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Conjugate m-ary partitions

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Abstract

Conjugation is a common tool in the field of integer partitions. However, in the study of m-ary partitions — partitions where all parts are a power of m — the notion of a conjugate partition is rarely used. This is because, in general, the conjugate of an m-ary partition is no longer m-ary. In this article, we consider m-ary partitions whose conjugate is also m-ary. We state conditions that an integer n must satisfy in order to have such partitions and explore properties of these rare, yet infinite, combinatorial objects.

1 Introduction

An m-ary partition of an integer n is a partition of n for which the parts are all powers of m. Properties of m-ary partitions have been studied by several authors. Early in the study of m-ary partitions, Mahler [12] gave asymptotic results for the number of m-ary partitions of an integer n. Work by de Bruijn [8] and Pennington [13] extended Mahler's results. Churchouse [6] initiated the study of congruence properties soon after. While Churchhouse considered binary partitions, several authors have extended this work to m-ary partitions such as in [1, 11, 14] and more recently, in [4]. Restricted m-ary partitions have also provided an interesting area of study such as in [5, 7, 9, 10].

Conjugation of partitions leads to some classic results in partition theory [2, 3]. For example, we can use conjugates to show that the number of partitions of an

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integer n into exactly r parts is equal to the number of partitions of n whose greatest part is r, or to show that the number of self-conjugate partitions of n is equal to the number of partitions of n with distinct odd parts. Ferrers diagrams provide an elegant visual proof of the latter statement as well as many other results about partitions.

However, conjugation does not appear in the study of m-ary partitions, because, in general, the conjugate of an m-ary partition is not m-ary. In this paper, we consider the exceptions — those m-ary partitions whose conjugate is also m-ary.

2 Conjugate m-ary partitions

Consider the partitions of $n = 15$ shown in Figure 1.

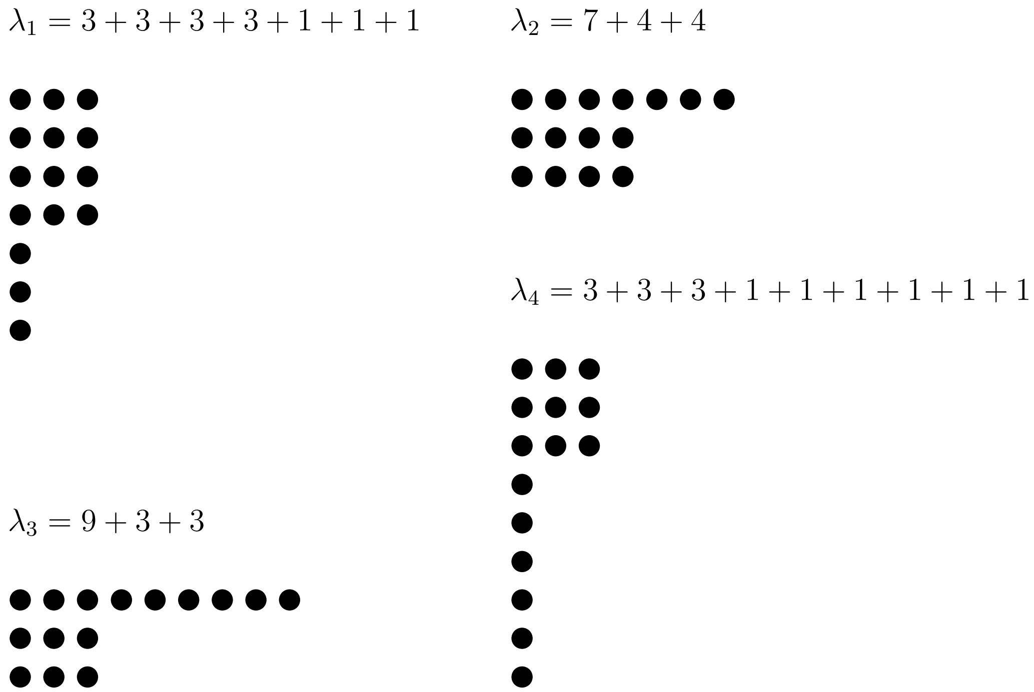


Figure 1: Four partitions of $n = 15$

We observe that λ_1 is a 3-ary partition with conjugate λ_2. As expected, λ_2 is not 3-ary. However, λ_3 and its conjugate λ_4 are both 3-ary.

Definition 2.1. *Let λ be a partition of n and let λ' be the conjugate of λ. We say that λ is a **conjugate m-ary partition** if λ and λ' are both m-ary partitions.*

In the example above, we see that λ_3 and λ_4 are conjugate m-ary partitions while λ_1 and λ_2 are not.

For convenience, we will refer to an integer partition with this property as a CMP (conjugate m-ary partition). It is clear that when λ is a CMP, λ' is as well.

Going forward, we will use the following convenient notation for representing m-ary partitions. Let λ be some m-ary partition of n and set $k = \lfloor \log_m(n) \rfloor$. Then

$$\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$$

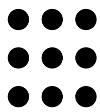
where $n = \sum_{i=0}^{k} a_i m^i$ and each $a_i \geq 0$. In other words, each a_i gives the number of times the part m^i appears in the given m-ary partition of n. We will refer to each a_i as the multiplicity of part m^i.

For example, with λ_3 and λ_4 from above, we write

$$\lambda_3 = (1, 2, 0)_3 \quad \text{and} \quad \lambda_3' = (0, 3, 6)_3 = \lambda_4.$$

It is well-known that an integer partition may be self-conjugate, where $\lambda = \lambda'$. We observe that it is also possible to have a self-conjugate CMP. For example, consider a "square" shaped Ferrers diagram such as

$$9 = 3 + 3 + 3$$



which is clearly both CMP and self-conjugate. Additionally, for $n = 117$ we notice that

$$(1, 8, 0, 18)_3' = (1, 8, 0, 18)_3.$$

which gives an example of a non-square self-conjugate CMP.

In order to address general properties of a CMP, we first consider an algorithmic approach to the process of conjugation as applied to an m-ary partition. For any integer partition λ, we find its conjugate λ' by reflecting the Ferrers diagram over its diagonal, so that the columns and rows of λ become the rows and columns of λ', respectively. In doing this, we find that the largest part of λ' is equal to the number of parts in λ, and the i^{th} part of λ' is equal to the number of parts in λ that are greater than or equal to i. In the case that λ is an m-ary partition, we can follow this algorithm to be even more specific about the form of the conjugate.

Consider the general m-ary partition λ of an integer n:

$$\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m.$$

Since the largest part of λ' is equal to the number of parts in λ, we find this by adding the multiplicities of all the parts of λ. Thus the largest part of λ' is $a_k + a_{k-1} + \cdots + a_1 + a_0$. To find the next largest part, we add all of the multiplicities of the parts of λ with exception of the smallest part of λ, that is, the multiplicities of the parts greater than m^0: $a_k + a_{k-1} + \cdots + a_1$. Similarly, to find the i^{th} largest part of λ', we sum the multiplicities of the parts of λ that are greater than or equal to m^{i-1}: $a_k + a_{k-1} + \cdots + a_{i-1}$.

Here, we define the function $f_m(r,\lambda)$ to be the sum of the multiplicities a_r through a_k, that is,

$$f_m(r,\lambda) = \sum_{i=r}^{k} a_i.$$

We will often refer to this summation as the partial sums function as it represents the partial sums of the multiplicities of the parts of a partition λ.

To find the multiplicity of each part of λ', we must find the number of columns in the Ferrers diagram of λ that have the same length. Since λ is an m-ary partition, each part of λ and thus each row of the corresponding Ferrers diagram will be a power of m. Therefore, the number of columns of the same length will be the difference of two powers of m: $m^i - m^j$. Observe that i and j need not be consecutive.

Thus we come to the following expression for the conjugate λ':

$$m^0 \cdot \sum_{i=0}^{k} a_i + (m-1)\cdot \sum_{i=1}^{k} a_i + \ldots + (m^{k-1} - m^{k-2}) \cdot \sum_{i=k-1}^{k} a_i + (m^k - m^{k-1}) \cdot a_k. \quad (1)$$

Alternatively, using the function f, we have that λ' is given by

$$m^0 \cdot f_m(0,\lambda) + \ldots + (m^{k-1} - m^{k-2}) \cdot f_m(k-1,\lambda) + (m^k - m^{k-1}) \cdot f_m(k,\lambda). \quad (2)$$

Note that these formulas give the conjugate as an expression including the difference of consecutive powers of m. In the case that some of the partial sums are equal, the terms will be combined, giving a difference of nonconsecutive powers of m. This will happen when there are powers of m that do not appear as parts in λ.

To illustrate this procedure for finding the conjugate of an m-ary partition, consider the 3-ary partition of $n = 3^6 = 729$,

$$\lambda = (1,0,2,24,0,216)_3.$$

Then its conjugate is

$$\begin{aligned}
\lambda' &= 3^0 f_3(0,\lambda) + (3-1) f_3(1,\lambda) + \ldots + (3^4 - 3^3) f_3(4,\lambda) + (3^5 - 3^4) f_3(5,\lambda) \\
&= 3^0 \cdot \sum_{i=0}^{5} a_i + (3-1)\cdot \sum_{i=1}^{5} a_i + \ldots + (3^4 - 3^3) \cdot \sum_{i=4}^{5} a_i + (3^5 - 3^4)\cdot a_5 \\
&= 3^0 \cdot (1+0+2+24+0+216) + (3-1)\cdot(1+0+2+24+0) \\
&\quad +(3^2 - 3)\cdot(1+0+2+24) + (3^3 - 3^2)\cdot(1+0+2) \\
&\quad +(3^4 - 3^3)\cdot(1+0) + (3^5 - 3^4)\cdot 1 \\
&= 1\cdot 3^5 + 2\cdot 3^3 + 6 \cdot 3^3 + 18\cdot 3^1 + 54 \cdot 3^0 + 162 \cdot 3^0 \\
&= 1\cdot 3^5 + 8 \cdot 3^3 + 18 \cdot 3^1 + 216 \cdot 3^0 \\
&= (1,0,8,0,18,216)_3.
\end{aligned}$$

Notice that λ is a CMP and λ' is a CMP since both are 3-ary partitions.

Based on the expression for the conjugate of an m-ary partition, we see that in order for an m-ary partition to be a CMP, the parts of its conjugate must be powers of m, that is, the partial sums found in the expression for λ' must all be powers of m. We now state our first theorem that follows directly from the comments above.

Theorem 2.2. *An m-ary partition λ is a conjugate m-ary partition if and only if $f_m(r, \lambda)$ is a power of m for $0 \le r \le k$, that is, if and only if the partial sums of the multiplicities of the parts of λ are powers of m.*

This theorem gives a method to quickly check if a given m-ary partition is a CMP. For example, for $n = 124$ consider the 4-ary partition $\lambda = (1, 3, 0, 12)_4$. The partial sums of the multiplicities are

$$f_4(3, \lambda) = 1\,; \quad f_4(2, \lambda) = 4\,; \quad f_4(1, \lambda) = 4 \;\; ; \quad f_4(0, \lambda) = 16.$$

Since these are all powers of 4, λ must be CMP. Using the algorithm above, we find $\lambda' = (0, 1, 15, 48)_4$, which has partial sums 1, 16, and 64.

3 A Counting Function

We now move to the natural question of enumerating the number of CMP for a particular value of n. To this end, for a positive integer n and for $m \ge 2$, we define $C_m(n)$ to be the number of conjugate m-ary partitions of n. We note that trivially $C_m(1) = 1$.

To illustrate this function, we consider $n = 117$ and $m = 3$. There are a total of 635 distinct 3-ary partitions of 117. Of these, five are CMP, so we write $C_3(117) = 5$. The conjugate 3-ary partitions of 117 are:

$$(1, 0, 2, 6, 0)_3, (3, 6, 72)_3, (1, 2, 24, 0)_3, (3, 0, 6, 18)_3, (1, 8, 0, 18)_3.$$

Observe that

$$(1, 0, 2, 6, 0)'_3 = (0, 0, 3, 6, 72)_3$$

and

$$(0, 1, 2, 24, 0)'_3 = (0, 3, 0, 6, 18)_3,$$

while

$$(0, 1, 8, 0, 18)'_3 = (0, 1, 8, 0, 18)_3.$$

In other words, we have two conjugate pairs and one self-conjugate CMP. We count both of the CMP in a conjugate pair and we count a self-conjugate CMP only once.

Of course, it is also possible for an integer n to have no CMP. For example, consider $n = 129$ and $m = 3$. There are 837 3-ary partitions of 129, but none of them are CMP. Thus $C_3(129) = 0$.

From Theorem 2.2, we know that the partial sums of the multiplicities of a CMP must be a power of m, including the sum of the multiplicities of all parts of a partition. This observation leads to the following proposition.

Proposition 3.1. *Let $m > 2$ and $n \geq 1$. If $C_m(n) > 0$, then $n \equiv 1 \pmod{(m-1)}$.*

Proof. Suppose $m > 2$ and there is $n \geq 1$ such that $C_m(n) > 0$. Then there is a conjugate m-ary partition of n, say λ, where $\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$. Thus $n = a_k m^k + a_{k-1} m^{k-1} + \ldots + a_1 m^1 + a_0 m^0$. Since λ is CMP, by Theorem 2.2 we know there exists a natural number ℓ such that $\sum_{i=0}^k a_i = m^\ell$. Then reducing n modulo $m-1$, we find

$$\begin{aligned}
n &= a_k m^k + a_{k-1} m^{k-1} + \ldots + a_1 m^1 + a_0 m^0 \\
&\equiv a_k \cdot 1^k + a_{k-1} \cdot 1^{k-1} + \ldots + a_1 \cdot 1 + a_0 \cdot 1 \pmod{(m-1)} \\
&\equiv \sum_{i=0}^{k} a_i \pmod{(m-1)} \\
&\equiv m^\ell \pmod{(m-1)} \\
&\equiv 1^\ell \pmod{(m-1)} \\
&\equiv 1 \pmod{(m-1)}.
\end{aligned}$$

□

Next, we observe that when a partition and its conjugate are both m-ary, a natural restriction is placed on the number of 1's that may appear as parts in the partitions. This observation is key in proving the result below.

Proposition 3.2. *Let $m > 2$ and $n > 1$. If $C_m(n) > 0$, then either $n \equiv 0 \pmod{m}$ or $n \equiv -1 \pmod{m}$.*

Proof. Assume $m > 2$, $n > 1$ and $C_m(n) > 0$. Then there is a partition of n that is CMP, say $\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$. If $a_0 = 0$, then

$$n = \sum_{i=1}^{k} a_i m^i \equiv 0 \pmod{m}$$

and the conclusion holds. Thus, suppose $a_0 > 0$ and notice $n \equiv a_0 \pmod{m}$.

If m^0 is the largest part of λ then a_0 is the largest part of λ'. Then $a_0 = m^\ell$ for some natural number ℓ. Thus $n \equiv a_0 \equiv 0 \pmod{m}$.

Now suppose there are two or more distinct parts, m^0 and one or more other powers of m. By Theorem 2.2 we know the partial sums of the multiplicities will be powers of m. In particular, the sum of multiplicities of parts greater than m^0 will be m^ℓ for some integer $\ell \geq 0$. Then the largest part of λ' will be

$$a_0 + m^\ell \equiv n + m^\ell \pmod{m}.$$

Suppose $\ell > 0$. Then

$$a_0 + m^\ell \equiv n + 0 \pmod{m}.$$

Since λ is CMP and $n > 1$, $a_0 + m^\ell$ must be a power of m, so $n \equiv 0 \pmod{m}$.

Now, suppose $\ell = 0$, then $a_0 + m^\ell \equiv n + 1 \pmod{m}$. Since λ is CMP, then we again know $a_0 + m^\ell$ is a power of m. Thus

$$a_0 + m^\ell \equiv n + 1 \equiv 0 \pmod{m}$$

and so $n \equiv -1 \pmod{m}$. □

These two propositions give necessary conditions for when the number of CMP for an integer n is nonzero. We can use the Chinese Remainder Theorem to combine these results, giving the following theorem.

Theorem 3.3. *Let $m > 2$ and $n > 1$. If $C_m(n) > 0$, then either*

$$n \equiv m \pmod{m(m-1)}$$

or

$$n \equiv 2m - 1 \pmod{m(m-1)}.$$

Proof. For $m > 2$, we have $\gcd(m, m-1) = 1$. Thus, by the Chinese Remainder Theorem, there must be a unique solution modulo $\operatorname{lcm}(m, m-1)$ to the system of congruences formed from Proposition 3.1 with the first part of Proposition 3.2 and also to the system formed from Proposition 3.1 with the second part of Proposition 3.2. Solving each system produces the desired result. □

We note here that binary (2-ary) partitions were excluded because Proposition 3.1 does not hold for $m = 2$. However, Theorem 3.3 is true for binary partitions as it would claim that $C_2(n) > 0$ implies n is either even or odd, which is certainly true. As we go forward we will allow that m may be 2, as in this next fact that extends a portion of Theorem 3.3.

Theorem 3.4. *For $m \geq 2$, suppose $n \equiv 2m - 1 \pmod{m(m-1)}$. Then,*

$$C_m(n) = \begin{cases} 2, & \text{if } n = m^s + m^t - 1 \text{ for } t \neq s, s, t \geq 1 \\ 1, & \text{if } n = 2 \cdot m^t - 1 \text{ for } t \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $n \equiv 2m - 1 \pmod{m(m-1)}$. From above, we know this means $n \equiv -1 \pmod{m}$. Suppose $C_m(n) > 0$ and let $\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$ be a partition of n that is CMP. Then, from the proof of Proposition 3.2, we must have $a_0 > 0$ and exactly one other part of λ, say m^t, with $t \geq 1$ and $a_t = 1$. Since λ is CMP, there must exist $s \geq 1$ such that the largest part of λ' is m^s meaning the total number of parts in λ is m^s. Thus, $a_0 = m^s - 1$ and we have $\lambda = 1 \cdot m^t + (m^s - 1) \cdot 1$. Using equation (1) or (2), we see that $\lambda' = 1 \cdot m^s + (m^t - 1) \cdot 1$.

Thus, for $n = m^s + m^t - 1$, we conclude that when $s \neq t$, the λ and λ' found above are the only possible CMP for n. So, $C_m(n) = 2$. Similarly, if $s = t$ then we must have $\lambda = \lambda'$, meaning λ is a self-conjugate CMP and $C_m(n) = 1$. □

As an example, consider $n = 3^2 + 3^1 - 1 = 11$. We have $C_3(11) = 2$ with $(1, 0, 2)'_3 = (0, 1, 8)_3$. The Ferrers diagrams for these partitions are shown in Figure 2. Notice that Theorem 3.4 gives a complete description of the value of $C_m(n)$ when $n \equiv 2m - 1 \pmod{m(m-1)}$ and every CMP in this case will look like a "corner" as in this example.

Figure 2: Ferrers diagrams for $(1, 0, 2)_3$ and $(0, 1, 8)_3$

Combining this with the results of Theorem 3.3, we see that for any $m \geq 2$, when $n \not\equiv m \pmod{m(m-1)}$, the value of $C_m(n)$ is known. It is almost always 0, unless n is such that it has 1 or 2 "corner" shaped CMP.

This leaves the case of $n \equiv m \pmod{m(m-1)}$ for us to explore.

4 A Case Study: $C_m(m^\ell)$

The behavior of $C_m(n)$ is quite predictable when $n \equiv 2m - 1 \pmod{m(m-1)}$. However, this is not the case for $n \equiv m \pmod{m(m-1)}$. The value of $C_m(n)$ for n in this congruence class is often 0 and when it is nonzero, the value appears to vary quite a bit. In this section we explore the case when $n = m^\ell$ in order to observe some behavior of the nonzero values of $C_m(n)$.

Consider $n = 3^\ell$, $\ell \geq 0$, as an example. The CMP for 3^0, 3^1, 3^2 and 3^3 are given in Table 1. Note that $n = 3^0 \not\equiv 3 \pmod 6$, but we include this power of 3 in the table for completeness.

Observe that for these powers of 3, each CMP contains only one distinct part. If we looked at the Ferrers diagram for each of these CMP, they would all have a rectangular appearance. In fact, the integer $n = 3^\ell$ has $\ell + 1$ rectangle CMP for any $\ell \geq 0$: $3^\ell \cdot 3^0, 3^{\ell-1} \cdot 3^1, 3^{\ell-2} \cdot 3^2, 3^{\ell-3} \cdot 3^3, \ldots, 3^0 \cdot 3^\ell$.

In general, the integer $n = m^\ell$ has at least one rectangle partition for all positive ℓ. These rectangle partitions give us a lower bound for $C_m(m^\ell)$.

Lemma 4.1. *For $m \geq 2$ and $\ell \geq 0$, $C_m(m^\ell) \geq \ell + 1$.*

n	$C_3(n)$	3-ary partitions of n
3^0	1	$(1)_3$
3^1	2	$(1,0)_3$ $(0,3)_3$
3^2	3	$(1,0,0)_3$ $(0,3,0)_3$ $(0,0,9)_3$
3^3	4	$(1,0,0,0)_3$ $(0,3,0,0)_3$ $(0,0,9,0)_3$ $(0,0,0,27)_3$

Table 1: CMP for small powers of 3

Proof. Observe that the partitions

$$m^\ell \cdot m^0, m^{\ell-1} \cdot m^1, m^{\ell-2} \cdot m^2, m^{\ell-3} \cdot m^3, \ldots, m^0 \cdot m^\ell$$

are all CMP of m^ℓ. Thus $C_m(m^\ell) \geq \ell + 1$. □

Returning to the example with $m = 3$, we see that the bound is tight for the first powers of 3 as shown in Table 1. However, for $n = 3^4$, we have $C_3(3^4) = 6$. The additional partition is $(1, 2, 6, 18)_3$, shown in Figure 3. This is a self-conjugate CMP for $n = 3^4$ that is not a rectangle partition.

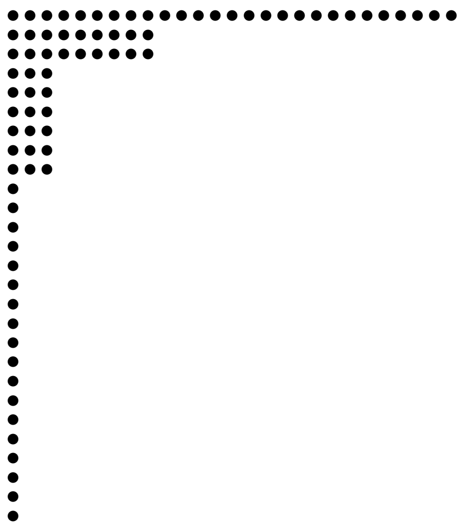


Figure 3: Non-rectangle CMP for 3^4

This pattern also generalizes. It appears that the rectangle CMP are the only CMP for powers of m up through m^{m+1}, that is $C_m(m^\ell) = \ell + 1$ for $0 \leq \ell \leq m$. For

$n = m^{m+1}$, we see a non-rectangular CMP that has a similar form as the example we see for $n = 3^4$.

Lemma 4.2. *The partition* $\lambda = (1, m-1, m^2-m, \ldots, m^m - m^{m-1})_m$ *is a self-conjugate CMP of* $n = m^{m+1}$.

Proof. We first observe that

$$\begin{aligned}
1 \cdot m^m + (m-1) \cdot m^{m-1} + (m^2-m) \cdot m^{m-2} + \cdots + (m^m - m^{m-1}) \cdot m^0 \\
&= 1 \cdot m^m + \sum_{i=1}^{m} (m^i - m^{i-1}) \cdot m^{m-i} \\
&= 1 \cdot m^m + \sum_{i=1}^{m} (m^m - m^{m-1}) \\
&= m^m + m^{m+1} - m^m \\
&= m^{m+1}.
\end{aligned}$$

Thus we see that $\lambda = (1, m-1, m^2-m, \ldots, m^m - m^{m-1})_m$ is an m-ary partition of $n = m^{m+1}$. From equation (2), we have that the conjugate of this partition is given by

$$m^0 \cdot f_m(0,\lambda) + (m-1) \cdot f_m(1,\lambda) + \ldots + (m^{m-1} - m^{m-2}) \cdot f_m(m-1,\lambda) + (m^m - m^{m-1}) \cdot f_m(m,\lambda).$$

Since $a_m = 1$ and $a_j = m^{m-j} - m^{m-j-1}$ for $0 \leq j < m$, then

$$f_m(m-j,\lambda) = \sum_{i=m-j}^{m} a_i = m^j$$

for $0 \leq j \leq m$. Thus by Theorem 2.2, λ is a CMP. Furthermore, the conjugate is

$$1 \cdot m^m + (m-1) \cdot m^{m-1} + \ldots + (m^{m-1} - m^{m-2}) \cdot m + (m^m - m^{m-1}) \cdot m^0,$$

and in weight form, $(1, m-1, m^2-m, \ldots, m^m - m^{m-1})_m$. Thus λ is self-conjugate. ☐

We will see that the existence of this non-rectangular CMP for m^{m+1} implies that every subsequent power of m will also have a non-rectangular CMP. For example, having $(1,2,6,18)_3$ as a CMP for 3^4 tells us that $(1,2,6,18,0)_3$ is a CMP for 3^5, $(1,2,6,18,0,0)_3$ is a CMP for 3^6, etc. (See Definition 5.1 and following in Section 5).

For now, we have a slightly better lower bound for $C_m(m^\ell)$ for sufficiently large powers of m. In particular, for $\ell \geq m+1$, $C_m(m^\ell) > \ell + 1$. However, computations indicate that the values increase more quickly than the bound increases. For example, consider the sequence

$$\left\{C_3(3^\ell)\right\}_{\ell \geq 0} = \{1, 2, 3, 4, 6, 8, 14, 32, 67, \ldots\}.$$

We see that the sequence not only begins to grow away from the lower bound of $\ell + 1$ but also appears to be strictly increasing.

In fact, it is true that the sequence $\left\{C_m(m^\ell)\right\}_{\ell \geq 0}$ is strictly increasing for $m \geq 2$. Rather than prove this here, we will first develop the necessary tools to prove a more general version of this result.

5 Raise and Shift

In the previous section we observed that $(1, 2, 6, 18)_3$ is a CMP of 3^4 and $(1, 2, 6, 18, 0)_3$ is a CMP of 3^5. The partition $(3, 6, 18, 54)_3$ is also a CMP of 3^5. Similarly, any m-ary partition of an integer n corresponds to two different partitions of the integer mn.

Let λ be an m-ary partition of n where $\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$. Then we know that

$$n = a_k m^k + a_{k-1} m^{k-1} + \ldots + a_1 m^1 + a_0 m^0.$$

To obtain a partition of mn, we multiply the terms of this sum by m. There are two ways to do this, either by multiplying the parts by m or multiplying the multiplicities by m. The results are two different m-ary partitions of the integer mn: $(a_k, a_{k-1}, \ldots, a_1, a_0, 0)_m$ and $(ma_k, ma_{k-1}, \ldots, ma_1, ma_0)_m$. We call these new partitions of mn the shift and the raise of λ, respectively.

Definition 5.1. *Let $\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$ be an m-ary partition of the integer n.*

(a) The ***raise*** *of λ is $R(\lambda) = (ma_k, ma_{k-1}, \ldots, ma_1, ma_0)_m$.*

(b) The ***shift*** *of λ is $S(\lambda) = (a_k, a_{k-1}, \ldots, a_1, a_0, 0)_m$.*

Observe that both R and S are one-to-one functions and are thus invertible. In addition, the CMP property is maintained under a raise and under a shift as we see in the following theorem.

Theorem 5.2. *Let $m \geq 2$ and $n \geq 1$. λ is CMP for n if and only if $R(\lambda)$ and $S(\lambda)$ are both CMP for mn.*

Proof. Let $m \geq 2$ and $n \geq 1$. By Theorem 2.2, $\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$ is CMP for n if and only if $f_m(r, \lambda)$ equals a power of m for $0 \leq r \leq k$. Then, for any such r,

$$f_m\left(r, R(\lambda)\right) = \sum_{i=r}^{k} m \cdot a_i = m \cdot \sum_{i=r}^{k} a_i = m \cdot f_m(r, \lambda).$$

So, $f_m\left(r, R(\lambda)\right)$ is a power of m if and only if $f_m(r, \lambda)$ is a power of m. Next, for $1 \leq r \leq k+1$, $f_m\left(r, S(\lambda)\right) = f_m(r-1, \lambda)$ is a power of m and

$$f_m\left(0, S(\lambda)\right) = f_m(0, \lambda) + 0 = f_m(0, \lambda).$$

Since the partial sums of multiplicities are equal, $f_m\left(r, S(\lambda)\right)$ is a power of m if and only if $f_m(r, \lambda)$ is a power of m. Thus, by Theorem 2.2, λ CMP is equivalent to both $R(\lambda)$ and $S(\lambda)$ being CMP. □

Consider a conjugate pair of m-ary partitions λ and λ'. We know that the multiplicity of the largest part of the conjugate λ' is equal to the smallest part of λ and the multiplicities of the remaining parts of λ' are equal to the difference of consecutive parts in λ. Finding the shift of the partition λ multiplies each of the parts of λ by

m, effectively multiplying each difference of consecutive parts by m as well. Finding the raise of the conjugate multiplies each of the multiplicities of λ' by m. Thus the multiplicities of $R(\lambda')$ remain equal to the difference of consecutive parts of $S(\lambda)$.

In addition, parts of λ' are equal to a partial sum of the multiplicities of λ. Finding the shift of the partition λ multiplies each part of λ by m, but does nothing to the multiplicities, leaving the partial sums of multiplicities the same. The raise of λ' leaves the parts the same. Thus the partial sums of multiplicities of $S(\lambda)$ are equal to the parts of $R(\lambda')$.

With the paragraphs above, we see that the conjugate of $S(\lambda)$ is $R(\lambda')$, that is, $S(\lambda)' = R(\lambda')$. Switching the role of λ and λ', we would see that $R(\lambda)$ and $S(\lambda')$ are also conjugates. These results are summarized in the theorem below.

Theorem 5.3. *Let λ and λ' be CMP for n. Then $R(\lambda)' = S(\lambda')$ and $S(\lambda)' = R(\lambda')$.*

Notice that Theorem 5.3 implies that when λ is self-conjugate for n, we have $R(\lambda)$ and $S(\lambda)$ are a conjugate pair for mn. In fact, we can say a bit more for self-conjugate CMP.

Theorem 5.4. *Let $\lambda = \lambda'$ be a self-conjugate CMP for n. Then $R\left(S(\lambda)\right) = S\left(R(\lambda)\right)$ is a self-conjugate CMP for m^2n.*

Proof. When λ is a CMP of n, it is clear from the definitions and Theorem 5.2 that $R\left(S(\lambda)\right) = S\left(R(\lambda)\right)$ and that this partition is CMP for m^2n. So, suppose λ is self-conjugate. Using $\lambda = \lambda'$ and the results of Theorem 5.3, we have

$$R\left(S(\lambda)\right)' = S\left(S(\lambda)'\right) = S\left(R(\lambda')\right) = S\left(R(\lambda)\right) = R\left(S(\lambda)\right).$$

Thus, $R\left(S(\lambda)\right)$ is self-conjugate. ☐

Theorem 5.4 illustrates one way to connect a CMP for n to related CMP for mn and m^2n. The next theorem gives a similar relationship.

Theorem 5.5. *Let $m \geq 2$ and $n \geq 1$. Let λ_1 and λ_2 each be a CMP for mn. Suppose $R(\lambda_1) = S(\lambda_2)$. Then there exists π, a CMP for n, such that $R(\pi) = \lambda_2$ and $S(\pi) = \lambda_1$. Equivalently, $\pi = S^{-1}(\lambda_1) = R^{-1}(\lambda_2)$.*

Proof. Suppose $\lambda_1 = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$ and $\lambda_2 = (b_k, b_{k-1}, \ldots, b_1, b_0)_m$ are both CMP for the integer mn such that $R(\lambda_1) = S(\lambda_2)$. We know that

$$R(\lambda_1) = (ma_k, ma_{k-1}, \ldots, ma_1, ma_0)_m$$

and

$$S(\lambda_2) = (b_k, b_{k-1}, \ldots, b_1, b_0, 0)_m.$$

Since $R(\lambda_1) = S(\lambda_2)$, we obtain the equalities below.

$$\begin{aligned} m \cdot a_0 &= 0 \\ m \cdot a_1 &= b_0 \\ m \cdot a_2 &= b_1 \\ &\vdots \\ m \cdot a_{k-1} &= b_{k-2} \\ m \cdot a_k &= b_{k-1} \\ 0 &= b_k \end{aligned}$$

The first equation implies that $a_0 = 0$. Thus $\lambda_1 = (a_k, a_{k-1}, \ldots, a_1, 0)_m$ must be the shift of a partition of n, meaning $S^{-1}(\lambda_1) = (a_k, a_{k-1}, \ldots, a_1)_m$. Also, we see that b_i is a multiple of m for $0 \le i \le k$. This implies $\lambda_2 = (b_k, b_{k-1}, \ldots, b_1, b_0)_m = (0, ma_k, ma_{k-1}, \ldots, ma_1)_m$ must be the raise of a partition of n. Thus $R^{-1}(\lambda_2) = (a_k, a_{k-1}, \ldots, a_1)_m$. Therefore, $S^{-1}(\lambda_1) = R^{-1}(\lambda_2)$. □

In this section, we have seen that once we identify a CMP for n, then a raise or a shift will give a CMP for mn. Consequently, we may conclude $m^\ell n$ will have partitions that are CMP for all $\ell \ge 0$.

6 Infinite Sequences of CMP

In Section 4, we claimed that $\{C_m(m^\ell)\}$ is a strictly increasing sequence. This is a consequence of a more general statement that we prove in this section. In particular, we will use the results of the previous section to show that given an n where $C_m(n)$ is positive, the sequence $\{C_m(m^\ell n)\}_{\ell \ge 0}$ must be strictly increasing.

We begin with a definition.

Definition 6.1. *A CMP $\lambda = (a_k, a_{k-1}, \ldots, a_1, a_0)_m$ is* ***simple*** *if and only if it is neither the shift nor the raise of another CMP. Equivalently, λ is simple if and only if $a_0 > 0$ (so it is not a shift) and $a_j = 1$ where $a_i = 0$ for all $i > j$ (so it is not a raise).*

We know from Theorem 5.3 that the conjugate of a non-simple CMP is always non-simple. Thus, each simple CMP must have a simple conjugate.

It is possible for an integer to be represented by both simple and non-simple CMP. Consider for example the simple CMP in Figure 3 along with the rectangular non-simple partitions of $81 = 3^4$ (see Section 4). However, certain special integers have only simple CMP.

Definition 6.2. *An integer n is* ***m-primitive*** *if $C_m(n) > 0$ and all CMP for n are simple. In this case, either $C_m(n/m) = 0$ or $n/m \notin \mathbb{Z}$.*

Theorem 3.4 gave a classification of all $n \equiv 2m-1 \pmod{m(m-1)}$ that have $C_m(n) > 0$. We now see that all of these "corner" partitions of the form $1 \cdot m^s + (m^t - 1) \cdot 1$ are simple. Thus, all integers represented by corner partitions will be m-primitive.

There are also integers in the m congruence class modulo $m(m-1)$ which are m-primitive. For example, 57 is 3-primitive (with 1 simple CMP) and 172 is 4-primitive (with 2 simple CMPs). However, not all integers in this congruence class are m-primitive. It is an open question to describe which n have this property for a given m.

The m-primitive integers give an ideal starting point for the sequences mentioned at the start of this section.

Theorem 6.3. *Let $m \geq 2$ and suppose $n \geq 1$ is m-primitive. Then, the sequence of positive integers $\left\{C_m(m^\ell n)\right\}_{\ell \geq 0}$ is strictly increasing.*

Proof. Since n is m-primitive, we have $C_m(n) > 0$. Now, for some $\ell \geq 0$, let $C_m(m^\ell n) = Y$. Thus, the CMP for $m^\ell n$ may be labeled as

$$\lambda_1, \lambda_2, \ldots, \lambda_Y. \tag{3}$$

We claim that at least one of $\lambda_1, \lambda_2, \ldots, \lambda_Y$ must have a non-zero multiplicity for the number of m^0 parts in the partition. Choose any of the CMP, say λ_1. If λ_1 has a non-zero multiplicity for the number of m^0 parts, the claim holds. Otherwise, we may write

$$\lambda_1 = (a_k, a_{k-1}, \ldots, a_z, 0, \ldots, 0)_m$$

where $1 \leq z \leq k-1$ is the number of trailing zeros in λ_1. Next, we compute $S^{-z}(\lambda_1) = \pi$. In other words, we do the inverse of the shift function z times to get π, a CMP for $m^{\ell - z} n$. By construction, π must have a_z as the multiplicity of m^0 parts and a_z is non-zero. Now, we raise π z times to get $R^z(\pi)$, a CMP for $m^\ell n$, which has a non-zero multiplicity of $m^z a_z$ for the number of m^0 parts. This must be one of the CMP listed in (3), so the claim holds. Relabel the CMP in (3) so this CMP is λ_Y.

Now, consider

$$S(\lambda_1), S(\lambda_2), \ldots S(\lambda_Y), R(\lambda_Y). \tag{4}$$

Since the original Y CMP were distinct, each shift is a distinct CMP for $m^{\ell+1} n$ (see Theorem 5.2). Also, by Definition 5.1(b), the first Y of the CMP in (4) will have no m^0 parts, meaning the multiplicity of the m^0 part must be 0 in all of them. Further, by Definition 5.1(a), λ_Y having a non-zero multiplicity of m^0 parts implies $R(\lambda_Y)$ is a CMP for $m^{\ell+1} n$ with a non-zero multiplicity of m^0 parts.

Therefore, the list (4) contains $Y + 1$ distinct CMP for $m^{\ell+1} n$ which implies $C_m(m^{\ell+1} n) > C_m(m^\ell n)$. Since this holds for any ℓ, the result follows. $\square$

One consequence of this theorem is that the sequence $\left\{C_m(m^\ell)\right\}$ is a strictly increasing sequence as we claimed in Section 4. Since $n = 1$ is m-primitive, setting $n = 1$ in Theorem 6.3 gives the result immediately.

In practice, the sequences often grow faster than suggested by the argument in the proof of Theorem 6.3. For example, $n = 17$ is 3-primitive and since $17 = 3^2 + 3^2 - 1$ there is a self-conjugate corner CMP. Computations give us the sequence

$$\left\{C_3(3^\ell \cdot 17)\right\}_{\ell \geq 0} = \{1, 4, 7, 12, 22, \ldots\}.$$

In addition to the CMP resulting from shifts and raises, additional simple CMP continue to appear as ℓ increases.

In conclusion, we may consider the existence of CMP from two perspectives. On one hand, CMP are rare. For a given m, integers in most of the congruence classes modulo $m(m-1)$ have no CMP. Then, in the two congruence classes where they are possible, many integers still have none. In fact, for those integers that are represented by at least one CMP, most of the m-ary partitions of that integer are not CMP (see examples in Section 3).

On the other hand, we know there are infinitely many CMP for a given m. Theorem 2.2 even suggests an algorithm to construct a simple CMP, λ: choose m; choose k and let m^k be the largest part of λ, setting $a_k = 1$; choose a_{k-1}, a_{k-2}, …, a_0 such that $f_m(r, \lambda)$ is a power of m for $0 \leq r \leq k$ and $a_0 > 0$. In this way we can create infinitely many distinct simple CMP. For each distinct CMP λ, we can use raises and shifts to generate an infinite family of CMP related to λ. In addition, if λ does not represent an m-primitive integer n, then the family of CMP arising from λ are only a subset of the infinitely many CMP that correspond with one of the sequences given in Theorem 6.3. This is a delightful property for an object that is as rare as stated above.

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