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Large parts of random plane partitions: a Poisson limit theorem

Ljuben Mutafchiev*

American University in Bulgaria
2700 Blagoevgrad
Bulgaria
ljuben@aubg.edu

Abstract

We propose an approach for asymptotic analysis of plane partition statistics related to counts of parts whose sizes exceed a certain suitably chosen level. In our study, we use the concept of conjugate trace of a plane partition of the positive integer n, introduced by Stanley in 1973. We derive generating functions and determine the asymptotic behavior of counts of large parts using a general scheme based on the saddle point method. In this way, we are able to prove a Poisson limit theorem for the number of parts of a random and uniformly chosen plane partition of n, whose sizes are greater than a function $m = m(n)$ as $n \to \infty$. An explicit expression for $m(n)$ is also given.

1 Introduction, Motivation and Statement of the Main Result

Plane partitions were originally introduced by Young [23] as a natural generalization of integer partitions in the plane. Enumerative problems for plane partitions were first studied via generating functions by MacMahon [13] (see also [14]). To describe the problem, we will introduce first the concept of a linear integer partition. For a positive integer n, by a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n, we mean the representation

$$n = \lambda_1 + \lambda_2 + \cdots + \lambda_k, \tag{1.1}$$

where $k \geq 1$ and the integers $\lambda_j, j = 1, 2, \ldots, k$, are arranged in non-increasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. The summands λ_j in (1.1) are usually called parts of λ. The

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Ferrers diagram of a partition is an array of boxes (or cells) in the plane, left-justified, with λ_j boxes in the jth row counting from the bottom. Reading consecutively the numbers of cells in the columns of the array of the partition λ, beginning from the leftmost column, we get the conjugate partition $\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_K^*)$, where $K = \lambda_1$. For example, the Ferrers diagram of the partition $\tilde{\lambda} = (5, 4, 3, 3, 2, 2, 2, 1)$ of $n = 22$ as $22 = 5 + 4 + 3 + 3 + 2 + 2 + 2 + 1$ and its conjugate partition $\tilde{\lambda}^* = (8, 7, 4, 2, 1)$ (that is, $22 = 8 + 7 + 4 + 2 + 1$) are presented in Figure 1.

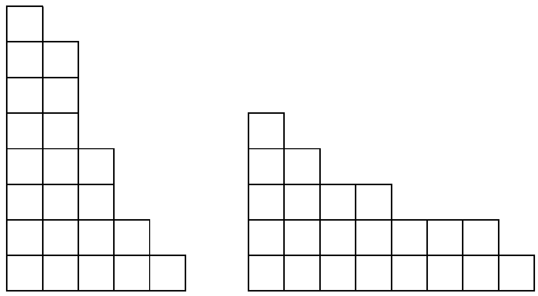

Figure 1: Ferrers diagrams of $\tilde{\lambda}$ and $\tilde{\lambda}^*$

Let $p(n)$ denote the total number of integer partitions of $n \geq 1$. For the generating function

$$P(x) = 1 + \sum_{n=1}^{\infty} p(n) x^n$$

of the sequence $\{p(n)\}_{n \geq 1}$, Euler established the following identity:

$$P(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-1}; \tag{1.2}$$

see, e.g., [2, Chapter 1]. Hardy and Ramanujan [9] developed the so called circle method and applied it to an asymptotic analysis for the coefficients of $P(x)$. In this way, they determined asymptotically the numbers $p(n)$ as follows:

$$p(n) \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi \sqrt{\frac{2n}{3}}\right), \quad n \to \infty.$$

For more details and a more precise asymptotic expansion for $p(n)$, we refer the reader to [18] and [2, Chapter 5].

The planar analogue of (1.1) is called a plane partition. A plane partition ω of the positive integer n is an array of non-negative integers

$$\begin{matrix} \omega_{1,1} & \omega_{1,2} & \omega_{1,3} & \cdots \\ \omega_{2,1} & \omega_{2,2} & \omega_{2,3} & \cdots \\ \cdots & \cdots & \cdots & \end{matrix} \tag{1.3}$$

that satisfy $\sum_{h,j \geq 1} \omega_{h,j} = n$, and the rows and columns in (1.3) are arranged in non-increasing order: $\omega_{h,j} \geq \omega_{h+1,j}$ and $\omega_{h,j} \geq \omega_{h,j+1}$ for all $h, j \geq 1$. The non-zero entries $\omega_{h,j} > 0$ are called parts of ω. If there are λ_h parts in the hth row, so that for some l, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > \lambda_{l+1} = 0$, then the (linear) partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of the integer $s = \lambda_1 + \lambda_2 + \cdots + \lambda_l$ is called the shape of ω. We also say that ω has l rows and s parts. Sometimes, for the sake of brevity, the zeros in array (1.3) are deleted. For instance, the abbreviation

$$\begin{array}{cccc} 5 & 4 & 1 & 1 \\ 3 & 2 & 1 & \\ 2 & 1 & & \end{array} \tag{1.4}$$

represents a plane partition $\tilde{\omega}$ of $n = 20$ with $l = 3$ rows and $s = 9$ parts. Any plane partition ω has an associated solid diagram $\Delta = \Delta(\omega)$ of volume n, which is considered as three-dimensional analogue of the Ferrers diagram of a linear integer partition. It is defined as a set of n integer lattice points $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{N}^3$, such that if $\mathbf{x} \in \Delta$ and $x'_j \leq x_j, j = 1, 2, 3$, then $\mathbf{x}' = (x'_1, x'_2, x'_3) \in \Delta$ too. (Here $\mathbb{N}$ denotes the set of all positive integers.) Indeed, the entry $\omega_{h,j}$ can be interpreted as the height of the column of unit cubes stacked along the vertical line $x_1 = h, x_2 = j$, and the solid diagram is the union of all such columns. Figure 2 represents the solid diagram of the plane partition in the example (1.4).

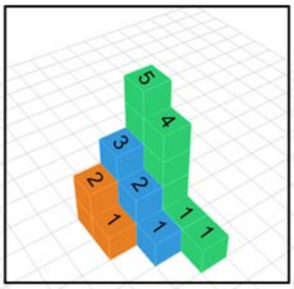

Figure 2: The solid diagram of plane partition (1.4)

Let $q(n)$ denote the total number of plane partitions of the positive integer n (or, the total number of solid diagrams of volume n). A basic generating function identity established by MacMahon [13] implies that the generating function of the sequence $\{q(n)\}_{n\geq 1}$,

$$Q(x) = 1 + \sum_{n=1}^{\infty} q(n)x^n,$$

satisfies

$$Q(x) = \prod_{j=1}^{\infty} (1 - x^j)^{-j} \tag{1.5}$$

(more details may be also found, e.g., in [19, Corollary 18.2] and [2, Corollary 11.3]). The asymptotic form of the numbers $q(n)$, as $n \to \infty$, has been obtained by Wright [22] (see also [16] for a little correction). It is given by the following formula:

$$q(n) \sim \frac{(\zeta(3))^{7/36}}{2^{11/36}(3\pi)^{1/2}} n^{-25/36} \exp\left(3(\zeta(3))^{1/3}(n/2)^{2/3} + 2\gamma\right), \tag{1.6}$$

where

$$\zeta(z) = \sum_{j=1}^{\infty} j^{-z}$$

is the Riemann zeta function and

$$\gamma = \int_0^\infty \frac{u \log u}{e^{2\pi u} - 1} du = \frac{1}{2}\zeta'(-1).$$

(The constant $\zeta'(-1) = -0.1654\ldots$ is closely related to the Glaisher-Kinkelin constant; see [5]).

Remark 1.1 In fact, Wright [22] proposed a variant of the circle method, which allows him to obtain an asymptotic expansion for $q(n)$.

Further, we need the concepts of conjugate trace and trace of a plane partition, introduced by Stanley [20].

Definition 1.1 *The conjugate trace of the plane partition ω, given by the array (1.3), is defined to be the number of parts $\omega_{h,j}$ of ω satisfying $\omega_{h,j} \geq h$.*

Hence the conjugate trace of the plane partition in example (1.4) is 6.

Definition 1.2 *The trace of a plane partition ω, given by (1.3), is defined to be the sum $\sum_h \omega_{h,h}$.*

Let $T^*_{lt}(n)$ ($T_{lt}(n)$) be the number of plane partitions of n with at most l rows, and conjugate trace (trace) t. Stanley [20] applied a bijection, established by Bender and Knuth [3], and showed that

$$T^*_{lt}(n) = T_{lt}(n). \tag{1.7}$$

Setting $T^*_t(n) = \lim_{l\to\infty} T^*_{lt}(n)$ and $T_t(n) = \lim_{l\to\infty} T_{lt}(n)$, he obtained the following identities:

$$1 + \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} T^*_t(n) y^t x^n = 1 + \sum_{n=1}^{\infty}\sum_{t=1}^{\infty} T_t(n) y^t x^n = \prod_{j=1}^{\infty}(1 - yx^j)^{-j}. \tag{1.8}$$

We notice that a linear (one-dimensional) partition λ has $2! = 2$ aspects—partition λ itself and its conjugate partition λ^*, while in the case of plane partitions, we observe $3! = 6$ aspects obtained from the six permutations of the three axes in the solid diagram. (Here we prefer to use MacMahon's term "aspect" [14, Section 427] rather than "conjugate" used by Stanley [20, p. 58].) Stanley [20, Section 3] showed that every plane partition ω of the positive integer n whose conjugate trace is t has exactly one aspect ω' with the same number of rows and trace equal to t. Suppose that the rows of ω and ω' are numbered from top to bottom by $1, 2, \ldots$. The unique partition

ω' is obtained from ω by taking the linear conjugate partition of row k of ω and then writing its parts in a non-increasing order on row k of ω'. This correspondence explains why both (1.7) and (1.8) hold. For example, using the conjugates of the partitions $11 = 5+4+1+1, 6 = 3+2+1$, and $3 = 2+1$ of the rows in the partition $\tilde{\omega}$ displayed by (1.4), we obtain $\tilde{\omega}'$ as

$$\begin{matrix} 4 & 2 & 2 & 2 & 1 \\ 3 & 2 & 1 & & \\ 2 & 1, & & & \end{matrix}$$

whose trace is obviously 6 and is equal to the conjugate trace of $\tilde{\omega}$.

Let $\Omega(n)$ be the set of all plane partitions of n, and let $\Lambda(n)$ be the set of all linear integer partitions of n. We introduce the uniform probability measures $\mathbb{P}$ and $\mathcal{P}$ on these two sets, respectively. That is, we assign the probability $1/q(n)$ to each plane partition of n and the probability $1/p(n)$ to each linear partition of n. In this way, each numerical characteristic of a plane partition from $\Omega(n)$ and of a linear partition from $\Lambda(n)$ becomes a random variable (or, a statistic in the sense of the random generation of plane or linear partitions of n). The analysis of linear integer partitions in terms of probabilistic limit theorems was initiated by Erdős and Lehner [4] who found an appropriate normalization for the largest part (for the number of parts, by the conjugation of the Ferrers diagram) in a random partition from $\Lambda(n)$ and established a weak convergence to the extreme value (Gumbel) distribution as $n \to \infty$. Subsequent work in this direction has been continued by many authors. Special interest in these studies was to determine the asymptotic behavior of counts of big part sizes of a random partition from $\Lambda(n)$ (say, part sizes greater than a certain suitable value $m = m(n)$). For typical results, we refer the reader to [21], [7] and [17]. In [7], among other important results, Fristedt proved the following Poisson limit theorem.

Theorem 1.1 *[7, p. 713]. Let $\xi_{m,n}$ be the number of parts greater than m in a random integer partition from the set $\Lambda(n)$, equipped with the uniform probability measure $\mathcal{P}$. Then, with respect to $\mathcal{P}$, $\xi_{m,n}$ has a limiting Poisson distribution with expectation e^{-c}, as $n \to \infty$ if, for any $c \in \mathbb{R}$,*

$$m = \frac{\sqrt{6n}}{\pi}\left(\log \frac{\sqrt{6n}}{\pi} + c\right). \tag{1.9}$$

The main purpose of this paper is to establish an analogue of Theorem 1.1 for plane partitions from the set $\Omega(n)$, equipped with the uniform probability measure $\mathbb{P}$. To introduce a statistic with an asymptotic behavior similar to that of $\xi_{m,n}$, in the two-dimensional case we need to take into account the order of the parts of a plane partition in both directions: from top to bottom (in rows) and from left to right (in columns). This idea is, in fact, what Stanley [20] stated in the definition of a conjugate trace of a plane partition (see Definition 1.1). The definition of the statistic that we propose as an analogue of $\xi_{m,n}$ is given below.

For $0 \leq m < n$, let $X_{m,n} = X_{m,n}(\omega)$ be the number of parts of $\omega \in \Omega(n)$ that satisfy the inequalities $\omega_{h,j} \geq h$ and $\omega_{h,j} > m$. In example (1.4), we have $X_{0,20}(\tilde{\omega}) = 6$, $X_{1,20}(\tilde{\omega}) = 4, X_{2,20}(\tilde{\omega}) = 3$ and $X_{3,20}(\tilde{\omega}) = 2$. Now, we state our main result.

Theorem 1.2 *With respect to the probability measure $\mathbb{P}$, the random variable $X_{m,n}$ has a limiting Poisson distribution with expectation $\frac{2}{3}e^{-c}$, as $n \to \infty$ if, for any $c \in \mathbb{R}$,*

$$m = \left(\frac{n}{2\zeta(3)}\right)^{1/3} \left(\log\left(\frac{n}{2\zeta(3)}\right)^{2/3} + \log\log n + c\right). \tag{1.10}$$

Remark 1.2 In [11] it is shown that the trace of a random plane partition of n, appropriately normalized, converges in distribution to a standard normal random variable as $n \to \infty$. We notice that the conjugate trace and the trace of a random plane partition of n have one and the same probability distribution with respect to the probability measure $\mathbb{P}$. This follows from Stanley's one-to-one correspondence [20], described above, and his identity (1.8). Hence, the limit theorem in [11] is also valid for the conjugate trace of a plane partition.

In the proof of Theorem 1.2 we follow a generating function approach. Let $\mathbb{E}$ denote the expectation taken with respect to the probability measure $\mathbb{P}$ on the space $\Omega(n)$. We observe that the generating function of the expectations $\{\mathbb{E}(y^{X_{m,n}})\}_{n\geq 1}$ satisfies an identity whose right-hand side is of the form $Q(x)f_m(x,y)$, where $Q(x)$ is given by (1.5) and the function $f_m(x,y)$ will be specified later. Then, we apply the saddle point method in a form given by Hayman [10] (see also, e.g., [6, Chapter VIII.5]).

Finally, we consider the simpler statistic $Z_{m,n} = Z_{m,n}(\omega)$ counting the number of parts which are greater than m in a randomly chosen $\omega \in \Omega(n)$. It is not difficult to show that if $m = m(n)$ is given by (1.10), then the difference $Z_{m,n} - X_{m,n}$ tends to 0 in probability as $n \to \infty$. Hence Theorem 1.2 implies the following corollary.

Corollary 1.1 *With respect to the probability measure $\mathbb{P}$, the random variable $Z_{m,n}$ has a limiting Poisson distribution with expectation $\frac{2}{3}e^{-c}$ as $n \to \infty$ if m satisfies (1.10).*

Our paper is organized as follows. In Section 2 we include the necessary generating function identities and the asymptotic results that will be used further. The proofs of Theorem 1.2 and Corollary 1.1 are given in Section 3. Some concluding remarks are given in Section 4.

2 Preliminary Results

Consider a plane partition $\omega \in \Omega(n)$, defined by the array (1.3). Let $L_n = L_n(\omega)$ be the largest part size of ω and let $R_n = R_n(\omega)$ be the number of rows in it.

Suppose that, for a certain ω, $R_n \le s$ and define the subsets of parts of ω by $\{\omega_{h,j} : \omega_{h,j} = k \ge h\}$ for $k = 1, 2, \dots, n$ (possibly some of the last subsets are empty). Let $Y_{k,n} = |\{\omega_{h,j} : \omega_{h,j} = k \ge h\}|$, where by $|A|$ we denote the cardinality of the set A. The next lemma extends the result of Theorem 2.2 from [20]. We state it in terms of probability generating functions. We assume there that the randomly chosen $\omega \in \Omega(n)$ is such that $R_n(\omega) \le s$ and $L_n(\omega) \le l$, for fixed s and l (i.e. the intersection $\{\omega : R_n \le s\} \cap \{\omega : L_n \le l\}$ is non-empty). For an arbitrary random variable $U_n = U_n(\omega), \omega \in \Omega(n)$, restricted on $\{\omega : R_n \le s\} \cap \{\omega : L_n \le l\}$, by $\mathbb{E}(U_n, R_n \le s, L_n \le l)$ we denote its expectation. (Obviously, after the two passages to the limit: $s \to \infty$ and $l \to \infty$, we will obtain the expectation of U_n, with respect to the probability measure $\mathbb{P}$ on the whole $\Omega(n)$, that is $\mathbb{E}(U_n)$.)

Lemma 2.1 *We have*

$$1 + \sum_{n=1}^{\infty} q(n)\mathbb{E}(y_1^{Y_{1,n}} y_2^{Y_{2,n}} \dots y_n^{Y_{n,n}}, R_n \le s, L_n \le l)x^n = \prod_{k=1}^{s}\prod_{j=1}^{l}(1 - y_j x^{k+j-1})^{-1}, \quad (2.1)$$

where $x, y_1, \dots, y_n$ are formal variables.

Sketch of the proof. We notice first that Definition 1.1 implies that $\sum_{j\ge1} Y_{j,n}$ equals the conjugate trace of a plane partition. Stanley [20, Theorem 2.2] showed that from (1.7) it follows that

$$1 + \sum_{n=1}^{\infty} q(n)\mathbb{E}(y^{\sum_{j\ge1} Y_{j,n}}, R_n \le s, L_n \le l)x^n = \prod_{k=1}^{s}\prod_{j=1}^{l}(1 - yx^{k+j-1})^{-1}, \quad (2.2)$$

where x and y are formal variables. Clearly, (2.1) is a slight extension of (2.2), in which the contribution of the parts k are separated by the variables y_k. The proof of (2.1) follows the same line of reasoning as in [20]. It is based on two bijections. The first one, due to Knuth [12], is as follows:

(K) There is a one-to-one correspondence between ordered pairs (ω_1, ω_2) of column strict plane partitions of the same shape and matrices $(b_{jk})_{j,k\ge1}$ of non-negative integers. In this correspondence, (i) the number k appears in ω_1 exactly $\sum_j b_{jk}$ times, and (ii) k appears in ω_2 exactly $\sum_j b_{kj}$ times.

(By a column strict plane partition we mean a plane partition whose non-zero entries are strictly decreasing in each column.)

The second bijection is given by Bender and Knuth [3].

(BK) There is a one-to-one correspondence between the plane partition $\omega \in \Omega(n)$ with $R_n(\omega) = s, L_n(\omega) = l$ and $Y_{j,n}(\omega) = t_j, j = 1, 2, \dots, n$, with $t = \sum_{j=1}^n t_j$, and pairs (ω_1, ω_2) of column strict partitions, so that the largest part of ω_2 is s, the largest part of ω_1 is l, the number of parts in the jth row of ω_1 or ω_2 is t_j and the conjugate trace t of ω equals the total number of parts $\sum_{j=1}^n t_j$ of ω_1 or ω_2. Moreover, if ω_k is a partition of $n_k, k = 1, 2$, then $n = n_1 + n_2 - t$; see also [20, p. 57].

In this way, using bijection (BK), we establish that the count $q(n)\mathbb{P}(Y_{j,n} = t_j, j = 1, 2, \dots, n, R_n \le s, L_n \le l)$ is equal to the number of pairs (ω_1, ω_2) of column strict plane partitions of the same shape satisfying: (i) the largest part of ω_1 is $\le s$, (ii) the largest part of ω_2 is $\le l$, (iii) the number of parts in row j of ω_1 or ω_2 is t_j, (iv) the number of parts of ω_1 or ω_2 is $t = \sum_{j=1}^n t_j$, (v) the sum of the parts of ω_1 and ω_2 is $n + t$. Then, one can obtain (2.1), using bijection (K) and following the same argument as in [20]. □

Remark 2.1 We notice that, in a similar way as Stanley did in [20], one can obtain from (2.1) as corollaries the following identities after both passages to the limit: $s \to \infty$ and $l \to \infty$. For instance, we have

$$1 + \sum_{n=1}^{\infty} q(n)\mathbb{E}(\prod_{k=1}^{\infty} y_k^{Y_{k,n}})x^n = \prod_{j=1}^{\infty}(1 - y_j x^j)^{-j}, \tag{2.3}$$

which generalizes Stanley's formula (6) in [20, p. 59]. Clearly, (1.8) follows from (2.3) after the substitution $y_j = y, j = 1, 2, \dots$.

Now, recall that $X_{m,n} = \sum_{j>m} Y_{j,n}$. Setting in (2.3) $y_1 = \cdots = y_{[m]} = 1$ and $y_j = y$ for $j > m$, where $[m]$ denotes the integer part of m, we obtain

$$\begin{aligned} 1 + \sum_{n=1}^{\infty} q(n)\mathbb{E}(y^{X_{m,n}}) &= \prod_{j \le m}(1 - x^j)^{-j} \prod_{j>m}(1 - yx^j)^{-j} \\ &= Q(x) f_m(x, y). \end{aligned} \tag{2.4}$$

Here $Q(x)$ is defined by (1.5) and

$$f_m(x, y) = \prod_{j>m} \left(\frac{1 - x^j}{1 - yx^j} \right)^j. \tag{2.5}$$

We are now ready to proceed with the preliminaries of our further asymptotic analysis. We have to study the behavior of the coefficient $[x^n]Q(x)f_m(x, y)$ of x^n in the Taylor expansion of the product $Q(x)f_m(x, y)$ as $n \to \infty$, where $Q(x)$ and $f_m(x, y)$ are defined by (1.5) and (2.5), respectively. We express this coefficient by means of Cauchy's integral formula using a suitably chosen closed curve around 0 as a contour of integration. Since the unit circle is a natural boundary for $Q(x)$, this contour lies inside the unit disk. We will estimate the Cauchy integral using a general theorem due to Hayman [10] whose proof is based on the saddle point method. We will describe next the wide class of functions to which Hayman's theorem apply. We employ the terminology given in [6, Chapter VIII.5].

Consider a function $G(x) = \sum_{n=0}^{\infty} g_n x^n$ that is analytic for $|x| < \rho, 0 < \rho \le \infty$. For $0 < r < \rho$, we set

$$a(r) = r\frac{G'(r)}{G(r)}, \tag{2.6}$$

$$b(r) = \frac{rG'(r)}{G(r)} + r^2\frac{G''(r)}{G(r)} - r^2\left(\frac{G'(r)}{G(r)}\right)^2. \tag{2.7}$$

We assume that $G(x) > 0$ for $x \in (\rho_0, \rho) \subset (0, \rho)$ and satisfies the following three conditions:

Capture condition. $lim_{r\to\rho} a(r) = \infty$ and $\lim_{r\to\rho} b(r) = \infty$.

Locality condition. For some function $\delta = \delta(r)$ defined over (ρ_0, ρ) and satisfying $0 < \delta < \pi$, one has

$$G(re^{i\theta}) \sim G(r)e^{i\theta a(r) - \theta^2 b(r)/2}$$

as $r \to \rho$, uniformly for $|\theta| \le \delta(r)$.

Decay condition.

$$G(re^{i\theta}) = o\left(\frac{G(r)}{\sqrt{b(r)}}\right)$$

as $r \to \rho$ uniformly for $\delta(r) < |\theta| \le \pi$.

Definition 2.1 *A function $G(x)$ which satisfies the capture, locality and decay conditions is called admissible in the sense of Hayman.*

Hayman's Theorem. *Let $G(x)$ be a Hayman admissible function and $r = r_n$ be the unique solution of the equation*

$$a(r) = n. \tag{2.8}$$

Then the Taylor coefficients g_n of $G(x)$ satisfy, as $n \to \infty$,

$$g_n \sim \frac{G(r_n)}{r_n^n\sqrt{2\pi b(r_n)}} \tag{2.9}$$

with $b(r_n)$ given by (2.7).

Hayman's theorem was applied in [15] to obtain a general asymptotic estimate for $[x^n]Q(x)F(x)$, where $Q(x)$ is defined by (1.5) and $F(x)$ is suitably restricted on its behavior in any neighborhood of $x = 1$. This result was then used to derive asymptotics of the expectations of several plane partition statistics. For linear integer partition statistics, a method for the asymptotic analysis of $[x^n]P(x)F(x)$, where $P(x)$ is defined by (1.2), was developed by Grabner et al. [8]. In our asymptotic analysis of $[x^n]Q(x)f_m(x, y)$ we are not able to apply directly the general result from [15, Theorem 1]. In fact, the function $f_m(x, y)$ (see (2.5)) is analytic only for $|x| < 1$, while $F(x)$ in [15] satisfies a more general assumption in a neighborhood of the point $x = 1$. In addition, $f_m(x, y)$ depends on the growth of n since the parameter $m = m(n)$ satisfies relation (1.10). Finally, the dependency on a second variable y in $f_m(x, y)$ plays an important role in our study since the estimates that we will explore further have to be uniform for $y \in [0, 1)$. Hence our further asymptotic analysis relies on the observations for the MacMahon's generating function $Q(x)$ established in [15].

The asymptotic behavior of $f_m(x,y)$, as $x \to 1$ and $|x| < 1$, is studied separately at the end of this section.

We continue with a lemma related to $Q(x)$ (see (1.5)). For its proof, we refer the reader to [15, pp. 261–265]. Further on, in (2.6), (2.7) and (2.9) we set $G(x) := Q(x)$, $\rho := 1$ and $g_n := q(n)$.

Lemma 2.2 *(i) The unique solution of (2.8) is $r = r_n = e^{-d_n}$, where, for large n, the sequence $\{d_n\}_{n\geq 1}$ has the following expansion:*

$$d_n = \left(\frac{2\zeta(3)}{n}\right)^{1/3} - \frac{1}{36n} + O(n^{-1-\beta}) \tag{2.10}$$

and $\beta > 0$ is a certain fixed constant. Moreover, as $n \to \infty$,

$$b(e^{-d_n}) \sim \frac{3n^{4/3}}{2\zeta^{1/3}(3)}, \tag{2.11}$$

where $b(r)$ is defined by (2.7). Hence (2.8) and (2.11) imply that $Q(x)$ satisfies Hayman's "capture" condition as $n \to \infty$.

(ii) With d_n and $b(e^{-d_n})$ as in part (i), we have

$$e^{-i\theta n}\frac{Q(e^{-d_n+i\theta})}{Q(e^{-d_n})} = e^{-\theta^2 b(e^{-d_n})/2}(1 + O(1/\log^3 n)) \tag{2.12}$$

as $n \to \infty$ uniformly for $|\theta| \leq \delta_n$, where

$$\delta_n = \frac{d_n^{5/3}}{\log n} = \frac{1}{\log n}\left(\frac{2\zeta(3)}{n}\right)^{5/9}(1 + O(n^{-2/3})). \tag{2.13}$$

(The last equality follows from (2.10).) In addition, (2.12) and (2.13) show that Hayman's "locality" condition holds for $Q(x)$ with $\delta_n := \delta(e^{-d_n})$.

(iii) For sufficiently large n, we have

$$\begin{aligned} |Q(e^{-d_n+i\theta})| &\leq Q(e^{-d_n})e^{-Cd_n^{-2/3}} \\ &\leq Q(e^{-d_n})e^{-C'n^{2/9}/\log^2 n} = o\left(\frac{Q(e^{-d_n})}{\sqrt{b(e^{-d_n})}}\right) \end{aligned} \tag{2.14}$$

uniformly for $\delta_n \leq |\theta| \leq \pi$, where $C, C' > 0$ are absolute constants and d_n and δ_n satisfy (2.10) and (2.13), respectively. By (2.14) $Q(x)$ satisfies also Hayman's "decay" condition with δ_n as in part (ii).

Remark 2.2 Clearly, for sufficiently large n, the arc $(-\delta_n, \delta_n)$ on the circle $x = e^{-d_n+i\theta}$, $-\pi < \theta \leq \pi$, becomes close to the main singularity $x = 1$ of MacMahon's generating function $Q(x)$. Furthermore, (2.12) and (2.14) show that $Q(e^{-d_n+i\theta})$ significantly changes its behavior when θ leaves the interval $(-\delta_n, \delta_n)$. Moreover, Lemma

2.2 and Definition 2.1 show that MacMahon's generating function $Q(x)$ is admissible in the sense of Hayman. Therefore, by Hayman's theorem we have

$$q(n) \sim \frac{e^{nd_n} Q(e^{-d_n})}{\sqrt{2\pi b(e^{-d_n})}}, \quad n \to \infty, \tag{2.15}$$

where d_n and $b(e^{-d_n})$ are given by (2.10) and (2.11), respectively. In the Appendix of [15] it is shown that (2.15) implies the corrected form of Wright's formula (1.6).

Our last task in this section is to study the behavior of $f_m(e^{-id_n+i\theta}, y)$; see (2.5). We obtain uniform estimates for the following two cases: θ belongs to a neighborhood of 0, and θ varies arbitrarily in the whole interval $(-\pi, \pi]$.

Lemma 2.3 *(i) If d_n and $m = m(n)$ satisfy (2.10) and (1.10), respectively, then*

$$\lim_{n\to\infty} \frac{f_m(e^{-id_n+i\theta}, y)}{f_m(e^{-d_n}, y)} = 1 \tag{2.16}$$

uniformly for $|\theta| \le \delta_n$ and $y \in [0, 1)$, where δ_n is given by (2.13).

(ii) Let d_n and m be the same as in part (i). Then, for any $\theta \in (-\pi, \pi]$ and sufficiently large n, we have

$$f_m(e^{-id_n+i\theta}, y) = O(1) \tag{2.17}$$

uniformly for $y \in [0, 1)$.

Proof. (i) We let $\log x$ denote the main branch of the logarithmic function, that is, we assume that $\log x < 0$ if $0 < x < 1$. Next, using (2.5), we represent the function $f_m(x, y)$ in the following way:

$$f_m(x, y) = \exp\,(\sum_{j>m} j g_j(x, y)), \tag{2.18}$$

where

$$g_j(x, y) = \log \frac{1 - x^j}{1 - yx^j}.$$

By the Taylor formula with $x = e^{-d_n+i\theta}$, we have

$$g_j(e^{-d_n+i\theta}, y) = g_j(e^{-d_n}, y) + O\left(\delta_n |\frac{\partial}{\partial x} g_j(x, y)|_{x=e^{-d_n}}\right), \tag{2.19}$$

since, for any $\theta_0 \in (-\delta_n, \delta_n)$, we have $|e^{i\theta_0} - 1| \le |\theta_0| < \delta_n$. A simple calculation shows that

$$\frac{\partial}{\partial x} g_j(x, y) = \frac{jx^{j-1}(y-1)}{(1-x^j)(1-yx^j)}. \tag{2.20}$$

Hence, using (2.18) - (2.20), we can write

$$\frac{f(e^{-d_n+i\theta},y)}{f(e^{-d_n},y)}=e^{S_{m,n}}, \tag{2.21}$$

where the sum $S_{m,n}$ of the remainder terms in the Taylor expansions of g_j satisfies the following estimate:

$$S_{m,n}=O\left(\delta_n(1-y)\sum_{j>m}\frac{j^2e^{-jd_n}}{(1-e^{-jd_n})(1-ye^{-jd_n})}\right) \tag{2.22}$$

uniformly for $|\theta|<\delta_n$ and $y\in[0,1)$. The sum on the right-hand side of (2.22) can be interpreted as a Riemann sum with step size d_n. At this moment, it is more convenient for us to express m as a function of d_n, given by (2.10). We set

$$m=d_n^{-1}(\log d_n^{-2}+\log\log n+c),\quad c\in\mathbb{R}. \tag{2.23}$$

By (2.23) the lower bound of the integral is $md_n=\log d_n^{-2}+\log\log n+c$. Thus we have

$$\begin{aligned}\sum_{j>m}\frac{j^2e^{-jd_n}}{(1-e^{-jd_n})(1-ye^{-jd_n})} &= O\left(d_n^{-3}\int_{\log d_n^{-2}+\log\log n+c}^{\infty}\frac{u^2e^{-u}}{(1-e^{-u})(1-ye^{-u})}du\right)\\ &= O\left(\frac{d_n^{-3}}{1-y}\int_{\log d_n^{-2}+\log\log n+c}^{\infty}\frac{u^2e^{-u}}{1-e^{-u}}du\right).\end{aligned} \tag{2.24}$$

The last integral is related to the Debye function of order 2. It is easy to see, using formula 27.1.2 in [1], that

$$\int_t^{\infty}\frac{u^2}{e^u-1}du=(t^2+2t+2)e^{-t}+O(t^2e^{-2t}),\quad t\to\infty. \tag{2.25}$$

We will also need asymptotic expansions for d_n^{-1}, d_n^{-2} and $\log d_n^{-2}$. From (2.10) it follows that

$$d_n^{-1}=\left(\frac{n}{2\zeta(3)}\right)^{1/3}+\frac{1}{36(2\zeta(3))^{2/3}n^{1/3}}+O(n^{-1/3-\beta}), \tag{2.26}$$

$$d_n^{-2}=\left(\frac{n}{2\zeta(3)}\right)^{2/3}+\frac{1}{36\zeta(3)}+O(n^{-\beta})$$

and

$$\log d_n^{-2}=\frac{2}{3}\log n-\frac{2}{3}\log\left(2\zeta(3)\right)+O(n^{-2/3}). \tag{2.27}$$

We notice here that from (2.26) and (2.27) it follows that

$$d_n^{-1}(\log d_n^{-2}+\log\log n+c)=\left(\frac{n}{2\zeta(3)}\right)^{1/3}\left(\log\left(\frac{n}{2\zeta(3)}\right)^{2/3}+\log\log n+c\right)+o(1),$$

i.e., the difference between the right-hand sides of (1.10) and (2.23) tends to 0 as $n \to \infty$, which justifies the value of m, given by (1.10). Moreover, from (2.25) and (2.27) we obtain

$$d_n^{-3} \int_{\log d_n^{-2} + \log\log n + c}^{\infty} \frac{u^2}{e^u - 1} du = O(d_n^{-1} \log n). \tag{2.28}$$

Combining (2.10), (2.13), (2.22) and (2.28), we see that

$$S_{m,n} = O(d_n^{2/3}) = O(n^{-2/9}),$$

which implies that the ratio in (2.21) tends to 1, as $n \to \infty$, uniformly for $|\theta| \le \delta_n$ and $y \in [0, 1)$. This completes the proof of part (i).

(ii) For any $\theta \in (-\pi, \pi]$, we observe that

$$\begin{aligned} |f_m(e^{-id_n + i\theta}, y)| &= \exp\left(\sum_{j>m} j \log \frac{|1 - e^{-jd_n\theta}|}{|1 - ye^{-jd_n\theta}|}\right) \\ &= \exp\left(\frac{1}{2} \sum_{j>m} j \log\left(\frac{1 - 2e^{-jd_n} \cos(j\theta) + e^{-2jd_n}}{1 - 2ye^{-jd_n} \cos(j\theta) + y^2 e^{-2jd_n}}\right)\right) \\ &\le \exp\left(\sum_{j>m} j \log\left(\frac{1 + e^{-jd_n}}{1 - e^{-jd_n}}\right)\right), \end{aligned} \tag{2.29}$$

where in the inequality on the last line we used that $y \in [0, 1)$ and that $-1 \le \cos(j\theta) \le 1$. Expanding the logarithm on the right hand side of (2.29) into powers of e^{-jd_n} with $j > m$ and using (2.23) and (2.27), we obtain

$$\begin{aligned} \log\left(\frac{1 + e^{-jd_n}}{1 - e^{-jd_n}}\right) &= 2e^{-jd_n} + 2\sum_{k\ge 2} \frac{e^{-j(2k-1)d_n}}{2k - 1} \\ &= 2e^{-jd_n} + O(e^{-3md_n}) \\ &= 2e^{-jd_n}(1 + O(e^{-2md_n})) \\ &= 2e^{-jd_n}(1 + e^{-(\frac{4}{3}\log n + 2\log\log n + O(1))}) \\ &= 2e^{-jd_n}(1 + O(n^{-4/3}/\log^2 n)). \end{aligned}$$

Substituting this expression into the right hand side of (2.29) and approximating again the underlying sum by a Riemann integral, from (2.26) and (2.27) we get the estimate:

$$\begin{aligned} |f_m(e^{-id_n + i\theta}, y)| &\le (1 + O(n^{-4/3}\log^2 n)) \exp\left(2\left(\left(\frac{n}{2\zeta(3)}\right)^{1/3} + O\left(\frac{1}{n}\right)\right) \sum_{j>m} (jd_n) e^{-jd_n}\right) \\ &\le \exp\left(C_0 n^{1/3} \int_{\frac{2}{3}\log n + \log\log n}^{\infty} u e^{-u} du\right) \end{aligned} \tag{2.30}$$

uniformly for $\theta \in (-\pi, \pi]$ and $y \in [0, 1)$, where $C_0 > 0$ is an absolute constant. Using the asymptotic formula for the incomplete gamma function [1, formula 6.5.32], we obtain

$$\int_{\frac{2}{3}\log n + \log\log n}^{\infty} ue^{-u} du = O(n^{-2/3}).$$

Replacing this estimate into the right hand side of (2.30), we complete the proof of part (ii). □

3 Proof of the Main Result

3.1 Proof of Theorem 1.2

First, we recall the expressions for d_n and δ_n given by (2.10) and (2.13), respectively. Next we apply the Cauchy coefficient formula to (2.4), using the circle $x = e^{-d_n + i\theta}$, $-\pi < \theta \le \pi$, as a contour of integration. Thus we obtain

$$\begin{aligned} [x^n]Q(x)f_m(x, y) &= \frac{e^{nd_n}}{2\pi} \int_{-\pi}^{\pi} Q(e^{-d_n+i\theta}) f_m(e^{-d_n+i\theta}, y) e^{-i\theta n} d\theta \\ &= J_{1,n} + J_{2,n}, \end{aligned} \tag{3.1}$$

where

$$J_{1,n} = \frac{e^{nd_n}}{2\pi} \int_{-\delta_n}^{\delta_n} Q(e^{-d_n+i\theta}) f_m(e^{-d_n+i\theta}) d\theta, \tag{3.2}$$

$$J_{2,n} = \frac{e^{nd_n}}{2\pi} \int_{\delta_n < |\theta| \le \pi} Q(e^{-d_n+i\theta}) f_m(e^{-d_n+i\theta}, y) d\theta. \tag{3.3}$$

The estimate of $J_{1,n}$ follows from parts (i) and (ii) of Lemma 2.2 and Lemma 2.3. First, in (3.2) we perform the following computation:

$$\begin{aligned} J_{1,n} &= \frac{e^{nd_n} Q(e^{-d_n}) f_m(e^{-d_n}, y)}{2\pi} \int_{-\delta_n}^{\delta_n} \left(\frac{Q(e^{-d_n+i\theta})}{Q(e^{-d_n})} \right) \left(\frac{f_m(e^{-d_n+i\theta}, y)}{f_m(e^{-d_n}, y)} \right) e^{-i\theta n} d\theta \\ &= \frac{e^{nd_n} Q(e^{-d_n}) f_m(e^{-d_n}, y)}{2\pi} \int_{-\delta_n}^{\delta_n} e^{-\theta^2 b(e^{-d_n})/2} \left(1 + O\left(\frac{1}{\log^3 n} \right) \right) (1 + o(1)) d\theta \\ &\sim \frac{e^{nd_n} Q(e^{-d_n}) f_m(e^{-d_n}, y)}{2\pi} \int_{-\delta_n}^{\delta_n} e^{-\theta^2 b(e^{-d_n})/2} d\theta. \end{aligned} \tag{3.4}$$

Note that in the second equality we applied (2.12) (that is, Hayman's "locality" condition) and (2.16) (i.e., Lemma 2.3(i)). Next, in the last integral of (3.4) we substitute $\theta = u/\sqrt{b(e^{-d_n})}$. We observe that

$$\int_{-\delta_n}^{\delta_n} e^{-\theta^2 b(e^{-d_n})/2} d\theta \sim \frac{1}{\sqrt{b(e^{-d_n})}} \int_{-\delta_n \sqrt{b(e^{-d_n})}}^{\delta_n \sqrt{b(e^{-d_n})}} e^{-u^2/2} du$$

$$\sim \frac{1}{\sqrt{b(e^{-d_n})}} \int_{-\infty}^{\infty} e^{-u^2/2} du$$
$$= \sqrt{\frac{2\pi}{b(e^{-d_n})}}, \quad n \to \infty,$$

since by (2.11) and (2.13)

$$\delta_n \sqrt{b(e^{-d_n})} \sim \sqrt{3}(2\zeta(3))^{7/18} \frac{n^{1/9}}{\log n} \to \infty, \quad n \to \infty.$$

Inserting the estimate of the last integral into (3.4), by Wright's formula (2.15) we obtain

$$\begin{aligned} J_{1,n} &= \frac{e^{nd_n} Q(e^{-d_n}) f_m(e^{-d_n}, y)}{\sqrt{2\pi b(e^{-d_n})}} \\ &= q(n) f_m(e^{-d_n}, y) + o(f_m(e^{-d_n}, y) q(n)) \\ &= q(n) f_m(e^{-d_n}, y) + o(q(n)), \end{aligned} \tag{3.5}$$

where in the last equality we have also used (2.17).

To estimate $J_{2,n}$, we use Lemma 2.2(iii) and Lemma 2.3(ii). We apply first the inequality given in (2.14) and combine it with (2.17). Thus we observe that

$$|Q(e^{-d_n+i\theta}) f_m(e^{-d_n+i\theta})| \le C'' Q(e^{-d_n}) e^{-C' n^{2/9} / \log^2 n} \tag{3.6}$$

uniformly for $\delta_n \le |\theta| \le \pi$, where $C'' > 0$ is a certain constant. From (3.3), (3.6), (2.11) and (2.15) it follows that

$$\begin{aligned} |J_{2,n}| &\le \frac{e^{nd_n}}{2\pi} \int_{\delta_n \le |\theta| < \pi} |Q(e^{-d_n+i\theta}) f_m(e^{-d_n+i\theta})| d\theta \\ &\le \frac{C'' e^{nd_n}}{\pi} Q(e^{-d_n}) e^{-C' n^{2/9} / \log^2 n} (\pi - \delta_n) \\ &= O\left(\frac{e^{nd_n} Q(e^{-d_n})}{\sqrt{2\pi b(e^{-d_n})}} n^{2/3} e^{-C' n^{2/9} / \log^2 n} \right) \\ &= O(q(n) n^{2/3} e^{-C' n^{2/9} / \log^2 n}). \end{aligned} \tag{3.7}$$

Substituting (3.5) and (3.7) into (3.1)–(3.3), we obtain

$$\frac{1}{q(n)} [x^n] Q(x) f_m(x, y) = f_m(e^{-d_n}, y) + o(1)$$

uniformly for $y \in [0, 1)$. Finally, going back to (2.4), we conclude that

$$\mathbb{E}(y^{X_{m,n}}) = f_m(e^{-d_n}, y) + o(1). \tag{3.8}$$

So, to complete the proof of the theorem it remains to study the asymptotic behavior of $f_m(e^{-d_n}, y)$, with m given by (1.10).

We will use an alternative representation for $f_m(x,y)$, which follows from (2.5). We have

$$\begin{aligned} f_m(x,y) &= \exp\,(\sum_{j>m} j(\log(1-x^j) - \log(1-yx^j))) \\ &= \exp\,((y-1)(\sum_{j>m} jx^j) + K_m(x,y)), \end{aligned} \tag{3.9}$$

where

$$K_m(x,y) = \sum_{j>m}\sum_{k\geq 2} \frac{j}{k}((yx^j)^k - x^{jk}). \tag{3.10}$$

The sum in the exponent of the right-hand side of (3.9) can be estimated using a Riemann sum approximation as in (2.30). Here we need a more precise estimate. Setting $x = e^{-d_n}$, we obtain

$$\begin{aligned} \sum_{j>m} je^{-jd_n} &\sim d_n^{-2}\int_{md_n}^{\infty} ue^{-u}du \\ &= d_n^{-2}\int_{\log d_n^{-2}+\log\log n+c}^{\infty} ue^{-u}du \\ &= -d_n^{-2}ue^{-u}|_{\log d_n^{-2}+\log\log n+c}^{\infty} + d_n^{-2}\int_{\log d_n^{-2}+\log\log n+c}^{\infty} e^{-u}du \\ &= (\log d_n^{-2} + \log\log n + c)\frac{e^{-c}}{\log n} + \frac{e^{-c}}{\log n} \\ &= \left(\frac{2}{3}\log n + \log\log n + O(1)\right)\frac{e^{-c}}{\log n} + O\left(\frac{1}{\log n}\right) \\ &= \frac{2}{3}e^{-c} + O\left(\frac{\log\log n}{\log n}\right), \end{aligned} \tag{3.11}$$

where in the fourth equality we have also used (2.27) and (2.23).

Finally, it remains to study the asymptotic behavior of the remainder term $K_m(e^{-d_n}, y)$ in (3.9). First, we change the order of summation in (3.10), and then we perform some algebraic computations in order to see that

$$\begin{aligned} K_m(e^{-d_n}, y) &= \sum_{k\geq 2} \frac{(y^k-1)e^{-kd_n}}{k}\left(\frac{me^{-mkd_n}}{1-e^{-kd_n}} + \frac{e^{-mkd_n}}{(1-e^{-kd_n})^2}\right) \\ &= K_{m,n}^{(1)} + K_{m,n}^{(2)}, \end{aligned} \tag{3.12}$$

where

$$K_{m,n}^{(1)} = \sum_{k\geq 2} \frac{(y^k-1)e^{-kd_n}me^{-mkd_n}}{k(1-e^{-kd_n})}, \tag{3.13}$$

$$K_{m,n}^{(2)} = \sum_{k\geq 2} \frac{(y^k-1)e^{-kd_n}e^{-mkd_n}}{k(1-e^{-kd_n})^2}.$$

From (1.10) it follows that

$$e^{-mkd_n} = \left(\frac{d_n^2 e^{-c}}{\log n}\right)^k,$$

$$1 - e^{-kd_n} = kd_n + O(k^2 d_n^2).$$

Replacing these two equalities in (3.13) and using (2.10), we conclude that

$$K_{m,n}^{(1)} = O\left(\sum_{k\geq 2} \frac{d_n^{2k-2}}{k^2}\right) = O(d_n^2) = O(n^{-2/3}) \tag{3.14}$$

uniformly for $y \in [0, 1)$. In the same way, we obtain the estimate for $K_{m,n}^{(2)}$. We have

$$K_{m,n}^{(2)} = O\left(\sum_{k\geq 2} \frac{d_n^{2k-2}}{k^3 \log^k n}\right) = O\left(\frac{d_n^2}{\log n}\right). \tag{3.15}$$

Combining (3.12), (3.14), (3.15) and (2.10), we get

$$K(e^{-d_n}, y) = O(d_n^2) = O(n^{-2/3}). \tag{3.16}$$

Thus (3.9), (3.11) and (3.16) imply that $f_m(e^{-d_n}, y)$ approaches $\exp\left(\frac{2}{3}e^{-c}(y-1)\right)$ as $n \to \infty$ uniformly for $y \in [0, 1)$. Now, Theorem 1.2 follows from the continuity theorem for probability generating functions and (3.8), since $\exp\left(\frac{2}{3}e^{-c}(y-1)\right)$ is the generating function of a Poisson distribution with expectation $\frac{2}{3}e^{-c}$. □

3.2 Proof of Corollary 1.1

We recall that the variables $X_{m,n}$ and $Z_{m,n}$ are defined on the set $\Omega(n)$ of all plane partitions of n equipped with the uniform probability measure $\mathbb{P}$. Let $\omega = (\omega_{h,j}) \in \Omega(n)$ be a plane partition defined by array (1.3). Since the restriction $\omega_{h,j} \geq h$ is removed in the definition of $Z_{m,n} = Z_{m,n}(\omega)$, then, for every $\omega \in \Omega(n)$, we have $Z_{m,n}(\omega) \geq X_{m,n}(\omega)$. Furthermore, we consider the following two events: $A_{m,n} = \{\omega = (\omega_{h,j}) \in \Omega(n) : Z_{m,n}(\omega) - X_{m,n}(\omega) > \epsilon\}$, with $\epsilon > 0$, and $B_{m,n} = \{\omega = (\omega_{h,j}) \in \Omega(n) : \text{there is a pair } (h_0, j_0) \text{ such that } m < \omega_{h_0,j_0} < h_0\}$. Hence, for any $\omega \in B_{m,n}$, the pair (h_0, j_0) satisfies the inequality $h_0 > m$. Since the columns of ω are non-increasing, for all $k \leq m$, we have $\omega_{k,j_0} \geq \omega_{h_0,j_0}$ and $\omega_{k,j_0} \geq k$, whence at least m parts of ω are greater than m. So, we observe that $A_{m,n} \subset B_{m,n} \subset \{X_{m,n} > m\}$ and thus

$$\mathbb{P}(A_{m,n}) \leq \mathbb{P}(X_{m,n} > m). \tag{3.17}$$

By Theorem 1.2, as $n \to \infty$, the upper bound in (3.17) deals with a right tail of a Poisson distribution. In addition, note that by (1.10) we have $m \to \infty$ as $n \to \infty$. Therefore we conclude that, under the assumptions of Theorem 1.2, the right hand side of (3.17) tends to 0. Hence $\mathbb{P}(A_{m,n}) \to 0$, which means that $Z_{m,n} - X_{m,n} \to 0$ in probability. Now, from the representation $Z_{m,n} = (Z_{m,n} - X_{m,n}) + X_{m,n}$ it follows that $Z_{m,n}$ and $X_{m,n}$ have one and the same limiting distribution as $n \to \infty$, which completes the proof of the corollary. □

4 Concluding Remarks

In this work, we have proposed an approach for analysis of statistics related to counts of large and small parts in a random and uniformly chosen plane partition. The concept of conjugate trace of a plane partition, introduced by Stanley [20], plays an important role in our study. We have decomposed the conjugate trace into the sum of the counts of parts $1, 2, \dots$ and then we have separated these counts by the formal variables $y_1, y_2, \dots$ in the underlying generating function. Cutting this sum until the mth row of the plane partition array shadowed by the conjugate trace, for suitable values of m (see (1.10)), we are able to study the asymptotic behavior of several particular statistics of counts of large part sizes. As an illustration, we have proved a Poisson approximation for a statistic, which has a natural analogue in the case of linear integer partitions. Finally, we have shown that the restriction given by the conjugate trace of a plane partition of n can be removed since it is bounded by a tail of a Poisson distribution and tends to 0 as $n \to \infty$. We believe that our approach could be also applied to further studies in this direction (for instance, to establish central and local limit theorems for other similar plane partition statistics).

Our last remark is related to the one-dimensional case of linear integer partitions. Fristedt's method of study of integer partition statistics [7] is purely probabilistic. It transfers the joint probability distribution of the part counts of a random and uniformly chosen one-dimensional partition into the joint conditional distribution of independent and geometrically distributed random variables. We are able to give an alternative proof of Theorem 1.1 based on generating function identities. Below, for the sake of completeness, we will briefly sketch this proof.

First, using general results from [2, Chapter 1], it is not difficult to show that

$$1 + \sum_{n=1}^{\infty} p(n)\mathcal{E}(y^{\xi_{m,n}})x^n = P(x)\varphi_m(x, y),$$

where

$$\varphi_m(x, y) = \prod_{j>m} \frac{1 - x^j}{1 - yx^j}.$$

We recall that $\mathcal{E}$ denotes the expectation with respect to the uniform probability measure $\mathcal{P}$ on the set $\Lambda(n)$ of linear integer partitions of n, and that $P(x)$, the generating function of the sequence $\{p(n)\}_{n\geq 1}$, satisfies (1.2). Then, one can apply the asymptotic scheme for analysis of linear integer partition statistics, proposed by Grabner et al. [8, Theorem 2.2] and based on the classical saddle point method, to show that, for

$$d'_n = \frac{\pi}{\sqrt{6n}} - \frac{1}{4n} + O(n^{-1-\alpha}), \quad \alpha > 0,$$

and m defined by (1.9), we have

$$\lim_{n\to\infty} \varphi_m(e^{-d'_n}, y) = \exp\left(e^{-c}(y - 1)\right)$$

uniformly for $y \in [0, 1)$. The last exponent is the probability generating function of the Poisson distribution with expectation e^{-c}, which completes the proof of Theorem 1.1.

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