every $i \geq m$ to simply finding a doubly chorded $m$-cycle:

**Lemma 3.2.** *Let $m \geq 4$. If $G$ is cycle extendable and every set of $k$ vertices is contained in a doubly chorded $m$-cycle, then $G$ is doubly chorded $(k, m)$-pancyclic.*

*Proof.* Let $m \geq 4$. Suppose that $G$ is cycle extendable and every set of $k$ vertices is contained in a doubly chorded $m$-cycle. Let $S$ be a set of $k$ vertices and $C_m$ a doubly chorded $m$-cycle containing $S$. Extending $C_m$ yields a cycle of length $m + 1$ containing $S$, and by Lemma 3.1, this extended cycle $C_{m+1}$ must have (at least) 2 chords. Repeating this process inductively, we obtain a doubly chorded $r$-cycle containing $S$ for any $m + 2 \leq r \leq |V(G)|$ by extending a doubly chorded $(r-1)$-cycle containing $S$. Thus $G$ is doubly chorded $(k, m)$-pancyclic. □

Furthermore, it will be important to distinguish between two possible types of cycle extensions. If a cycle $C_k$ is extended on $x_i$ and $x_j$ to $C_{k+1}$, then it may be the case that the edge $x_ix_j \in E(C_k)$. The next lemma addresses the case in which $x_ix_j \notin E(C_k)$.

**Lemma 3.3.** *For any $k \geq 4$, if an extendable cycle $C_k$ cannot be extended using two adjacent vertices of $C_k$, then $C_k$ must be chorded.*

*Proof.* Suppose $C = x_1x_2 \dots x_kx_1$ is an extendable cycle of length $k$ that cannot be extended on two vertices adjacent on $C$. This implies that $C$ is extendable on two vertices $x_i$ and $x_j$ non-adjacent on $C$. Let $C'$ be an extended cycle of $C$ such that $V(C') = V(C) \cup \{v\}$ for some $v \in V(G) - V(C)$ and $vx_i, vx_j \in E(C')$. Now assume towards contradiction that $C$ is not chorded. Note that there must be exactly two edges on $C'$ incident to each vertex in $V(C')$. Consider the vertices $x_{i-1}, x_i$, and $x_{i+1}$. By assumption, $vx_i$ is on $C'$. However, $x_{i-1}x_i$ and $x_{i+1}x_i$ must also be in $E(C')$ since $C$ is chordless and by assumption $vx_{i-1}, vx_{i+1} \notin E(G)$. But then we have three edges all incident to $x_i$ that are in $E(C')$, which is a contradiction. Thus it must be that $C$ is chorded. □

### 3.1 The pair $\{K_{1,3}, P_4\}$

Now we extend results about graphs with $K_{1,3}$ and $P_4$ as forbidden subgraphs. In particular, we build off of Crane's work on $\{K_{1,3}, P_4\}$-free graphs, stated in the following theorem.

**Theorem 3.2.** *(Crane [3]). Let $G$ be a 2-connected, $\{K_{1,3}, P_4\}$-free graph of order $n \geq 10$. Then each of the following hold:*
*(i) $G$ is $(1, 4)$-pancyclic;*
*(ii) $G$ is $(k, k + 2)$-pancyclic for $k \geq 2$.*

We extend this pancyclicity result to the following theorem on chorded pancyclicity.

**Theorem 3.3.** *Let $G$ be a 2-connected, $\{K_{1,3}, P_4\}$-free graph of order $n \geq 10$. Then each of the following hold:*
*(i) $G$ is doubly chorded $(1, 5)$-pancyclic;*
*(ii) $G$ is doubly chorded $(2, 5)$-pancyclic;*
*(iii) $G$ is doubly chorded $(k, k + 2)$-pancyclic for $k \geq 3$.*

*These results are sharp.*

*Proof.* Suppose $G$ is a 2-connected, $\{K_{1,3}, P_4\}$-free graph of order $n \geq 10$. Parts (i), (ii), and (iii) each follow from Theorem 3.2 and part (i) of Lemma 2.1, which tells us that every cycle of length $m \geq 5$ must be doubly chorded under these conditions. □

To show that these results are best possible, construct a graph $G'$ from $G = K_{n-1}$ as follows. Remove an edge between two vertices $u, v \in V(G)$. Now add a vertex $w$ to

$G$ with edges $wu$ and $wv$. The resulting graph $G'$ is 2-connected and $\{K_{1,3}, P_4\}$-free, but the vertex $w$ is not contained in any doubly chorded 4-cycle. Therefore, parts (i) and (ii) are sharp with respect to $m = 5$. Similarly, part (iii) is sharp with respect to $k \geq 3$.

### 3.2 The pair $\{K_{1,3}, P_5\}$

Next we extend the following result about $\{K_{1,3}, P_5\}$-free graphs.

**Theorem 3.4.** *(Crane [4]). Let $G$ be a 2-connected $\{K_{1,3}, P_5\}$-free graph on $n \geq 5$ vertices. Then the following hold:*
*(i) $G$ is $(1, 5)$-pancyclic;*
*(ii) $G$ is $(k, 3k)$-pancyclic for all $k \geq 2$.*
*These results are best possible under the given conditions.*

We extend this pancyclicity result to the following theorem on chorded pancyclicity.

**Theorem 3.5.** *Let $G$ be a 2-connected, $\{K_{1,3}, P_5\}$-free graph of order $n \geq 5$. Then each of the following hold:*
*(i) $G$ is doubly chorded $(1, 6)$-pancyclic.*
*(ii) $G$ is doubly chorded $(k, 3k)$-pancyclic for all $k \geq 2$.*

*These results are best possible.*

*Proof.* Let $G$ be a 2-connected, $\{K_{1,3}, P_5\}$-free graph of order $n \geq 5$. By part (ii) of Lemma 2.1, under these conditions any cycle of length $m \geq 6$ is doubly chorded. Thus, parts (i) and (ii) follow from Theorem 3.4. □

The sharpness of the condition on $k$ in part (ii) follows from the sharpness of Crane's Theorem 3.4. To show that part (i) is best possible, construct a graph $G'$ from $G = K_{n-2}$ as follows. Remove an edge between two vertices $x, v \in V(G)$. Now add two vertices $a, b$ and the edges $ax, bz, ab$. Any 5-cycle that contains $\{a, b\}$ must also contain $\{x, z\}$. But for any $y \in V(G') - \{a, b, x, z\}$, the 5-cycle $abzyxa$ is not chorded. We note that $G'$ is $\{K_{1,3}, P_5\}$-free, but $a$ is not in a chorded 5-cycle.

### 3.3 The pair $\{K_{1,3}, P_6\}$

Next we build off of the following result about $\{K_{1,3}, P_6\}$-free graphs.

**Theorem 3.6.** *(Crane [5]). Let $G$ be a 2-connected $\{K_{1,3}, P_6\}$-free graph on $n \geq 7$ vertices. Then $G$ is $(k, 3k + 4)$-pancyclic for all $k \geq 1$. This result is best possible under the given conditions.*

We extend this pancyclicity result to the following theorem on chorded pancyclicity.

**Theorem 3.7.** *Let $G$ be a 2-connected, $\{K_{1,3}, P_6\}$-free graph of order $n \geq 7$. Then $G$ is doubly chorded $(k, 3k+4)$-pancyclic for all $k \geq 1$. This result is best possible.*

*Proof.* If $G$ is a 2-connected, $\{K_{1,3}, P_6\}$-free graph of order $n \geq 7$, then by part (iii) of Lemma 2.1, we have that every cycle of length $m \geq 7$ is doubly chorded. Thus Theorem 3.6 extends to doubly chorded $(k, 3k+4)$-pancyclicity for $k \geq 1$. □

Since Theorem 3.6 is best possible under the given conditions, Theorem 3.7 is also best possible.

### 3.4 The pair $\{K_{1,3}, Z_1\}$

Turning our attention to $\{K_{1,3}, Z_1\}$-free graphs, we again build off of a result due to Crane.

**Theorem 3.8.** *(Crane [3]). Let $G \neq C_n$ be a 2-connected graph of order $n \geq 5$. If $G$ is $\{K_{1,3}, Z_1\}$-free, then each of the following hold:*
*(i) $G$ is $(1,3)$-pancyclic;*
*(ii) $G$ is $(k,4)$-pancyclic for $k \in \{2,3\}$;*
*(iii) $G$ is $(k,k)$-pancyclic for each integer $k \geq 4$.*

To extend this theorem, we will use the following result due to Faudree and Gould.

**Theorem 3.9.** *(Faudree and Gould [14]). If $G$ is a connected $\{K_{1,3}, Z_1\}$-free graph with a vertex of degree at least three, then $G$ is a complete graph or a complete graph minus a matching.*

In particular, we use this result to show that Crane's Theorem can be extended to doubly chorded pancyclicity results as follows.

**Theorem 3.10.** *Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 10$. Then each of the following hold:*
*(i) $G$ is doubly chorded $(1,4)$-pancyclic;*
*(ii) $G$ is doubly chorded $(2,5)$-pancyclic;*
*(iii) $G$ is doubly chorded $(3,5)$-pancyclic.*

*These results are best possible.*

*Proof.* Suppose $G \neq C_n$ is a 2-connected, $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 10$. We first prove part (i).

Note that $G$ must contain a vertex of degree at least 3 because $G$ is 2-connected and $G \neq C_n$, and $C_n$ is the only 2-connected graph on $n$ vertices with a maximum degree of 2. So by Theorem 3.9, $G$ is isomorphic to $K_n$ minus at most a perfect matching. We claim that every vertex is contained in a doubly chorded 4-cycle. To see this, let $v_1 \in V(G)$, and consider seven other vertices $v_2, v_3, \ldots, v_8$ in $G$. Since $G = K_n$ minus at most a matching, without loss of generality, the only edges that may be missing among these eight vertices are $v_1v_2, v_3v_4, v_5v_6$, and $v_7v_8$. Then $\{v_1, v_3, v_5, v_7\}$ induce a doubly chorded 4-cycle in $G$. Since $G$ is cycle extendable by Theorem 3.1, it follows from Lemma 3.2 that $G$ is doubly chorded $(1, 4)$-pancyclic.

To prove part (ii) and (iii), let $k \in \{2, 3\}$, and let $S$ be a set of $k$ vertices. By Theorem 3.8, $S$ is in a 4-cycle, which extends to a chorded 5-cycle containing $S$ by Lemma 3.1. Then by Lemma 2.2, any chorded 5-cycle must also be doubly chorded under these conditions. So every set of $k$ vertices is contained in doubly chorded 5-cycle, which implies doubly chorded $(k, 5)$-pancyclicity by Lemma 3.2. □

To show that parts (ii) and (iii) are best possible, construct a graph $G'$ from $G = K_n$ by removing an edge between two vertices $u, v \in V(G)$. Then $G'$ is 2-connected and $\{K_{1,3}, Z_1\}$-free, but the set $\{u, v\}$ is not contained in a doubly chorded 4-cycle. Note, however, that $\{u, v\}$ are still in a chorded 4-cycle.

Although the above conditions are not sufficient to guarantee *doubly* chorded $(2, 4)$-pancyclicity, *singly* chorded $(2, 4)$- and $(3, 4)$-pancyclicity do hold:

**Theorem 3.11.** *Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 10$. Then $G$ is chorded $(2, 4)$-pancyclic and chorded $(3, 4)$-pancyclic.*

*Proof.* Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 10$. By Theorem 3.10, $G$ is doubly chorded $(3, 5)$-pancyclic, so it suffices to show that every set of three vertices in $G$ is contained in a chorded 4-cycle. To do this, begin with a set of three vertices that is contained in a doubly chorded 5-cycle $C = v_1v_2v_3v_4v_5v_1$. We now divide the proof into the two cases (up to symmetry) of the location of the two chords in $C$.

*Case* 1. Suppose $C$ contains the chords $v_1v_3$ and $v_1v_4$. Then avoiding an induced $Z_1$ on $v_1, v_2, v_4$, and $v_5$, either the edge $v_2v_4$ or $v_2v_5$ must be present. If $v_2v_5 \in E(G)$, every set of three vertices in $C$ is contained in a chorded 4-cycle. So suppose $v_2v_4 \in E(G)$, and notice that every set of three vertices in $C$ is now contained in a chorded 4-cycle except for $\{v_2, v_3, v_5\}$. So, now observe that the edge $v_2v_5$ or $v_3v_5$ must be present so that the set $\{v_2, v_3, v_4, v_5\}$ does not induce a $Z_1$. When either edge is added, the set $\{v_2, v_3, v_5\}$ is now in a chorded 4-cycle.

*Case* 2. Suppose $C$ contains the chords $v_2v_4$ and $v_3v_5$. Then the edge $v_1v_3$ or $v_1v_4$ must be present to avoid an induced $Z_1$ on $\{v_1, v_3, v_4, v_5\}$. Adding either edge, we obtain a graph that simplifies the argument to Case 1.

Therefore, $G$ is chorded $(3, 4)$-pancyclic, which also implies chorded $(2, 4)$-pancyclicity. □

Further, we can use Theorem 3.8 to show the following.

**Theorem 3.12.** *Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 10$. Then each of the following hold:*
*(i) $G$ is doubly chorded $(k, k + 1)$-pancyclic, for all $k \geq 4$.*
*(ii) $G$ is doubly chorded $(k, k)$-pancyclic, for all $k \geq 5$.*

*These results are best possible.*

*Proof.* Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_1\}$-free graph of order $n \geq 10$. To prove part (i), we first note by Theorem 3.8 that $G$ is $(k,k)$-pancyclic for all $k \geq 4$. Also, note that $G$ is cycle extendable by Theorem 3.1. For any $k \geq 4$, by Lemma 3.1, we have that every set of $k$ vertices is contained in a chorded $m$-cycle for all $m \geq k+1$. Then by Lemma 2.2, any such chorded $m$-cycle must also be doubly chorded. So $G$ is doubly chorded $(k, k+1)$-pancyclic for all $k \geq 4$.

To prove part (ii), we first note by Theorem 3.8, that $G$ is $(k,k)$-pancyclic, so any set of $k$ vertices for $k \geq 5$ in $V(G)$ must be contained in a $k$-cycle $C$. We will show that $C$ must be chorded; it then follows by Lemma 2.2 that any chorded cycle must also be doubly chorded. If $k = n$, then $C$ must contain at least one chord because $G \neq C_n$. If $k < n$, then since $G$ is cycle extendable, we can extend $C$ to a $(k+1)$-cycle.

Suppose $C = x_1 x_2 \dots x_k x_1$ is extended on $x_1$ and $x_2$ to $C'$ such that $x_1 x_2 \in E(C)$, and let $v$ be the additional vertex in the extended cycle, $C'$. Notice that $\{v, x_1^-, x_1, x_2\}$ and $\{v, x_1, x_2, x_2^+\}$ each cannot induce a $Z_1$. If $x_1^- x_2 \in E(G)$ or $x_2^+ x_1 \in E(G)$, then $C$ is chorded and we are done. So, assume $x_1^- v \in E(G)$ and $x_2^+ v \in E(G)$. Then avoiding an induced $Z_1$ on $\{x_1^-, x_1, v, x_2^+\}$, $x_1^- x_2^+$ or $x_2^+ x_1$ must be in $E(G)$. In either case, $C$ is now chorded.

Otherwise, suppose $C$ is extendable only on nonadjacent vertices of $C$. Then by Lemma 3.3, $C$ must be chorded. Thus every set of $k$ vertices is contained in a doubly chorded $k$-cycle for $k \geq 5$. □

To show that the statement in (i) is best possible, begin with a complete graph $G = K_n$ and form $G'$ by removing two vertex-disjoint edges $xy$ and $uv$. Then $G'$ is 2-connected and $\{K_{1,3}, Z_1\}$-free, but the set of vertices $\{x, y, u, v\}$ is not contained in a chorded 4-cycle. This graph also shows that part (ii) is sharp with respect to $k$.

### 3.5 The pair $\{K_{1,3}, Z_2\}$

Our results in this section extend the various types of pancyclicity in the following result on $\{K_{1,3}, Z_2\}$-free graphs to various types of doubly chorded pancyclicity.

**Theorem 3.13.** *(Crane [3]). Let $G \neq C_n$ be a 2-connected graph of order $n \geq 10$. If $G$ is $\{K_{1,3}, Z_2\}$-free, then each of the following hold:*
*(i) $G$ is $(1,4)$-pancyclic;*
*(ii) $G$ is $(k, 3k)$-pancyclic for each integer $k \geq 2$.*

We first extend part (i) to a doubly chorded result. Our proof is analogous to the proof of part (i) in [3]; instead of avoiding $C_3$ and $C_4$, we avoid $C_3$ and a chorded $C_4$.

**Theorem 3.14.** *Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n \geq 10$. Then $G$ is doubly chorded $(1,5)$-pancyclic. This result is best possible.*

*Proof.* Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n \geq 10$. Let $w \in V(G)$. By the cycle extendability of $G$ and Lemma 3.1, it is sufficient to show that $w$ is contained in a triangle or a chorded 4-cycle. Thus, we assume this is not the case.

Since $G$ is doubly chorded pancyclic by Theorem 2.7, there must exist a doubly chorded 4-cycle in $G$. Recall that $G$ is 2-connected. Let $s$ be the smallest integer such that there exists a doubly chorded 4-cycle $H$ in $G$ and a pair of vertex disjoint paths from $w$ to $H$, one of which has length $s$. Now let $t$ be the smallest integer such that there exists a doubly chorded 4-cycle $C$ in $G$, a path $P$ of length $s$ from $w$ to $C$, and a path $Q$ of length $t$ from $w$ to $C$ that is vertex disjoint from $P$ (except for $w$). We note $s \leq t$. Let $C = xyzvx$.

Without loss of generality, suppose that $P = xx_1x_2 \dots x_s$ and $Q = yy_1y_2 \dots y_t$, where $x_s = y_t = w$. Note that $P$ and $Q$ are disjoint paths from $w$ to $H$. We will show via contradiction that $s = 1$ and $t = 1$. Suppose $s \geq 2$. We will show that the vertices $\{z, v, x, x_1, x_2\}$ induce a $Z_2$. To avoid this $Z_2$, we must have one edge of $\{x_2x, x_2v, x_2z, x_1v, x_1z\}$ in $E(G)$. However, note that $x_2x$, $x_2v$, or $x_2z \notin E(G)$, for otherwise we violate the minimality of $s$ by creating a path from $w$ to $C$ of length $s - 1$ that is disjoint from $Q$.

Note that the cases $x_1v \in E(G)$ and $x_1z \in E(G)$ are the same by symmetry. Without loss of generality, assume $x_1v \in E(G)$. Then $x_1z \notin E(G)$, since otherwise $w$ is connected to the doubly chorded 4-cycle $x_1vzxx_1$ by a path (that is disjoint from $Q$) of length $s - 1$, which contradicts the minimality of $s$. But now we must have $s = 2$, since otherwise $\{v, x, x_1, x_2, x_3\}$ induces a $Z_2$ by the minimality of $s$. If $t \geq 3$, then $\{v, x, x_1, w, y_{t-1}\}$ induces a $Z_2$ (using the fact that $w$ is not contained in a triangle, and using the minimality of $t$). Thus $t = 2$.

Avoiding an induced claw on $\{v, y_1, x_1, z\}$,we must have one edge of$\{x_1y, y_1z, x_1z\}$ $\in E(G)$. Note that $x_1y_1 \notin E(G)$, since $w$ is not contained in a triangle. Now if $y_1z \notin E(G)$, then $y_1v \in E(G)$ since $\{v, z, y, y_1, w\}$ cannot induce a $Z_2$. Therefore $y_1z \in E(G)$. Since $\{v, x, x_1, w, y_1\}$ cannot induce a $Z_2$, we must have $y_1x \in E(G)$ or $y_1v \in E(G)$. If $y_1x \in E(G)$, then $y_1xzyy_1$ is a doubly chorded 4-cycle that violates the minimality of $t$. So $y_1v \in E(G)$. But now $y_1yvzy_1$ is a doubly chorded 4-cycle that violates the minimality of $t$ when we replace $P$ with the path $P' = vx_1w$ of length $s$. This contradiction shows that $x_1v \notin E(G)$. By symmetry, we have that $x_1z \notin E(G)$. So the vertices $\{z, v, x, x_1, x_2\}$ must induce a $Z_2$, and we have a contradiction. Therefore, $s = 1$.

Assume $t = 2$. Then the vertices $\{v, z, y, y_1, w\}$ induce a $Z_2$, which cannot be avoided without creating a doubly chorded 5-cycle containing $w$. Now suppose $t \geq 3$. To avoid an induced $Z_2$ on the vertices $\{v, z, y, y_1, y_2\}$, we must have one of the edges $\{y_2v, y_2z, y_2y, y_1z, y_1v\} \in E(G)$. Adding any one of the edges $\{y_2v, y_2z, y_2y\}$ violates the minimality of $t$. Adding the edge $y_1z$ yields a $Z_2$ induced by $\{z, y, y_1, y_2, y_3\}$. So $y_1z \notin E(G)$, and by symmetry, $y_1v \notin E(G)$. Thus, $t = 1$, and $w$ is contained in a doubly chorded 5-cycle. □

To see that Theorem 3.14 is sharp, construct a graph $G'$ from $G = K_{n-1}$ by removing an edge $xy$ from $G$, and then adding a vertex $w$ of degree 2 such that $wx, wy \in E(G')$. Then $G'$ is a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n \geq 10$, but $w$ is not contained in a chorded 4-cycle.

Next we extend part (ii) of Theorem 3.13 in the following way.

**Theorem 3.15.** *Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n \geq 10$. Then each of the following hold:*
*(i) $G$ is doubly chorded $(2, 7)$-pancyclic;*
*(ii) $G$ is doubly chorded $(k, 3k)$-pancyclic, for all $k \geq 3$.*

*This result is best possible.*

*Proof.* Let $G \neq C_n$ be a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n \geq 10$. To prove part (i), we first note that by Theorem 3.13, any set of two vertices in $G$ must be contained in a 6-cycle $C$. Since $G$ is cycle-extendable, we can extend $C$ to a 7-cycle $C'$, which by Lemma 3.1 must contain a chord. Then by Lemma 2.2, $C'$ must also be doubly chorded. This suffices to show that $G$ is doubly chorded $(2, 7)$-pancyclic by Lemma 3.2.

To prove part (ii), consider a set of $k$ vertices in $V(G)$ for any $k \geq 3$. This set must be contained in some $(3k)$-cycle $C$ since $G$ is $(k, 3k)$-pancyclic by Theorem 3.13. We will show that $C$ is chorded and then apply Lemmas 2.2 and 3.2 to obtain the desired result. If $3k = n$, then $C$ is chorded since $G \neq C_n$. So suppose $3k < n$. Since $G$ is cycle extendable by Theorem 3.1, $C$ can be extended to a $(3k+1)$-cycle. If $C$ is only extendable on two vertices that are nonadjacent on $C$, then by Lemma 3.3, $C$ must be chorded. Otherwise, extend $C$ to a $(3k+1)$-cycle $C'$, where $V(C') = V(C) \cup \{v\}$ for some $v \in V(G) - V(C)$, such that $v$ is adjacent to a pair of vertices $x_1, x_2$ which are adjacent on $C$.

Avoiding an induced $Z_2$ on $\{v, x_2, x_1, x_1^-, x_1^{--}\}$ and a chord on $C$, either the edge $vx_1^-$ or $vx_1^{--}$ must be present. Similarly, avoiding an induced $Z_2$ on $\{v, x_1, x_2, x_2^+, x_2^{++}\}$ and a chord on $C$, either the edge $vx_2^+$ or $vx_2^{++}$ must be present.

If $vx_1^{--}, vx_2^{++} \in E(G)$, avoiding a claw induced by $\{v, x_1^{--}, x_2^{++}, x_2\}$, there must be a chord on $C$. If $vx_1^{--}, vx_2^+ \in E(G)$, avoiding a claw induced by $\{v, x_1^{--}, x_2^+, x_1\}$, there must be a chord on $C$. The case where $vx_1^-, vx_2^{++} \in E(G)$ is symmetric to the previous case. Lastly, suppose $vx_1^-, vx_2^+ \in E(G)$. Avoiding an induced $Z_2$ on $\{x_1^{--}, x_1^-, v, x_2, x_2^+\}$ and a chord on $C$, the edge $vx_1^{--}$ must be present. Avoiding an induced claw on $\{v, x_1^{--}, x_1, x_2^+\}$, one of three edges must be present, all of which add a chord to $C$.

Therefore, $C$ is chorded, which by Lemma 2.2 is sufficient to show that $C$ is doubly chorded. So every set of $k$ vertices is contained in a doubly chorded $(3k)$-cycle. Then applying Lemma 3.2, we have that $G$ is doubly chorded $(k, 3k)$-pancyclic. □

To see that part (i) of Theorem 3.15 is sharp, let the graph $G$ be defined by $V(G) = V(C) \cup V(K_m)$ and $E(G) = E(C) \cup E(K_m)$, where $C = axbcyda$ is a 6-cycle and $K_m$ is a complete subgraph on $m \geq 4$ vertices. Also, ensure that $C$ and $K_m$ are vertex disjoint. Then, create a graph $G'$ from $G$ as follows. For each $v \in \{a, b, c, d\}$, add edges connecting $v$ to every vertex in the subgraph $K_m$. Now, $G'$ is a 2-connected, $\{K_{1,3}, Z_2\}$-free graph of order $n = 6 + m \geq 10$, but $\{x, y\}$ is a set of two vertices not contained in a chorded 6-cycle.

Crane showed in [3] that part (ii) of Theorem 3.13 is sharp with respect to $m = 3k$, which implies that part (ii) of Theorem 3.15 is sharp with respect to $m = 3k$ as well.

## 4 Degree Sum Conditions

Faudree et al. established sharp degree sum bounds that guarantee a graph to be $(k, m)$-pancyclic for certain $k$ and $m$. This work was extended by Crane, who specifically examined claw-free graphs in Theorem 4.1 [6]. In this section, we further extend the work of Crane to establish degree sum bounds that guarantee chorded and doubly chorded $(k, m)$-pancyclicity for claw-free graphs. Cream, Gould, and Hirohata established $\sigma_2$ conditions for (doubly) chorded vertex pancyclicity (e.g. Theorem 4.2). We extend Theorem 4.2 through the addition of a claw-free condition.

**Theorem 4.1.** *(Crane, [6]) Let $k$ be an integer, and let $G$ be a claw-free graph of order $n \geq 3$.*
*(i) If $\sigma_2(G) \geq n$ then $G$ is $(k, n)$-pancyclic, for $k \geq 1$.*
*(ii) If $\sigma_2(G) \geq n + 1$ then $G$ is $(1, 3)$-pancyclic.*
*(iii) If $\sigma_2(G) \geq 2n - 3$ then $G$ is $(2, 3)$-pancyclic.*
*(iv) If $\sigma_2(G) \geq 2n - k$ then $G$ is $(k, k)$-pancyclic for $3 \leq k \leq n$.*
*(v) If $\sigma_2(G) \geq \frac{4n-2k-5}{3}$ then $G$ is $(k, k + 2)$-pancyclic for $3 \leq k \leq \frac{n-5}{2}$.*
*(vi) If $\sigma_2(G) \geq n$ then $G$ is $(k, k + 2)$-pancyclic for $k > max\{\frac{n-5}{2}, 2\}$ or $k = 2$.*
*(vii) If $\sigma_2(G) \geq n$ then $G$ is $(k, k + 3)$-pancyclic for $k \geq 3$.*

*All of the $\sigma_2(G)$ bounds are sharp.*

Our first result is an extension of part (ii) of Theorem 4.1 and the following theorem of Cream, Gould, and Hirohata [7].

**Theorem 4.2.** *(Cream, Gould, and Hirohata [7]) Let $G$ be a graph of order $n \geq 4$. If $\sigma_2(G) \geq n + 1$, then $G$ is chorded vertex $(1, 5)$-pancyclic.*

**Theorem 4.3.** *Let $G$ be a claw-free graph of order $n \geq 3$. If $\sigma_2(G) \geq n + 1$, then $G$ is chorded $(1, 4)$-pancyclic.*

*Proof.* Suppose $G$ is a claw-free graph of order $n \geq 3$ with $\sigma_2(G) \geq n + 1$. By Theorem 4.2, every vertex is contained in a chorded $m$-cycle for all $m \geq 5$. Thus it remains to show that every vertex is contained in a chorded 4-cycle.

Consider a vertex $v_1 \in V(G)$. The case when $n = 3$ is trivial, so let $n \geq 4$. By part (ii) of Theorem 4.1, $v_1$ must be contained in a 4-cycle, which we label $C = v_1v_2v_3v_4v_1$ for some $v_2, v_3, v_4 \in V(G)$. Suppose $C$ is chordless, otherwise we are done. Now consider the case when $n = 4$. Note that $v_1v_3 \notin E(G)$, so $deg_G(v_1)+deg_G(v_3) \geq n+1$ by our assumption. Then $v_1$ and $v_3$ must have at least three common neighbors. But $v_1$ and $v_3$ share the only other possible neighbors of $v_2$ and $v_4$. This is a contradiction.

Consider the case when $n \geq 5$. It follows by the same logic from the previous case that $v_1$ and $v_3$ must have at least one common neighbor not on the 4-cycle. Suppose the vertex $w \in V(G) - V(C)$ is a common neighbor of $v_1$ and $v_3$. Now the set $\{v_1, v_2, v_4, w\}$ cannot induce a claw centered at $v_1$. Then $v_2w \in E(G)$ or $v_4w \in E(G)$. In each case, it is easy to verify that $v_1$ is in a chorded 4-cycle. □

The bound on $\sigma_2(G)$ in Theorem 4.3 is sharp due to the sharpness of part (ii) of Theorem 4.1, since chorded $(1, 4)$-pancyclicity implies $(1, 3)$-pancyclicity.

**Theorem 4.4.** *Let $G$ be a claw-free graph of order $n \geq 5$. If $\sigma_2(G) \geq n+1$, then $G$ is doubly chorded $(1, 5)$-pancyclic. This result is best possible.*

*Proof.* Suppose $G$ is a claw-free graph of order $n \geq 5$ with $\sigma_2(G) \geq n + 1$. We have by Theorem 4.3 that $G$ is chorded $(1, 4)$-pancyclic. We assume that $n \geq 9$, since if $5 \leq n \leq 8$, then it can easily be verified that the $\sigma_2(G)$ condition guarantees that every vertex is contained in a doubly chorded $m$-cycle for all $5 \leq m \leq n$.

Claim: Every vertex is in a doubly chorded 5-cycle.

Let $x \in V(G)$. Assume there exists a vertex $y \in V(G)$ satisfying $xy \notin E(G)$. Otherwise, it is easy to see that since $G$ is Hamiltonian, $x$ must be contained in a doubly chorded 5-cycle. Since $\sigma_2(G) \geq n + 1$, and $x$ and $y$ are nonadjacent, there exist at least three distinct common neighbors $a, b, c$ of $x$ and $y$. Avoiding an induced claw on $\{x, a, b, c\}$, we can assume without loss of generality that $ab \in E(G)$. Then we have the doubly chorded 5-cycle $abycxa$ (with chords $bx$ and $ay$) containing $x$.

Claim: Every vertex is in a doubly chorded $m$-cycle for $m \geq 6$.

Let $x \in V(G)$. By Theorem 4.3, $x$ is contained in a chorded $m$-cycle $C$, for $m \geq 6$. We write $C = v_1v_2 \ldots v_mv_1$ where $x = v_j$ for some $1 \leq j \leq m$. Assume that $C$ has exactly one chord; otherwise the claim holds. We have two cases depending on the position of the chord in $C$.

*Case* 1. The chord on $C$ does not form a triangle with three vertices of $C$.

Without loss of generality, suppose the chord is the edge $v_1v_i$, where $4 \leq i \leq m-2$. Avoiding a claw induced by $\{v_1, v_2, v_m, v_i\}$, one edge of $\{v_2v_m, v_mv_i, v_2v_i\}$ must be in $E(G)$. Any of these edges is a second chord of $C$. Thus $C$ is a doubly chorded $m$-cycle containing $x$.

*Case* 2. The chord on $C$ forms a triangle with three vertices of $C$.

Without loss of generality, assume the chord is the edge $v_2v_m$. We have two subcases:

*Subcase* 2.1 ($x \neq v_1$). Since $v_3v_5 \notin E(G)$ by assumption, the $\sigma_2(G)$ condition ensures that $v_3$ and $v_5$ have at least three common neighbors, at least one of which we can assume is not on $C$, for otherwise $C$ is doubly chorded and we are done. Thus, let $w_1 \notin V(C)$ be a common neighbor of $v_3$ and $v_5$. Note that $\{v_3, v_2, v_4, w_1\}$ must not induce a claw centered at $v_3$. To avoid this claw, note that if $v_2v_4 \in E(G)$, then $C$ is doubly chorded and we are done. If $v_2w_1 \in E(G)$, then there is a doubly chorded $m$-cycle, $v_2w_1v_3v_4v_5 \ldots v_mv_2$ with chords $v_2v_3$ and $v_5w_1$, containing $x$ since $x \neq v_1$. If $v_4w_1 \in E(G)$, there is a doubly chorded $m$-cycle, $v_2v_3w_1v_4v_5 \ldots v_mv_2$ with chords $v_3v_4$ and $v_5w_1$, also containing $x$.

*Subcase* 2.2 ($x = v_1$). Suppose $m = 6$. Since $v_4v_6 \notin E(G)$, and $\sigma_2(G) \geq n + 1$, the vertices $v_4$ and $v_6$ must share some common neighbor $w_1 \in V(G) - V(C)$. Now, notice that the vertices $\{v_6, v_2, v_5, w_1\}$ must not induce a claw centered at $v_6$. The edge $v_5v_2$ would add a second chord to $C$. The edge $w_1v_2$ would create the doubly chorded 6-cycle $v_1v_2v_3v_4w_1v_6v_1$ with chords $v_2v_6$ and $w_1v_2$. So assume that $v_5w_1 \in E(G)$. Now, again using the $\sigma_2(G)$ condition, the nonadjacent vertices $v_1$ and $v_5$ must share a common neighbor $w_2$ where $w_2 \notin V(C) \cup \{w_1\}$. Now we have that $x = v_1$ is contained in the doubly chorded 6-cycle $v_1v_6w_1v_4v_5w_2v_1$ with the chords $v_5v_6$ and $v_5w_1$.

Suppose $m \geq 7$. Since $v_2v_4 \notin E(G)$, by the $\sigma_2(G)$ condition, $v_2$ and $v_4$ must share at least one common neighbor $w_1 \notin V(C)$. Similarly, since $v_3v_7 \notin E(G)$, $v_3$ and $v_7$ must share at least one common neighbor $w_2$ where $w_2 \notin V(C) \cup \{w_1\}$. Now, $x = v_1$ is contained in the doubly chorded $m$-cycle $v_1v_2w_1v_4v_3w_2v_7 \dots v_mv_1$ with the chords $v_2v_3$ and $v_2v_m$.

This completes the proof of the claim and hence of the theorem. □

To show that Theorem 4.4 is best possible with respect to $m$, we construct a graph $G'$ from $G = K_{n-1}$ as follows. Add a vertex $w$ to $G$ such that $w$ is adjacent to three other vertices. Let $x$, $y$, and $z$ be these three neighbors of $w$. Then remove the edge $xy$. The resulting graph $G'$ is claw-free and satisfies $\sigma_2 \geq n + 1$, but the vertex $w$ is not contained in a doubly chorded 4-cycle.

The following theorem is a restatement of part (iii) of Theorem 4.1, given by observing that $(2, 3)$-pancyclicity and doubly chorded $(4, 4)$-pancyclicity are both equivalent to a graph being complete. The sharpness of Theorem 4.5 with respect to the $\sigma_2(G)$ bound follows immediately from the sharpness of Theorem 4.1.

**Theorem 4.5.** *Let $G$ be a claw-free graph of order $n \geq 3$. If $\sigma_2(G) \geq 2n - 3$, then $G$ is doubly chorded $(4, 4)$-pancyclic. This result is best possible.*

We now extend part (iv) of Theorem 4.1.

**Theorem 4.6.** *Let $G$ be a claw-free graph of order $n \geq 5$. If $\sigma_2(G) \geq 2n - k$, then $G$ is doubly chorded $(k, k)$-pancyclic for $5 \leq k \leq n$.*

*This result is best possible.*

*Proof.* Suppose $G$ is a claw-free graph of order $n \geq 5$ with $\sigma_2(G) \geq 2n-k$. Consider a set $S$ of $k$ vertices in $V(G)$ for any $5 \leq k \leq n$. By part (iv) of Theorem 4.1, $G$ is $(k, k)$-pancyclic for all $3 \leq k \leq n$, so $S$ is contained in an $m$-cycle $C_m$ for all $m \geq k$. We will show that $C_m$ must be doubly chorded. So, assume towards a contradiction that $C_m$ has at most one chord. Then we can choose two vertices $x, y \in V(C_m)$ satisfying $xy \notin E(G)$ such that neither $x$ nor $y$ is an endpoint of the chord on $C_m$ (if such a chord exists). Then, $deg(x) \leq n - m + 2$, and likewise $deg(y) \leq n - m + 2$. So, since $m \geq k$ and $k > 4$, we have that $deg(x) + deg(y) \leq 2n + 4 - 2m \leq 2n + 4 - 2k < 2n - k \leq \sigma_2(G)$. But this is a contradiction to the $\sigma_2(G)$ condition because $x$ and $y$ are nonadjacent. Therefore, it must be that $C_m$ has two or more chords, and hence $G$ is doubly chorded $(k, k)$-pancyclic for $5 \leq k \leq n$. □

The same sharpness example associated with part (ii) and (iii) of Theorem 3.10 can be applied to to show that Theorem 4.6 is sharp with respect to $k$. The sharpness of Theorem 4.6 with respect to $\sigma_2(G)$ follows from the sharpness of Theorem 4.1 part (iv).

Next we introduce a series of lemmas that will be instrumental in extending the results of Theorem 4.1. We will use Lemma 4.1 to prove Lemma 4.2, a result that does not rely on any forbidden subgraph or degree sum conditions, but rather explores when $(k, m)$-pancyclicity can directly imply doubly chorded $(k, m + 1)$-pancyclic results.

**Lemma 4.1.** *For any $k, l \geq 0$, if $G$ is $(k, k + l)$-pancyclic, then any set $S$ of $k$ vertices satisfies $|E(\langle S\rangle)| \geq k - l$.*

*Proof.* Let $S$ be a set of $k$ vertices in $V(G)$. We know that $S$ is contained in a cycle $C$ of length $k+l$ since $G$ is $(k, k+l)$-pancyclic. Since $|V(C) - S|$ is $l$, the maximum number of edges on $C$ incident to a vertex in $V(C) - S$ is $2l$. Furthermore, any edge in $C$ that is not incident to any vertex in $V(C) - S$ must lie between two vertices of $S$. So, since $C$ has $k + l$ edges, it follows that there are at least $(k + l) - 2l = k - l$ edges in $E(\langle S\rangle)$. □

**Lemma 4.2.** *For any $k \geq 3$ and $0 \leq l < k$, if $G$ is $(k, k + l)$-pancyclic, then $G$ is doubly chorded $(k, k + l + 1)$-pancyclic.*

*Proof.* Let $k \geq 3$ and $0 \leq l < k$, and assume $G$ is $(k, k + l)$-pancyclic. Since we would like to show doubly chorded $(k, k+l+1)$-pancyclicity, note that by definition, $k + l < n$. Consider a set $S$ of $k$ vertices, and let $r \in \{k + l + 1, k + l + 2, \ldots, n\}$. By assumption, there exists a cycle $C$ of length $r$ containing $S$. We will show that $C$ must be doubly chorded, so assume towards contradiction that $C$ has at most one chord. Label the vertices of $C = v_1v_2 \ldots v_{k+l+1} \ldots v_rv_1$ such that $v_r$ is an endpoint of the chord (if a chord exists), and note that $r \geq k + l + 1 \geq 4$.

In order to apply Lemma 4.1, consider the set of $k$ vertices $T = V(C[v_1, v_{k+l+1}])-R$, where $R$ is a set of $l + 1$ vertices that minimizes the number of edges in $\langle T\rangle$.

First we prove the case of $r > k + l + 1$. Notice that $|E(C[v_1, v_{k+l+1}])| = k + l$. By construction of the set $R$, we have $|E(\langle T\rangle)| = |E(\langle V(C[v_1, v_{k+l+1}]) - R\rangle)| \leq (k + l) - (2l + 1) = k - l - 1$. Note that we subtract $2l + 1$ because there are $l + 1$ vertices in $R$, each with two incident edges that lie in $E(C[v_1, v_{k+l+1}]) - E(\langle T\rangle)$, with the possible exception of one vertex ($v_1$ or $v_{k+l+1}$) which may only have one such incident edge. But the above inequality contradicts Lemma 4.1, which says that the set $T$ must induce at least $k - l$ edges. Therefore, since we assumed $C$ has at most one chord, we now have that $C$ is doubly chorded.

Next, we prove the case of $r = k+l+1$. Making a similar argument as the previous case, we have $|E(\langle T\rangle)| = |E(\langle V(C) - R\rangle)| \leq (k + l + 1) - (2l + 2) = k - l - 1$. But Lemma 4.1 says that $T$ must induce at least $k - l$ edges. So $C$ is doubly chorded. Thus $G$ is doubly chorded $(k, k + l + 1)$-pancyclic. □

Lemma 4.2 allows us to extend the results of Theorem 4.1 to be doubly chorded as follows:

**Theorem 4.7.** *Let $G$ be a claw-free graph of order $n \geq 5$. Then each of the following hold:*

*(i) If $\sigma_2(G) \geq \frac{4n-2k-5}{3}$, then $G$ is doubly chorded $(k, k+3)$-pancyclic for $3 \leq k \leq \frac{n-5}{2}$.*
*(ii) If $\sigma_2(G) \geq n$, then $G$ is doubly chorded $(k, k+3)$-pancyclic for $k > \max\{\frac{n-5}{2}, 2\}$.*
*(iii) If $\sigma_2(G) \geq n$, then $G$ is doubly chorded $(k, k+4)$-pancyclic for $k \geq 3$.*

*Proof.* Parts (i) and (ii) follow immediately from parts (v) and (vi), respectively, of Theorem 4.1 and Lemma 4.2. For part (iii), Theorem 4.1 part (vii) and Lemma 4.2 give us the result for all $k \geq 4$. We now prove the $k = 3$ case separately:

Let $G$ be a claw-free graph of order $n \geq 5$ satisfying $\sigma_2(G) \geq n$. By Lemma 4.2, $G$ is doubly chorded $(4, 8)$-pancyclic. Then it only remains to show that every set of three vertices is in a doubly chorded 7-cycle. So let $S$ be a set of three vertices, which by Theorem 4.1 must be contained in a 7-cycle $C$.

Assume that $C$ has at most one chord, otherwise we are done. Note that there must exist a pair of vertices in $V(C) - S$ that are adjacent on the cycle $C$. (This can be verified by checking the four distinct distributions of three vertices on a 7-cycle.) Label the vertices of $C$ such that $C = v_1 \ldots v_5 a b v_1$ with $a, b \notin S$ and $v_1v_4, v_3v_5 \notin E(G)$. Such a labeling must exist, for otherwise $C$ would be doubly chorded.

Since $\sigma_2(G) \geq n$, there exists a common neighbor $w_1 \neq v_4$ of $v_3$ and $v_5$. If $w_1 \in V(C)$, then $C$ is either immediately doubly chorded, or $C$ has one chord and the claw-free condition then guarantees a second chord. So assume $w_1 \notin V(C)$. Next consider the nonadjacent vertices $v_1$ and $v_4$, which must have at least two common neighbors. If two common neighbors are on $C$, then $C$ is doubly chorded. So, assume $v_1$ and $v_4$ have at least one common neighbor that is not on $C$.

For the first case, suppose that the only common neighbor of $v_1$ and $v_4$ not on $C$ is the vertex $w_1$. Then $v_1$ and $v_4$ also have one common neighbor $x \in V(C)$. If $x \in \{v_5, a, b\}$, then $C$ is either immediately doubly chorded, or doubly chorded after applying the claw-free condition. Thus $x \in \{v_2, v_3\}$. If $x = v_2$, then $v_2v_4 \in E(G)$, and either $v_1v_3$, $v_3v_5$, or $v_1v_5$ must also be an edge in $G$, otherwise $\{w_1, v_1, v_3, v_5\}$ induces a claw. This makes $C$ doubly chorded. So suppose $x = v_3$. Then $v_1v_3 \in E(G)$. Note that $v_4$ and $a$ must be nonadjacent, for otherwise $C$ is doubly-chorded. Now apply the degree sum condition to $v_4$ and $a$. If the only common neighbors of $v_4$ and $a$ are contained in $V(C) \cup \{w_1\}$, then $S$ is contained in a doubly chorded 7-cycle. So there exists a vertex $w_2 \notin V(C) \cup \{w_1\}$ such that $v_4w_2, aw_2 \in E(G)$. We must avoid an induced claw on $\{v_4, v_3, v_5, w_2\}$. If $v_3w_2$ is an edge in $G$, then $v_1v_2v_3w_2v_4v_5w_1v_1$ is a doubly chorded 7-cycle containing $S$. If $v_5w_2$ is an edge in $G$, then $v_1w_1v_5w_2v_4v_3v_2v_1$ is a doubly chorded 7-cycle containing $S$.

For the second case, assume there exists a vertex $w_2 \notin V(C)$ such that $v_1w_2, v_4w_2 \in E(G)$ and $w_2 \neq w_1$. Notice that $v_1v_2v_3w_1v_5v_4w_2v_1$ is a 7-cycle containing $S$ with a single chord $v_3v_4$. To avoid the induced claw on $\{v_3, v_2, v_4, w_1\}$, we must add one of three possible edges, any of which form a second chord on the 7-cycle containing $S$. This completes the argument that $S$ must be contained in a doubly chorded 7-cycle. □

*Remark.* We now construct a graph to show that part (i) of Theorem 4.7 is sharp with respect to $m$ when $k = 3$, i.e. doubly chorded $(3, 6)$-pancyclicity is best possible under these conditions. In particular, note that $\sigma_2(G) \geq n$ guarantees $(2, 5)$-pancyclicity according to Part (vi) of Theorem 4.1. The following example also shows that Part (vi) of Theorem 4.1 cannot be extended to chorded $(2, 5)$-pancyclicity.

We construct a graph on $n$ vertices as follows. Label three vertices $u$, $v$, and $z$, and add the edge $uv$. Then partition the other $n - 3$ vertices into three vertex-disjoint complete subgraphs $A$, $B$, and $C$, where $|V(A)| = |V(B)| = |V(C)| = \frac{n-3}{3}$ if $n \equiv 0 \pmod 3$. If $n \not\equiv 0 \pmod 3$, then let $A$ be the set with one more or one less vertex than $B$ and $C$. Now add additional edges to the graph as follows. For every $a \in V(A)$ and for every $x \in V(B) \cup V(C)$, add the edge $ax$. For every $x \in V(A) \cup V(B)$, add the edge $ux$, and for every $x \in V(A) \cup V(C)$, add the edge $vx$. For every $x \in V(B) \cup V(C)$, add the edge $zx$. Then the graph is claw-free, and $\sigma_2(G) \geq \frac{4n-10}{3} \geq \frac{4n-2k-5}{3}$ when $k = 3$. However, the set $\{u, v, z\}$ is not contained in a chorded 5-cycle.

The set $\{z, u\}$ in the above graph also shows that Part (vi) of Theorem 4.1 cannot be extended to chorded $(2, 5)$-pancyclicity, even if we require a higher bound of $\sigma_2(G) \geq \frac{4n-10}{3}$.

We do not yet have sharpness examples for part (i) of Theorem 4.7 when $k \geq 4$. Thus, a question for future work is whether or not $m = k + 3$ in part (i) of Theorem 4.7 can be lowered when $k \geq 4$.

**Lemma 4.3.** *Let $G$ be a claw-free graph of order $n$ satisfying $\sigma_2(G) \geq n - 3$. Then for all $m \geq 8$, every $m$-cycle in $G$ is doubly chorded.*

*Proof.* It is shown in [12] that the independence number $\alpha(G)$ in a claw-free graph satisfies $\alpha(G) \leq \frac{4n}{\sigma_2(G)+4}$. Thus we have $\alpha(G) \leq \frac{4n}{(n-3)+4} < \frac{4n}{n} = 4$. Let $C = v_1v_2 \ldots v_mv_1$ be a cycle of length $m \geq 8$. Then $\{v_1, v_3, v_5, v_7\}$ cannot be an independent set, otherwise $\alpha(G) < 4$ is contradicted. So, there must be at least one edge among this set of vertices, and any such edge is a chord on $C$. Similarly, $\{v_2, v_4, v_6, v_8\}$ cannot be an independent set, and thus must induce at least one edge, making $C$ doubly chorded. □

We now use this lemma to prove a stronger version of Part (i) of Theorem 4.7 for sufficiently large $k \leq \frac{n-5}{2}$:

**Theorem 4.8.** *Let $G$ be a claw-free graph of order $n$. Then*

*(i) If $\sigma_2(G) \geq \frac{4n-2k-5}{3}$, then $G$ is doubly chorded $(k, k + 2)$-pancyclic for all $6 \leq k \leq \frac{n-5}{2}$.*

*(ii)* *If* $\sigma_2(G) \geq n$ *and* $n \geq 15$*, then* $G$ *is doubly chorded* $(k, k+2)$*-pancyclic for all* $k > \frac{n-5}{2}$.

*(iii)* *If* $\sigma_2(G) \geq n$ *and* $n \geq 5$*, then* $G$ *is doubly chorded* $(k, k+3)$*-pancyclic for all* $k \geq 5$.

*Proof.* For part (i), by part (v) of Theorem 4.1, under these conditions $G$ is $(k, k+2)$-pancyclic for $6 \leq k \leq \frac{n-5}{2}$. Since $k \leq \frac{n-5}{2}$, we have $\frac{4n-2k-5}{3} \geq n-3$, and thus we can apply Lemma 4.3, which ensures that every cycle in $G$ of length at least 8 is doubly chorded. Therefore, $G$ is doubly chorded $(k, k+2)$-pancyclic for $6 \leq k \leq \frac{n-5}{2}$. Similarly, parts (ii) and (iii) follow from parts (vi) and (vii) of Theorem 4.1 respectively and Lemma 4.3. □

## Acknowledgements

We are grateful to Ron Gould and the anonymous referees, whose detailed mathematical and stylistic comments helped us improve this paper. This work has been supported by NSF grant #1560222 at the Lafayette College REU. We thank Derek Smith and Liz McMahon for their support at the program.

## Appendix

In this appendix, we give an example of the steps of Algorithm 1, the algorithm for forbidden subgraph subcases. We use the example of a graph $G \neq C_n$ on $n = 6$ vertices such that $G$ is $K_{1,3}$ and $Z_1$-free. As detailed in the proof of Theorem 2.5, it is known that $G$ must contain a chorded Hamiltonian cycle. Consider the subgraph $C$ of $G$ that is a chorded Hamiltonian cycle $C = v_1v_2 \dots v_6v_1$. We will use the algorithm to examine the case when $v_1v_3$ is the chord of the Hamiltonian cycle, so the first edge in the algorithm is $v_1v_3$. The algorithm systematically adds and removes edges until $G$ is claw-free, $Z_1$-free, and does not contain a doubly chorded 4-cycle. Then false is returned, and we obtain the sharpness example shown in Figure 4. Whenever an edge is added, we use an indentation to show the steps that belong to the same subcase, i.e. the subcase determined by choosing one edge of a set of possible edges that eliminate a certain forbidden subgraph.

Algorithm Steps:

Add $v_1v_3$ to $E(G)$.
No induced claws are found.
The induced $Z_1$ on $\{v_1, v_2, v_6, v_3\}$ is found.
The edge $v_2v_6$ is added to eliminate the $Z_1$ on $\{v_1, v_2, v_6, v_3\}$.
 No induced claws are found.
 The induced $Z_1$ on $\{v_3, v_4, v_2, v_1\}$ is found.
 The edge $v_2v_4$ is added to eliminate the $Z_1$ on $\{v_3, v_4, v_2, v_1\}$.
  No induced claws are found.
  The induced $Z_1$ on $\{v_2, v_4, v_1, v_6\}$ is found.
  The edge $v_4v_1$ is added to eliminate the $Z_1$ on $\{v_2, v_4, v_1, v_6\}$.
   A doubly chorded 4-cycle is found on $\{v_3, v_2, v_1, v_4\}$.
   The edge $v_4v_1$ is removed.
  The edge $v_4v_6$ is added to eliminate the $Z_1$ on $\{v_2, v_4, v_1, v_6\}$.
   No induced claws are found.
   The induced $Z_1$ on $\{v_4, v_3, v_5, v_2\}$ is found.
    The edge $v_3v_5$ is added to eliminate the $Z_1$ on $\{v_4, v_3, v_5, v_2\}$.
    No induced claws are found.
    The induced $Z_1$ on $\{v_3, v_2, v_5, v_1\}$ is found.
    The edge $v_2v_5$ is added to eliminate the $Z_1$ on $\{v_3, v_2, v_5, v_1\}$.
     A doubly chorded 4-cycle is found on $\{v_3, v_2, v_5, v_4\}$
     The edge $v_2v_5$ is removed.
    The edge $v_1v_5$ is added to eliminate the $Z_1$ on $\{v_3, v_2, v_5, v_1\}$.
     The conditions have been met and no doubly chorded 4-cycle exists.

## References


[1] J. A. Bondy, Pancyclic graphs, *J. Combin. Theory Ser. B* 11 (1971), 80–84.

[2] G. Chen, R. J. Gould, X. Gu and A. Saito, Cycles with a chord in dense graphs, *Discrete Math.* 341 (2018), 2131–2141.

[3] C. B. Crane, Forbidden Pairs and $(k, m)$-pancyclicity, *Discuss. Math. Graph Theory* 37 (2017), 649–663.

[4] C. B. Crane, Hamiltonian Type Properties in Claw-Free $P_5$-Free Graphs, *Graphs Combin.* 32 (2016), 1817–1828.

[5] C. B. Crane, Pancyclic Type Properties of Claw-Free $P_6$-Free Graphs, *Australas. J. Combin.* 72 (2018), 185–200.

[6] C. B. Crane, Generalized Pancyclic Properties in Claw-free Graphs, *Graphs Combin.* 31 (2015), 2149–2158.

[7] M. Cream, R. J. Gould and K. Hirohata, Extending vertex and edge pancyclic graphs, *Graphs Combin.* 34 (2018), 1691–1711.

[8] M. Cream, R. J. Gould and K. Hirohata, A note on extending Bondy's meta-conjecture, *Australas. J. Combin.* 67 (2017), 463–469.

[9] M. Cream, R. J. Gould and V. Larsen, Forbidden Subgraphs for Chorded Pancyclicity, *Discrete Math.* 340 (2017), 2878–2888.

[10] R. J. Faudree, E. Flandrin and Z. Ryjacek, Claw-free graphs—A survey, *Discrete Math.* 164 (1997), 87–147.

[11] R. J. Faudree, R. J. Gould, M. S. Jacobson and L. Lesniak, Generalizing pancyclic and k-ordered graphs, *Graphs Combin.* 20 (2004), 291–309.

[12] R. J. Faudree, R. J. Gould, M. S. Jacobson, L. Lesniak and T. E. Lindquester, On independent generalized degrees and independence numbers in $K(1, m)$-free graphs, *Discrete Math.* 103 (1992), 17–24.

[13] S. Goodman, and S. Hedetniemi, Sufficient Conditions for a Graph to be Hamiltonian, *J. Comb. Theory. Ser. B* 16 (1974), 175–180.

[14] R. J. Gould and R. J. Faudree, Characterizing forbidden pairs for Hamiltonian properties, *Discrete Math.* 173 (1997), 45–60.

[15] R. J. Gould, *Graph Theory*, Dover Publications Inc., Mineola, NY, 2012.

(Received 16 Dec 2020; revised 5 Feb 2022, 14 June 2022)