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On paths with three blocks $P(k, 1, l)$

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Abstract

A path $P(k, r, l)$ is an oriented path consisting of k forward arcs, followed by r backward arcs, and then by l forward arcs. We prove the existence of any $P(k, 1, l)$ of length $n-1$ in any $(2n+2)$-chromatic digraph. Moreover, if D is an n-chromatic digraph containing a Hamiltonian directed path, then it contains any $P(k, 1, l)$ of length $n - 1$.

1 Introduction

Digraphs considered in this paper are finite, having no loops, multiple edges or circuits of length 2. Let D be a digraph. We denote by $E(D)$ the arc set, $V(D)$ the vertex set of D and $v(D)$ the number of vertices of D. We say that D contains a digraph H if H is isomorphic to a subdigraph of D. Let $K \subseteq V(D)$; the subdigraph of D induced by K is denoted by $D[K]$. We denote by D^c the digraph obtained from D after reversing the orientations of all arcs in $E(D)$. For every $v \in V(D)$, $N^+_H(v)$ (respectively, $N^-_H(v)$) denotes the outneighborhood (respectively, inneighborhood) of v in a subdigraph H of D. For short, we write $N^+(v)$ (respectively, $N^-(v)$) instead of $N^+_D(v)$ (respectively, $N^-_D(v)$). The underlying graph of D is denoted by $G[D]$ and the maximum degree of vertices in $G[D]$ is denoted by $\Delta(D)$.

The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. The chromatic number of a digraph D is the chromatic number of its underlying graph. A digraph D is said to be k-chromatic if $\chi(D) = k$.

An *oriented path* is a digraph whose underling graph is a path. An oriented path is said to be directed if all its arcs are of the same orientation. Let $P = x_1x_2 \dots x_n$ be an oriented path; we denote by $P_{[x_i,x_j]}$ the oriented subpath $x_ix_{i+1} \dots x_j$, for every $1 \le i < j \le n$. A *block of* P is a maximal directed subpath of P. Assuming that P has l blocks of consecutive lengths $k_1, k_2, \dots, k_l$, in which the first block is made of forward arcs, then we write $P = P(k_1, k_2, \dots, k_l)$. An oriented path P in D is said to be *Hamiltonian* if $V(P) = V(D)$.

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The notions of oriented cycles are defined in a similar way as those of oriented paths. Likewise the notation for an oriented cycle is defined similarly. Thus, an oriented cycle consisting of l blocks of consecutive lengths $k_1, k_2, ..., k_l$ is denoted by $C(k_1, k_2, ..., k_l)$, and a directed cycle is called a circuit. A digraph D is said to be *Hamiltonian* if it contains a *Hamiltonian* circuit.

An *outbranching* (respectively, *inbranching*) is an oriented tree in which a unique vertex has its indegree (respectively, outdegree) zero, and the other vertices have indegree (respectively, outdegree) one. The vertex of indegree (respectively, outdegree) zero in an outbranching (respectively, inbranching) is called a *root*. An *outforest* (respectively, *inforest*) is a digraph whose connected components are outbranchings (respectively, inbranchings). Let F be an outforest and let v be a vertex in F. We denote by $P_F(v)$ the unique directed path in F joining v with the root of the outbranching containing v. The level of v, denoted by $l_F(v)$, is the order of the path $P_F(v)$. Set $L_i(F) = \{v \in V(F); l_F(v) = i\}$. In a similar way, we define $P_F(v)$, $l_F(v)$, and $L_i(F)$ in case F is an inforest and v is a vertex in F.

Let D be a digraph; then D contains a spanning outforest and a spanning inforest. An arc $(u, v) \in E(D)$ is said to be a forward arc with respect to a spanning outforest (respectively, inforest) F of D whenever $l_F(u) < l_F(v)$ (respectively, $l_F(u) > l_F(v)$), otherwise it is called a backward arc.

A *maximal* outforest (respectively, inforest) of D is a spanning outforest (respectively, inforest) of D such that $\sum_{v\in V(D)} l_F(v)$ is maximal. In [5], El Sahili and Kouider, after introducing the notion of maximal outforest, proved that in a maximal outforest, $L_i(F)$ is stable in D for every i. In a similar way, one can easily notice that in a maximal inforest, $L_i(F)$ is stable in D for every i.

Digraphs contained in any n-chromatic digraph are called n-universal. Our focus in this paper is on studying the universality of oriented paths. In 2015, El Sahili [7] conjectured that for $n \geq 8$, every oriented path of order n is n-universal.

Regarding oriented paths in general, there is no better result than the one given by Burr [3], that is, every oriented path of length $n-1$ is $(n-1)^2$-universal. Gallai [8] and Roy [12] proved that every directed path of order n is n-universal. However in tournaments, Havet and Thomassé [9] proved that, except for three particular cases, every tournament of order n contains every oriented path of order n. Addario-Berry et al. [1] used strongly connected digraphs to prove El Sahili's conjecture for paths with two blocks. El Sahili et al. [6] gave a new elementary proof without using strongly connected digraphs.

After the case of two blocks was solved, the case of three blocks remained open. In the general case of paths with three blocks, the first linear bound was given by El Joubbeh [4] who proved that any $P(k, r, l)$ of length $n-1$ is contained in any $(4.6n)$-chromatic digraph. Mourtada et al. [10] proved that the path $P(n-3, 1, 1)$ is contained in any $(n+1)$-chromatic digraph. In addition, Mourtada et al. [11] proved that any $(2k+1)$-chromatic digraph contains a $P(1, k, 1)$.

In this paper, we are interested in studying the existence of the path $P(k, 1, l)$ of length $n-1$. We prove its existence in any $(2n+2)$-chromatic digraph. Then, we study the case of an n-chromatic digraph containing a Hamiltonian directed path.

2 The existence of $P(k,1,l)$ in a $(2n+2)$-chromatic digraph

In this section, we study the chromatic number of digraphs containing any $P(k,1,l)$ of length $n-1$ and we get the following result:

Theorem 2.1 *Let D be a $(2n+2)$-chromatic digraph; then D contains any $P(k,1,l)$ of length $n-1$ with $k,\ l \in \mathbb{N}^*$ where $\mathbb{N}^*$ is the set of positive integers.*

Proof. Let D be a $(2n+2)$-chromatic digraph and suppose to the contrary that there exist $k,\ l \in \mathbb{N}^*$ such that D contains no $P(k,1,l)$ of length $n-1$. Divide D into two induced subdigraphs D_1 and D_2 such that $\chi(D_1) = \chi(D_2) = n+1$.

Let F_1 be a maximal spanning outforest of D_1 and F_2 be a maximal spanning inforest of D_2. Let $x \in D_1$ and $y \in D_2$ with $l_{F_1}(x) \geq k+1$ and $l_{F_2}(y) \geq l+1$; then $(x,y) \in E(D)$ whenever x and y are neighbors in $G[D]$. In fact, if $(y,x) \in E(D)$, then $P_{F_1}(x) \cup (y,x) \cup P_{F_2}(y)$ contains a $P(k,1,l)$, a contradiction.

Set $B = \{y \in \bigcup_{i\geq l+1} L_i(F_2);\ y$ has an inneighbor $x \in \bigcup_{i\geq k+1} L_i(F_1)\}$. There exists no backward arc in $D[B]$ with respect to F_2. Actually, if (y_1,y_2) is a backward arc with respect to F_2 in $D[B]$, then $P_{F_1}(x) \cup (x,y_2) \cup (y_1,y_2) \cup P_{F_2}(y_1)$ contains a $P(k,1,l)$, where x is an inneighbor of y_2 in $\bigcup_{i\geq k+1} L_i(F_1)$, a contradiction. It follows that $D[B]$ contains no circuits. Moreover, $d^+_{D[B]}(y) \leq 1$ for every $y \in B$, since if there exists $y \in B$ such that $d^+_{D[B]}(y) \geq 2$, let y_1 and y_2 be two outneighbors of y in B with $l_{F_2}(y_1) \geq l_{F_2}(y_2)$; then $P_{F_1}(x) \cup (x,y_1) \cup (y,y_1) \cup (y,y_2) \cup P_{F_2}(y_2)$ contains a $P(k,1,l)$, where x is an inneighbor of y_1 in $\bigcup_{i\geq k+1} L_i(F_1)$, a contradiction. Thus, $D[B]$ contains no cycles and so it is a bipartite graph. Color the first k levels of F_1 and the first l levels of F_2 by $k+l$ colors by giving each level a color distinct from the others. Color $(\bigcup_{i\geq k+1} L_i(F_1)) \cup (\bigcup_{i\geq l+1} L_i(F_2) - B)$ by $n+1$ new colors and B by 2 colors. All colorings are done properly, so the obtained coloring is a $(2n+1)$-proper coloring, a contradiction. □

3 The existence of $P(k,1,l)$ in digraphs containing a Hamiltonian directed path

The bound $2n+2$ may be strongly improved if the digraph contains a Hamiltonian directed path. First, we prove the existence of any $P(k,1,l)$ of length $n-1$ in any n-chromatic Hamiltonian digraph. Then, depending on this result, we prove the existence of any $P(k,1,l)$ of length $n-1$ in any n-chromatic digraph with a Hamiltonian directed path.

In our proof, we take advantage of Brooks' theorem [2] which states that:

Theorem 3.1 *For every connected graph G that is neither an odd cycle nor a complete graph, $\chi(G) \leq \Delta(G)$.*

Since in tournaments with n vertices, the existence of any $P(k,1,l)$ is already proved [9], it follows that all the digraphs considered in this section are not tournaments.

In order to reach the proofs of our main results, we need the following lemma:

Lemma 3.2 *Let D be a Hamiltonian digraph with $\chi(D) \geq n$ such that D contains no $P(k,1,l)$ of length $n-1$ for some $k,\ l \in \mathbb{N}^*$. Then $\Delta(D) = n$.*

Proof. Let D be a Hamiltonian digraph with $\chi(D) \geq n$ such that D contains no $P(k,1,l)$ for some $k,\ l \in \mathbb{N}^*$. Let $v(D) = m$ and $C = v_1 v_2 \dots v_m$ be a Hamiltonian circuit in D. Since $n = k+1+l+1 > 3$, we get that D is not an odd cycle. Besides, as D is not a tournament, then by Theorem 3.1, $\chi(D) \leq \Delta(D)$.

For every $v_t \in V(D)$, we define the vertices a_t and a'_t and the set A_t such that:

$$l(C_{[v_t,a_t]}) = k+1,\ l(C_{[a't,v_t]}) = l+1, \text{and} A_t = V(C_{[a_t,a'_t]}).$$

Set $v_{t+1} = v_1$ for $t = m$, and $v_{t-1} = v_m$ for $t = 1$. We are going to show that $|N(v_t) \cap A_t| \leq 2$.

If v_t has two inneighbors in the set A_t, say v_i and v_j with $i < j$, then $C_{[v_{t+1},v_i]} \cup (v_i, v_t) \cup (v_j, v_t) \cup C_{[v_j,v_{t-1}]}$ contains a $P(k,1,l)$, a contradiction. So v_t has at most one inneighbor in the set A_t.

If v_t has two outneighbors in the set A_t, say v_i and v_j with $i < j$, then $C_{[v_{t+1},v_i]} \cup (v_t, v_i) \cup (v_t, v_j) \cup C_{[v_j,v_{t-1}]}$ contains a $P(k,1,l)$, a contradiction. Thus v_t has at most one outneighbor in the set A_t. Hence, we have

$$|N(v_t)| \leq |N(v_t) \cap A_t| + |V(C) - A_t| \leq 2 + k + l = n \qquad (*)$$

Therefore $\Delta(D) = n$. $\square$

As a direct conclusion from Lemma 3.2, we get that an $(n+1)$-chromatic Hamiltonian digraph D contains any $P(k,1,l)$, since otherwise we have $\Delta(D) = n$, and so $\chi(D) \leq n$, a contradiction.

One can easily see that a cycle of type $C(1,r)$ with $1 + r \geq n$ contains any $P(k,1,l)$ of length $n-1$.

We remark that if D is an n-chromatic digraph with $v(D) = n+1$, then D contains an n-tournament. Let $S_1, \dots, S_n$ be n stable sets covering $V(D)$. Since $v(D) = n+1$, there exists $i_0 \in \{1, \dots, n\}$ such that $|S_{i_0}| = 2$ and $|S_i| = 1$ for every $i \neq i_0$. We have $D[\bigcup_{i \neq i_0} S_i]$ is an $(n-1)$-tournament T_{n-1} since if there exists $\{i, j\} \subset \{1, \dots, n\}$ such that $S_i \cup S_j$ is stable in D, then $\chi(D) \leq n-1$, which is a contradiction. There exists $x \in S_{i_0}$ such that x is adjacent to all the vertices in $\bigcup_{i \neq i_0} S_i$, since otherwise the vertices of S_{i_0} can be added to $\bigcup_{i \neq i_0} S_i$ forming $n-1$ stable sets in D covering it, contradicting $\chi(D) = n$. Therefore $D[T_{n-1} \cup \{x\}]$ is an n-tournament.

Theorem 3.3 *Let D be an n-chromatic Hamiltonian digraph. Then D contains any $P(k,1,l)$ of length $n-1$.*

Proof. Let $C = v_1v_2\dots v_m$ be a Hamiltonian circuit in D. Suppose, without loss of generality, that $d(v_1) = \Delta(D)$. The proof proceeds by induction on $v(D) = m \geq n+1$. It is true for $m = n+1$, since in this case, D contains a tournament T of order n which contains any $P(k,1,l)$. Let us prove it for $m \geq n+2$, assuming that it is true up to $m-1$.

Suppose to the contrary that D contains no $P(k,1,l)$ for some k, $l \in \mathbb{N}^*$; then using Lemma 3.2, we have $d(v_1) = n$, and so by (*), we get $V(C_{[v_2,v_{k+1}]}) \cup V(C_{[v_{m-l+1},v_m]}) \subset N(v_1)$, and v_1 has one inneighbor and another outneighbor in the set A_1.

To continue our proof, we need to consider two cases concerning the values of k and l.

• Case 1: $k > 2$.
We have $(v_1, v_3) \in E(D)$, since otherwise $(v_3, v_1) \cup C_{[v_3,v_1]}$ is a $C(1, m-2)$, a contradiction.

Now consider the vertex v_2; using (*), we have

$$|N(v_2)| \leq |V(C_{[v_3,v_{k+2}]}) \cup V(C_{[v_{m-l+2},v_1]})| + 2.$$

We are going to prove that $v_4 \notin N(v_2)$. Indeed, $(v_4, v_2) \cup C_{[v_4,v_2]}$ is a $C(1, m-2)$ if $(v_4, v_2) \in E(D)$, and $(v_1, v_3) \cup (v_2, v_3) \cup (v_2, v_4) \cup C_{[v_4,v_1]}$ is a $C(1, m-1)$ if $(v_2, v_4) \in E(D)$, and so, in both cases, D contains a $P(k, 1, l)$, a contradiction. Also, $v_5 \notin N(v_2)$, since otherwise $(v_5, v_2) \cup C_{[v_5,v_2]}$ is a $C(1, m-3)$ if $(v_5, v_2) \in E(D)$, and $(v_1, v_3) \cup (v_2, v_3) \cup (v_2, v_5) \cup C_{[v_5,v_1]}$ is a $C(1, m-2)$ if $(v_2, v_5) \in E(D)$, and so, in both cases, D contains a $P(k, 1, l)$, a contradiction. Thus, $d(v_2) \leq n-2$.

Now consider the digraph $D' = D - \{v_2\}$. Note that $\chi(D') = \chi(D) = n$ and $C_{[v_3,v_1]} \cup (v_1, v_3)$ is a Hamiltonian circuit in D'. Thus, by induction, D' contains a $P(k, 1, l)$, a contradiction.

We omit the case when $l > 2$, since it is done analogously by proving that $d(v_m) \leq n-2$, and then applying the induction hypothesis on $D' = D - \{v_m\}$.

• Case 2: $k \leq 2$ and $l \leq 2$.
In this case, $P(k,1,l) \in \{P(1,1,1), P(1,1,2), P(2,1,1), P(2,1,2)\}$. We are going to deal with the existence of each type in D.

1. The existence of a $P(1, 1, 1)$:
   Let v_j be the outneighbor of v_1 in $A_1 = V(C_{[v_3,v_{m-1}]})$. We have $j = 3$, since otherwise $(v_{j-1}, v_j) \cup (v_1, v_j) \cup (v_1, v_2)$ is a $P(1, 1, 1)$, a contradiction. Moreover, $N^-(v_2) = \{v_1\}$, since otherwise $(w, v_2) \cup (v_1, v_2) \cup (v_1, v_3)$ is a $P(1, 1, 1)$, where w is an inneighbor of v_2 other than v_1, a contradiction. We have $N^+(v_2) = \{v_3\}$, since otherwise $(v_1, v_3) \cup (v_2, v_3) \cup (v_2, w)$ is a $P(1, 1, 1)$, where w is an outneighbor of v_2 other than v_3, a contradiction. Consequently, $d(v_2) = 2 = n-2$. Let $D' = D - \{v_2\}$; then D' is an n-chromatic Hamiltonian digraph and $C_{[v_3,v_1]} \cup (v_1, v_3)$ is a Hamiltonian circuit in D'. Thus, by induction, D' contains a $P(1, 1, 1)$, a contradiction.

2. The existence of a $P(1,1,2)$:
   $(v_{m-1}, v_1) \in E(D)$, since otherwise a $C(1, m-2)$ appears in D, a contradiction. We have $v_{m-2} \notin N(v_m)$, since otherwise either $(v_{m-2}, v_m) \cup (v_{m-1}, v_m) \cup (v_{m-1}, v_1) \cup (v_1, v_2)$ is a $P(1,1,2)$ or $(v_m, v_{m-2}) \cup C_{[v_m, v_{m-2}]}$ is a $C(1, m-2)$, a contradiction.
   No outneighbor of v_m exists in $A_m = V(C_{[v_2, v_{m-3}]})$, since otherwise $(v_{m-1}, v_1) \cup (v_m, v_1) \cup (v_m, v_i) \cup (v_i, v_{i+1})$ is a $P(1,1,2)$ with v_i an outneighbor of v_m in A_m, a contradiction.
   Using (*), we get $d(v_m) \leq |\{v_1\} \cup \{v_{m-1}\}| + 1 = 3 = n-2$. Let $D' = D - \{v_m\}$; then D' is an n-chromatic Hamiltonian digraph with $C_{[v_1, v_{m-1}]} \cup (v_{m-1}, v_1)$ a Hamiltonian circuit in D'. Thus, by induction, D' contains a $P(1,1,2)$, a contradiction.

3. The existence of a $P(2,1,1)$:
   We proved that D contains a $P(1,1,2)$. Applying this result on D^c, which is an n-chromatic Hamiltonian digraph, we get that D^c contains a $P(1,1,2)$, and so D contains a $P(2,1,1)$, a contradiction.

4. The existence of a $P(2,1,2)$:
   By (*), we have $|N(v_2)| \leq |\{v_3, v_4\} \cup \{v_m, v_1\}| + 2$. We use a similar argument to that used in the beginning of case 1, to prove that neither v_4 nor v_m is a neighbor of v_2, so $d(v_2) \leq 4 = n-2$. Let $D' = D - \{v_2\}$; then, as in the previous cases, D' contains a $P(2,1,2)$.

□

Theorem 3.4 *Let D be an n-chromatic digraph containing a Hamiltonian directed path. Then D contains any $P(k,1,l)$ of length $n-1$ with $n \geq 5$ and $k,\ l \in \mathbb{N}^*$.*

Proof. We proceed by induction on $v(D)$. For $v(D) = n+1$, D contains an n-tournament which contains any $P(k,1,l)$. Let us prove it for $m \geq n+2$, assuming that it is true up to $m-1$.

Let $P = v_1 \ldots v_m$ be a Hamiltonian directed path in D. Suppose that D contains no $P(k,1,l)$ for some $k,\ l \in \mathbb{N}^*$ and, without loss of generality, suppose that $k \geq l$. We have $k + l = n-2$ with $n \geq 5$, and $k \geq l$ so it follows that $k \geq 2$.

Note that $v_1 v_m \notin E(G[D])$, since otherwise either D is a Hamiltonian digraph if $(v_m, v_1) \in E(D)$, or $(v_1, v_m) \cup P$ is a $C(1, m-1)$ if $(v_1, v_m) \in E(D)$, and so, in both cases, D contains a $P(k,1,l)$, a contradiction.

We are going to prove now that $v_2 v_m \notin E(G[D])$. If $(v_2, v_m) \in E(D)$, then $(v_2, v_m) \cup P_{[v_2, v_m]}$ is a $C(1, m-2)$, a contradiction. Suppose that $(v_m, v_2) \in E(D)$. Then v_1 has no outneighbors in $P_{[v_3, v_{m-n+3}]}$, since otherwise $(v_m, v_2) \cup (v_1, v_2) \cup (v_1, w) \cup P_{[w, v_m]}$ is a $C(1, r)$ with $r \geq n-1$, where $w \in N^+(v_1) \cap P_{[v_3, v_{m-n+3}]}$, a contradiction. Besides, v_1 has at most one inneighbor in $P_{[v_3, v_{m-n+3}]}$, since otherwise $(v_m, v_2) \cup P_{[v_2, v_i]} \cup (v_i, v_1) \cup (v_j, v_1) \cup P_{[v_j, v_m]}$ is a $C(1, r)$ with $r \geq n$, where v_i and v_j, $i < j$, are two inneighbors of v_1 in $P_{[v_3, v_{m-n+3}]}$, a contradiction. Thus $d(v_1) \leq n-2$. Hence $D - \{v_1\}$

is an n-chromatic digraph containing a Hamiltonian circuit $C = (v_m, v_2) \cup P_{[v_2,v_m]}$. Then using Theorem 3.3, $D - \{v_1\}$ contains a $P(k, 1, l)$, a contradiction.

Set $A = V(P_{[v_{k+1},v_{m-l-1}]})$. Then, by following a similar argument to that introduced in Lemma 3.2, we get that v_m has at most one inneighbor and at most one outneighbor in the set A. Thus $|N(v_m) \cap A| \leq 2$, and so $d(v_m) \leq |N_{P_{[v_1,v_k]}}(v_m)| + |N_{P_{[v_{m-l},v_{m-1}]}}(v_m)| + 2 \leq (k-2) + l + 2 = n - 2$.

Therefore $\chi(D - \{v_m\}) = n$ and $P_{[v_1,v_{m-1}]}$ is a Hamiltonian directed path in $D - \{v_m\}$. Then, by induction, $D - \{v_m\}$ contains a $P(k, 1, l)$, a contradiction. □

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