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# A theorem about vectors in $\mathbb{R}^2$ and an algebraic proof of a conjecture of Erdős and Purdy

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**Abstract**

Let $V$ be a set of $n$ vectors in $\mathbb{R}^2$. Assume that for every distinct $v$, $v'$ and $v''$ in $V$, the vectors $v + v'$ and $v + v''$ are linearly independent. We show that in such a case the set of vectors $\{v + v' \mid v, v' \in V,\ v \neq v'\}$ contains at least $n$ vectors every two of which are linearly independent, unless $n = 2$, $n = 4$, $n = 6$, or $n \geq 8$ is even and $O$, the origin is in $V$. In the latter case the other $n - 1$ vectors are (up to a linear transformation) the set of vertices of a regular $(n - 1)$-gon centered at $O$. We use this result to provide a short algebraic proof of an old conjecture of Erdős and Purdy: Let $P$ be a set of $n$ points in general position in the plane. Suppose that $R$ is a set of red points disjoint from $P$ such that every line determined by $P$ passes through a point in $R$. Then $|R| \geq n$, unless $n = 2$ or $n = 4$.

## 1 Introduction

A classical theorem of De-Bruijn and Erdős [3] implies that any non-collinear set of $n$ points in the Euclidean plane determines at least $n$ distinct lines.

In 1970 Scott [13] asked the same question about the minimum possible number of distinct directions of these lines. Scott conjectured that $n$ non-collinear points in the plane determine at least $n$ lines with pairwise distinct directions if $n$ is even and $n - 1$ distinct directions if $n$ is odd. This conjecture of Scott was proved by Ungar in 1982.

If we assume in addition that $P$ is in general position in the sense that no three points of $P$ are collinear, then things are slightly different and much simpler. In such

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a case $P$ determines at least $n$ lines with pairwise distinct directions regardless of whether $n$ is even or odd. To see this, consider the leftmost point in $P$ and denote it by $p_0$. Then denote the other points in $P$ by $p_1, \ldots, p_{n-1}$ according to the increasing slope of the lines $p_0p_1, \ldots, p_0p_{n-1}$. Then these lines have pairwise distinct directions and these $n-1$ directions are all different from the direction of the line $p_1p_{n-1}$.

The bound $n$ is best possible in this problem as can be seen by taking $P$ to be the set of vertices of a regular $n$-gon. It is shown in [7] that up to a linear transformation this is the only example in which $n$ points in general position determine precisely $n$ distinct directions.

An equivalent way of formulating the problem of Scott is to consider a set $V$ of $n$ vectors in $\mathbb{R}^2$ with affine dimension equal to 2 (corresponding to the points of $P$ being not collinear). Let $D = \{v - v' \mid v, v' \in V, \quad v \neq v'\}$ be the set of the pairwise differences of vectors in $V$. We are interested in the minimum number of distinct lines (through $O$) spanned by vectors in $D$. If we add the condition that no three of the vectors in $V$ are affinely dependent this will correspond to the points of $P$ being in general position. Adding this condition makes the problem simpler because for every $v \in V$ the $n-1$ vectors $\{v - v' \mid v' \in V, \quad v' \neq v\}$ span distinct lines.

What if we change in the definition of $D$ the differences into sums? That is, let $S(V) = \{v + v' \mid v, v' \in V, \quad v \neq v'\}$ and once again we would like to know what is the minimum possible number of distinct lines spanned by vectors in $S$. Equivalently, we would like to show that $S(V)$ contains many vectors, every two of which are linearly independent. As far as we know this problem has not received attention. Here it is not true anymore that for every $v \in V$ every pair of the $n-1$ vectors $\{v + v' \mid v' \in V, \quad v' \neq v\}$ are linearly independent even if we assume that every three vectors in $V$ are affinely independent. This new problem seems to be very interesting and non-trivial even in the case where every three vectors in $V$ are affinely independent. In this paper we will address (and solve) this problem under the assumption that for every $v \in V$ every two of the $n-1$ vectors $\{v+v' \mid v' \in V, \; v' \neq v\}$ are linearly independent. We will show that under this assumption one can always find at least $n$ vectors in $S(V)$, every two of which are linearly independent, unless $n = 2, 4$, or 6, or $n \geq 8$ is even and one of $v_1, \ldots, v_n$ is equal to 0. In the latter case the other $n-1$ vectors must be the set of vertices of a regular $(n-1)$-gon, up to a linear transformation of $\mathbb{R}^2$.

Clearly, a lower bound of $n-1$ is trivial in this problem, as we assume that for every $v \in V$ every two of the $n-1$ vectors $\{v + v' \mid v' \in V, \quad v' \neq v\}$ are linearly independent. If $n$ is odd, then it is very easy to improve this lower bound by one unit to be $n$ (in which case this bound is tight). Indeed, observe that if $v_1, v_2$ and $v_3, v_4$ are two different pairs of vectors such that $v_1 + v_2$ and $v_3 + v_4$ are linearly dependent, then the vectors $v_1, v_2, v_3$, and $v_4$ must be distinct. Therefore, at most $\lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$ of different pairs of vectors may be pairwise dependent. As there are $\binom{n}{2} = \frac{n(n-1)}{2}$ different pairs of vectors, it follows that there must be at least $n$ sums of pairs of vectors no two of which are linearly dependent. When $n$ is odd the bound $S(V) \geq n$ is also best possible. This can easily be seen by taking $V$ to be the set of vertices of a regular $n$-gon.

The case where $n$ is even turns to be much more challenging. This is our main result:

**Theorem 1.1.** *Let $n \geq 8$ be even. Let $V$ be a set of $n$ vectors in the $\mathbb{R}^2$. Assume that for every distinct $v, v', v'' \in V$ the vectors $v + v'$ and $v + v''$ are linearly independent. Then the set of vectors $S(V) = \{v+v' \mid v, v' \in V, \ \ v \neq v'\}$ contains at least $n$ vectors every two of which are linearly independent, unless $0 \in V$ and the nonzero vectors in $V$ are (up to a linear transformation) the set of vertices of a regular $(n-1)$-gon centered at the origin.*

Clearly, Theorem 1.1 is false when $n = 2$ because then $|S(V)| = 1$. Theorem 1.1 is false also for $n = 4$. To see this just take any four vectors, satisfying the conditions of Theorem 1.1, whose sum is equal to 0. Very surprisingly, the result in Theorem 1.1 fails to be true also for $n = 6$. Here it is much more challenging to find counterexamples. Although the case $n = 6$ just by itself may have only limited importance, there is a very nice elementary mathematics behind it and we will address this case in detail in a separate section at the end of this paper.

We remark that the bound $S(V) \geq n$ in Theorem 1.1 is best possible up to at most one unit. To see this consider the set of vertices of a regular $(n+1)$-gon minus one point. The bound $S(V) \geq n$ in Theorem 1.1 is enough for our main application and we leave it open whether it can be improved by one unit, or not.

We will now introduce our main application in which, as we will see, the assumption in Theorem 1.1 that for every distinct $v, v', v'' \in V$ the vectors $v + v'$ and $v + v''$ are linearly independent comes naturally. Our main application is a short algebraic proof of a conjecture of Erdős and Purdy about line blockers for sets of points in general position in the plane.

Let $P$ be a set of $n$ points in the projective plane. A set of points $R$ disjoint from $P$ is called a *line blocker* for $P$ if every line through two (or more) points of $P$ passes also through a point in $R$. Erdős and Purdy asked the following question in [5]. How small can be the cardinality of a line blocker for a set $P$ of $n$ points in the plane? Clearly, if $P$ is contained in a line, then $R$ may consist of just one point. Therefore, the question of Erdős and Purdy is about sets $P$ that are not collinear. The best known lower bound for this question is given in [10], where it is shown that $|R| \geq n/3$.

In [5] Erdős and Purdy considered also the case in which $P$ is in *general position* in the sense that no three points of $P$ are collinear. In this case there is a simple construction showing that $|R|$ can be as small as $n$. To observe this let $P$ be the set of vertices of a regular $n$-gon and let $R$ be the set of $n$ points on the line at infinity that correspond to the $n$ possible directions of the edges and diagonals of $P$.

To get a lower bound for $|R|$, notice that every point in $R$ may be incident to at most $\lfloor \frac{n}{2} \rfloor$ lines determined by $P$. Because there are $\binom{n}{2}$ lines determined by $P$, then if $n$ is odd we get $|R| \geq n$ (which is tight) and if $n$ is even we get $|R| \geq n - 1$. This easy lower bound for $|R|$ is in fact sharp in the cases $n = 2$ and $n = 4$, as can be

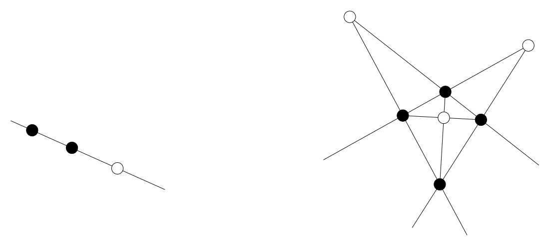

Figure 1: Counterexamples for $n = 2, 4$ in the primal version (Theorem 1.2). The points in $P$ are colored black while the points in $R$ are colored white.

seen in Figure 1. Is the bound $|R| \geq n-1$ sharp also for larger values of $n$? The answer is 'NO' as follows from the next theorem, proving a conjecture of Erdős and Purdy in [5].

**Theorem 1.2.** *Let $P$ be a set of $n \geq 5$ points in general position in the projective plane. Suppose that $R$ is a line blocker for $P$. Then $|R| \geq n$.*

Theorem 1.2 was first proved in [1] (see Theorem 8 there), as a special case of the solution of the Magic Configurations conjecture of Murty [9]. The proof in [1] contains a topological argument based on Euler's formula for planar maps and the discharging method. An elementary (and long) proof of Theorem 1.2 was given by Milićević in [8]. Probably the "book proof" of the Theorem 1.2 can be found in [11]. Theorem 1.2 was proved also over $\mathbb{F}_p$ by Blokhuis, Marino, and Mazzocca [2]. In this paper we provide an algebraic proof for Theorem 1.2 as an application of Theorem 1.1 for $n \neq 6$. Although Theorem 1.2 is valid in the case $n = 6$, we will not be able to conclude this case from Theorem 1.1 because of the surprising fact that Theorem 1.1 is not true for $n = 6$.

The approach in [2] to proving Theorem 1.2, although uses a slightly different language and deals with geometries over finite fields, has things in common to the approach we present in this paper. In particular equation (2) in [2] is essentially identical to the observation that $v_i + v_j$ is in the direction of some $u_k$ at the end of the proof of Theorem 1.2 in this paper. However, the two proofs are different. While our proof works over the reals, the proof in [2] is over the finite fields $\mathbb{Z}_p$.

## 2 Proof of Theorem 1.1

Denote by $v_1, \ldots, v_n$ the vectors in $V$. As we already observed, the set $S(V)$ of all sums of pairs of vectors in $V$ contains at least $n-1$ vectors, every two of which are linearly independent. Let $u_1, \ldots, u_{n-1}$ be $n-1$ such vectors in $S(V)$. We need to show that if every vector in $S(V)$ is proportional to one of $u_1, \ldots, u_{n-1}$, then $0 \in V$ and the other $n-1$ vectors in $V$ are (up to a linear transformation) the set of vertices of a regular $(n-1)$-gon.

Assume that every vector in $S(V)$ is proportional to one of $u_1, \ldots, u_{n-1}$. For every fixed $i$, every vector in $\{v_i + v_j \mid j \neq i\}$ is proportional to a different vector in

$\{u_1, \ldots, u_{n-1}\}$. Therefore, if we couple vectors in $v_1, \ldots, v_n$ whose sum is parallel to say $u_j$, we will get a perfect matching. This implies that $\sum_{i=1}^{n} v_i$ is a vector parallel to $u_j$. Since this is true for every $j$, we conclude that $\sum_{i=1}^{n} v_i = 0$.

Let $Q$ denote the convex hull of $V \cup -V$ (where $-V = \{-v \mid v \in V\}$). Observe that $Q$ is centrally symmetric. Fix $1 \leq i \leq n-1$. A line $\ell$ parallel to $u_i$ passes through a point in $V$ if and only if it passes through a point of $-V$. Indeed, assume $\ell$ passes through $v_k$ (similarly if it passes through $-v_k$), then it passes through the unique point $-v_j$ such that $v_k + v_j$ is proportional to $u_i$. It follows from here that $Q$ has two edges parallel to $u_i$. As this is true for every $i = 1, \ldots, n-1$, we conclude that $Q$ has at least $2(n-1)$ edges, and therefore, at least $2(n-1)$ vertices.

We claim that $Q$ has exactly $2(n-1)$ edges. This is to say that it is not possible that all the points in $V \cup (-V)$ are vertices of $Q$. Indeed, assume to the contrary that $Q$ has $2n$ vertices (notice that this is the contrary assumption, as $Q$ is centrally symmetric). We claim that it must be that there are two vertices of $Q$ in $V$ consecutive along the boundary of $Q$. Indeed, otherwise the vertices in $V$ and $-V$ appear alternately on the boundary of $Q$ and this is impossible because $n$ is even (easy exercise!). The contradiction now follows from the following claim:

**Claim 2.1.** *It is not possible that two vertices of $V$ appear consecutively on the boundary of $Q$.*

**Proof.** Assume that $v$ and $v'$ are both in $V$ and they are two consecutive vertices of $Q$. Consider the point $-v_k$ such that the angle $\measuredangle(-v_k)vv'$ is minimum. There must be a point $-v_t$ such that the line connecting $-v_t$ to $v'$ is parallel to the line connecting $v$ and $(-v_k)$. This is impossible because then the angle $\measuredangle(-v_t)vv'$ is smaller than $\measuredangle(-v_k)vv'$, contradicting the minimality of $\measuredangle(-v_k)vv'$. □

Having established the fact that $Q$ has precisely $2(n-1)$ vertices, we rename these vertices and denote them by $a_0, \ldots, a_{m-1}$, where $m = 2(n-1)$, indexed in correspondence to their clockwise order on the boundary of $Q$. Without loss of generality we have that $\{a_0, a_2, a_4, \ldots, a_{m-2}\}$ are all in $V$ and $\{a_1, a_3, a_5, \ldots, a_{m-1}\}$ are all in $-V$. We may assume this because, by Claim 2.1, it is not possible that two consecutive vertices along the boundary of $Q$ belong to $V$.

**Claim 2.2.** *For every $i$ and $1 \leq k < n/2 - 1$ the line through $a_{i-k}$ and $a_{i+1+k}$ is parallel to the line through $a_i$ and $a_{i+1}$ (here all the indices are taken modulo $m$).*

We remark that once we prove Claim 2.2 it will follow from the symmetry of $Q$ that for every fixed $i$ the lines $\{a_{i-k}a_{i+1+k} \mid 0 \leq k < n-1, \ \ k \neq n/2-1\}$ are pairwise parallel.

**Proof of Claim 2.2.** We prove the claim by induction on $k$. Observe that for every $i$ and $1 \leq k < n/2-1$, the line through $a_{i-k}$ and $a_{i+1+k}$ is parallel to one of the edges of $Q$. This is because one of $a_{i-k}$ and $a_{i+1+k}$ belongs to $V$ and the other belongs to $-V$, and we cannot *not* have $a_{i-k} = -a_{i+1+k}$ (if $k = n/2 - 1$, then $a_{i-k} = -a_{i+1+k}$, which is the reason we assume $k < n/2 - 1$).

For $k = 1$ the line through $a_{i-1}$ and $a_{i+2}$ must be parallel to the edge $a_i a_{i+1}$ (or to the opposite and parallel edge $a_{i-n/2}a_{i+1+n/2}$) of $Q$, as it cannot be parallel to any other edge of $Q$.

As for the induction step, assume the claim is true for $k-1$ and we prove it for $k$. Consider the line through $a_{i-k}$ and $a_{i+1+k}$. By the induction hypothesis the line through $a_{i-k}$ and $a_{i+k-1}$ is parallel to the edge $a_{i-1}a_i$. The line through $a_{i-k+2}$ and $a_{i+1+k}$ is parallel to the edge $a_{i+1}a_{i+2}$. Therefore, the only possible edge of $Q$ that may be parallel to the line through $a_{i-k}$ and $a_{i+1+k}$ is the edge $a_i a_{i+1}$ (or its reflection through the origin $O$, namely the edge $a_{i+n-1}a_{i+n}$). This completes the induction step. □

The next step is to show that the vertices of $Q$ lie on a quadric. Here we will use the assumption that $2(n-1) > 10$, that is, $n > 6$ (as we shall see, this is false for $n = 6$). Let $C$ be a quadric passing through $a_0, a_1, a_2, a_3$, and $a_4$. We will show that $C$ must pass through $a_5$. Then repeating this argument we conclude that all the vertices of $Q$ lie on $C$.

For every $0 \leq i < j \leq 5$ denote by $\ell_{ij}$ the line, considered also as polynomial of degree 1 in $x$ and $y$, through $a_i$ and $a_j$. The lines $\ell_{03}$ and $\ell_{12}$ are parallel and meet at a point $A$ on the line at infinity. The lines $\ell_{14}$ and $\ell_{05}$ are parallel and meet at a point $B$ on the line at infinity (here we use the fact that $n > 6$). The lines $\ell_{34}$ and $\ell_{25}$ are parallel and meet at a point $C$ on the line at infinity.

Consider the two triples of lines $\ell_{03}, \ell_{14}, \ell_{25}$ and $\ell_{05}, \ell_{12}, \ell_{34}$. These two triples of lines meet at nine points: $a_0, \ldots, a_5$ and $A, B, C$.

We will use a generalization of Pappus theorem called Chasles theorem [4]. This classical result states that if three lines intersect three other lines in nine points, then any cubic curve passing through 8 of the intersection points must pass also through the ninth. See Theorem 4.1 in [6] for more details about the history of this result and more references. Let $C(x, y)$ denote the quadric $C$ as a polynomial in $x$ and $y$. Let $\ell^*$ denote the line at infinity. Therefore, the polynomial $C(x, y)\ell^*$ passes through all eight points $a_0, \ldots, a_4$ and $A, B, C$. Therefore, by Chasles theorem, $C(x, y)\ell^*$ passes also through $a_5$. Because $a_5$ does not lie on $\ell^*$ we conclude that $C$ passes through $a_0, \ldots, a_5$, as desired.

Having shown that the points in $V \cup (-V)$ lie on a quadric $C$ we claim that $C$ is an ellipse. Indeed, notice that $C$ and $-C$ intersect in $m = 2(n-1)$ points. For $n > 3$ this is possible only if $C = -C$. This shows that $C$ is not a parabola. If it is a hyperbola, then $O$ must be the center of it but then the points of $V \cup (-V)$ cannot lie in convex position for $n > 3$. For a similar reason $C$ cannot be a union of two lines.

Therefore, $C$ must be an ellipse. By applying a linear transformation, we may assume that $C$ is a circle. Because for every $i$ the edge $a_i a_{i+1}$ is parallel to the line through $a_{i-1}$ and $a_{i+2}$ we conclude that the distance between $a_i$ and $a_{i-1}$ is equal to the distance between $a_{i+1}$ and $a_{i+2}$. This, together with the fact that $Q$ is centrally symmetric, imply that all the distances $a_i a_{i+1}$ are equal. Hence $Q$ is a regular polygon centered at the origin and consequently $V$ is the set of vertices of a regular $(n-1)$-gon, centered at the origin. Therefore, we have $\sum_{i=1}^{n-1} v_i = 0$. Recall that $\sum_{i=1}^{n} v_i = 0$. From here we conclude $v_n = 0$, as desired.

This concludes the proof of Theorem 1.1. □

## 3 Proof of Theorem 1.2

In this section we provide a short algebraic proof to the following theorem conjectured by Erdős and Purdy:

**Theorem 3.1.** *Let $P$ be a set of $n > 6$ points in general position in the projective plane. Suppose that $R$ is a line blocker for $P$. Then $|R| \geq n$.*

Notice that here we excluded the case $n = 6$ compared to Theorem 1.2, as we remarked in the introduction.

It will be more convenient for us to consider the dual theorem using standard duality of points and lines in the plane.

**Theorem 3.2.** *Let $L$ be a set of $n > 6$ lines in general position in the projective plane. Suppose that $R$ is a set of red lines, different from the lines in $L$ such that every intersection point of two lines in $L$ is incident to a line in $R$. Then $|R| \geq n$.*

As we observed already in the introduction, Theorem 1.2 (and consequently also Theorem 3.2) is easily seen to be true if $n$ is odd. Therefore, the challenge in the proof of Theorem 3.2 is the case when $n$ is even. We shall therefore assume in the proof that $n \geq 8$ is even.

**Proof of Theorem 3.2.** Denote the lines in $L$ by $\ell_1, \ldots, \ell_n$. We may assume that no two of the lines in $L \cup R$ are parallel (for example by applying a generic projective transformation). We think of each $\ell_i$ as a linear polynomial $\ell_i(x, y) = a_i x + b_i y + c_i$, in the variables $x$ and $y$, whose set of zeroes is the line represented by $\ell_i$. We remark that we use affine coordinates although we sometimes refer to the projective plane.

Assume to the contrary that $|R| = n - 1$ and denote by $r_1, \ldots, r_{n-1}$ the lines in $R$, again considered as linear polynomials in the two variables $x$ and $y$. Specifically, we write $r_i = r_i(x, y) = e_i x + f_i y + g_i$.

With a slight abuse of notation, we denote by $R$ the polynomial

$$R = R(x, y) = r_1(x, y) r_2(x, y) \cdots r_{n-1}(x, y)$$

and observe that the degree of $R$ is $n - 1$. Similarly, let $P$ denote the polynomial

$$P = P(x, y) = \ell_1(x, y) \ell_2(x, y) \cdots \ell_n(x, y).$$

For every $i = 1, \ldots, n$, we denote by $P_i(x, y)$ the polynomial $P/\ell_i$, that is, the product of all the polynomials $\ell_1, \ldots, \ell_n$ except for $\ell_i$. We note that the degree of every $P_i$ is equal to $n - 1$.

Fix $1 \leq i \leq n$. Consider the polynomial $P_i$ restricted to the line $\ell_i$ and notice that it vanishes at all intersection points of $\ell_i$ with the other lines in $L$. Notice that also the polynomial $R$ restricted to $\ell_i$ vanishes on the same $n-1$ intersection points. Because both polynomials $P_i$ and $R$ are of degree $n-1$ we conclude that there is a nonzero $\alpha_i$ such that $\alpha_i P_i$ and $R$ are identical if restricted to the line $\ell_i$. It follows that $\ell_i$ is a factor of $\alpha_i P_i - R$. We now observe that $\ell_i$ must also be a factor of $(\sum_{i=1}^n \alpha_i P_i) - R$ (simply because $\ell_i$ is a factor of every $P_j$ for $j \neq i$). Because this is true for $i = 1, \ldots, n$ and because the degree of $(\sum_{i=1}^n \alpha_i P_i) - R$ is smaller than or equal to $n-1$, we conclude that $(\sum_{i=1}^n \alpha_i P_i) - R = 0$.

Consider any two distinct lines from $L$, say $\ell_i$ and $\ell_j$. Let $A = A_{ij}$ denote the intersection point of the two lines $\ell_i$ and $\ell_j$. Let $k$ be the index such that $r_k$ is the line in $R$ passing through $A$. Consider the polynomial equation

$$\alpha_1 P_1 + \cdots + \alpha_n P_n - R = 0. \tag{1}$$

The partial derivatives (with respect to $x$ and with respect to $y$) of the left hand side of (1), must be equal to 0, at any point. This is true in particular for the point $A$. Notice that $\frac{\partial}{\partial y} P_t(A) = 0$ and $\frac{\partial}{\partial x} P_t(A) = 0$ for every $t$ different than $i$ and $j$. Denote by $P_{ij}$ the polynomial that is the product of all polynomials $\ell_1, \ldots, \ell_n$ except for $\ell_i$ and $\ell_j$. Denote by $R_k$ the polynomial $R/r_k$.

Recall that $\ell_i(x, y) = a_i x + b_i y + c_i$, $\ell_j(x, y) = a_j x + b_j y + c_j$, and $r_i(x, y) = e_i x + f_i y + g_i$. We have

$$\begin{aligned} \frac{\partial}{\partial x} P_i(A) &= a_j P_{ij}(A), \\ \frac{\partial}{\partial x} P_j(A) &= a_i P_{ij}(A), \\ \frac{\partial}{\partial x} R(A) &= e_k R_k(A). \end{aligned}$$

Therefore, taking the partial derivative in the direction of the $x$-axis of the left hand side of (1) and equating it to 0 we get

$$\alpha_i a_j P_{ij}(A) + \alpha_j a_i P_{ij}(A) - R_k(A) e_k = 0. \tag{2}$$

Similarly, by considering the partial derivative in the direction of the $y$-axis of the left hand side of (1) and equating it to 0 we get

$$\alpha_i b_j P_{ij}(A) + \alpha_j b_i P_{ij}(A) - R_k(A) f_k = 0. \tag{3}$$

Observe that $P_{ij}(A) \neq 0$ and $R_k(A) \neq 0$. We recall that both $\alpha_i$ and $\alpha_j$ are nonzero.

Dividing both equations (2) and (3) by $\alpha_i \alpha_j P_{ij}(A)$ we get

$$\begin{aligned} \frac{1}{\alpha_j} a_j + \frac{1}{\alpha_i} a_i &= \frac{R_k(A)}{\alpha_i \alpha_j P_{ij}(A)} e_k, \\ \frac{1}{\alpha_j} b_j + \frac{1}{\alpha_i} b_i &= \frac{R_k(A)}{\alpha_i \alpha_j P_{ij}(A)} f_k. \end{aligned}$$

This analysis is valid for every $i \neq j$. For $i = 1, \ldots, n$ denote by $v_i$ the vector $\frac{1}{\alpha_i}(a_i, b_i)$. For $i = 1, \ldots, n-1$ denote by $u_i$ the vector $(e_i, f_i)$. Observe that because we assume that no two lines among $\ell_1, \ldots, \ell_n$ and $r_1, \ldots, r_{n-1}$ are parallel, then every pair of vectors from $v_1, \ldots, v_n$ and $u_1, \ldots, u_{n-1}$ are linearly independent.

For every $i, j, k$ such that $i \neq j$ and $\ell_i$ and $\ell_j$ meet at a point that is incident to $r_k$, we have that $v_i + v_j$ is a nonzero vector in the linear span of $u_k$. Moreover, if $j' \neq j$, then $v_i + v_{j'}$ is in the direction of some $u_{k'}$ different from $u_k$. The contradiction now follows from Theorem 1.1. □

## 4 The case $n = 6$ in Theorem 1.1.

In this section we will show that Theorem 1.1 cannot be extended to $n = 6$. Surprisingly, it is quite challenging to find a counterexample for the case $n = 6$ in Theorem 1.1.

The only place in the proof of Theorem 1.1 that fails to be true for $n = 6$ is where we need to show that the vertices of $Q$ (the convex hull of $V \cup (-V)$) lie on a quadric. In fact, a positive answer to the following statement could be enough to conclude also the case $n = 6$:

Suppose $a_0, a_1, \ldots, a_9$ are 10 vertices of a centrally symmetric convex polygon $Q$, indexed according to their clockwise order on the boundary of $Q$. Assume that for every $0 \leq i \leq 9$ that the diagonal $a_{i-1}a_{i+2}$ is parallel to $a_i a_{i+1}$ (and therefore also to $a_{i+5}a_{i-4}$ and to $a_{i+4}a_{i-3}$ because $Q$ is centrally symmetric). Does this imply that $a_0, \ldots, a_9$ lie on a quadric (in fact an ellipse)?

A little surprisingly (at least to the author) it turns out that the answer to this question is NO. This was communicated to me by Francisco L. Santos [12] who was able to construct a counterexample using Geogebra.

As we will show in this section, even more surprisingly, not only does the proof of Theorem 1.1 fail for the case $n = 6$, but also the statement is not true for $n = 6$. We will construct a counterexample for the case $n = 6$. Our construction is explicit and in this sense it could be enough to introduce a counterexample of six vectors $v_0, \ldots, v_5$ that satisfy the conditions of Theorem 1.1 but at the same time $S(V)$ does not have more than five vectors each two of which are linearly independent. We will do this at the end of this section. Nevertheless, we choose to present here the way in which we found these counterexamples, together with some very nice observations and claims of independent interest. We will be able to generate infinitely many (essentially different) such counterexamples.

When coming to analyze the case $n = 6$, and in particular if we wish to find a counterexample, we do have some information from the proof of Theorem 1.1, where we assumed (to the contrary) that a counterexample exists. Recall that if $V = \{v_0, \ldots, v_5\}$ is a set of six vectors that can serve as a counterexample, then for every distinct $i, j_1, j_2$, the vectors $v_i + v_{j_1}$ and $v_i + v_{j_2}$ are linearly independent. We defined the polygon $Q$, which is the convex hull of $V \cup -V$. Under the contrary assumption,

$Q$ is a 10-gon whose vertices are without loss of generality $\pm v_0, \ldots, \pm v_4$. Again without loss of generality we may assume that $v_0, \ldots, v_4$ appear in this clockwise cyclic order on the boundary of $Q$. As we have seen in the proof of Theorem 1.1, if $V$ is indeed a counterexample for the case $n = 6$, then we must have that for $i = 0, \ldots, 4$ the pair of vectors $v_i + v_{i+1}$ and $v_{i+2} + v_{i+4}$ are linearly dependent. The sum of indices here is modulo 5.

One key observation in the way to find a counterexample for the case $n = 6$ is the following.

**Claim 4.1.** *Assume $v_0, \ldots, v_4 \in \mathbb{R}^2$ satisfy the following conditions:*

- *For $i = 0, \ldots, 4$ the pair of vectors $v_i + v_{i+1}$ and $v_{i+2} + v_{i+4}$ are linearly dependent.*
- *No two of $v_0, v_1, v_2, v_3, v_4, -(v_0 + v_1 + v_2 + v_3 + v_4)$ are linearly dependent (in particular, $-(v_0 + v_1 + v_2 + v_3 + v_4) \neq 0$) .*

*Then $V = \{v_0, v_1, v_2, v_3, v_4, -(v_0 + v_1 + v_2 + v_3 + v_4)\}$ forms a counterexample for the case $n = 6$ in Theorem 1.1.*

**Proof.** Set $v^* = -(v_0 + v_1 + v_2 + v_3 + v_4)$. Consider the following five perfect matchings of the vectors in $V$. For $i = 0, 1, 2, 3, 4$ we let $M_i$ be the perfect matching $M_i = \{\{v_i, v_{i+1}\}, \{v_{i-1}, v_{i+2}\}, \{v^*, v_{i+3}\}\}$, where the summation of indices is taken modulo 5. It is easy to observe that these are five disjoint perfect matchings that together contain all pairs of vectors in $V$. For every $i = 0, 1, 2, 4$ we have that the three sums $v_i + v_{i+1}$, $v_{i+2} + v_{i+4}$, and $v^* + v_{i+3}$ are pairwise proportional. This is because by our assumption $v_i + v_{i+1}$ and $v_{i+2} + v_{i+4}$ are linearly dependent, while $v^* + v_{i+3} = -(v_0 + v_1 + v_2 + v_3 + v_4) + v_{i+3} = -(v_i + v_{i+1}) - (v_{i+2} + v_{i+4})$. For $i = 0, \ldots, 4$ we denote by $m_i$ the line through the origin that contains all three sums $v_i + v_{i+1}$, $v_{i+2} + v_{i+4}$, and $v^* + v_{i+3}$.

We conclude that in $S(V)$ there are at most five vectors, each two of which are linearly independent. This is because, by the pigeonhole principle, out of every six pairs of vectors, two pairs must belong to the same matching $M_i$ and then their sums are proportional.

It is left to show that for distinct $i, j_1, j_2$ the sums $v_i + v_{j_1}$ and $v_i + v_{j_2}$ are not proportional. Clearly, $\{v_i, v_{j_1}\}$ and $\{v_i, v_{j_2}\}$ belong to two different matchings $M_x$ and $M_y$, respectively. We observe that the union of every two matchings and in particular $M_x \cup M_y$ must be a cycle of length 6. If we assume to the contrary that $v_i + v_{j_1}$ and $v_i + v_{j_2}$ are linearly dependent, then the six sums of pairs of matched vectors in $M_x \cup M_y$ are all proportional to one another and to a fixed vector $u$. This implies that $v_0, \ldots, v_4$, and $v^*$ must lie on two (parallel) lines $\ell_1$ and $\ell_2$, equidistant from the origin and parallel to the line spanned by $u$. This is because the line parallel to $u$ must pass through the midpoints of the segments connecting the two vectors in every pair in $M_x \cup M_y$.

From here it is not difficult to verify that the only possibility is that the vectors $v_0, \ldots, v_4$ and $v^*$ are arranged centrally symmetrically on the two parallel lines yielding a contradiction as we assume that no two of the vectors $v_0, \ldots, v_4, v^*$ are proportional. We present the details of this argument now.

Denote arbitrarily by $a_1, a_2, a_3$ the three points (vectors) among $v_0, \ldots, v_4$, and $v^*$ that lie on $\ell_1$. For $1 \leq i < j \leq 3$ let $c_{ij} = \frac{a_i + a_j}{2}$. Notice that $c_{12}, c_{13}, c_{23}$ are pairwise distinct.

Denote arbitrarily by $b_1, b_2$, and $b_3$ the three points (vectors) among $v_0, \ldots, v_4$, and $v^*$ that lie on $\ell_2$. For $1 \leq i < j \leq 3$ let $d_{ij} = \frac{b_i + b_j}{2}$. Notice that $d_{12}, d_{13}, d_{23}$ are pairwise distinct.

The points $\{c_{12}, c_{13}, c_{23}\} \cup \{d_{12}, d_{13}, d_{23}\}$ must lie on the union of the three lines $m_i$ different from $m_x$ and $m_y$. Because $\ell_1$ and $\ell_2$ are parallel and equidistant from $O$, we conclude that $\{d_{12}, d_{13}, d_{23}\} = -\{c_{12}, c_{13}, c_{23}\}$. Because $b_1, b_2, b_3$ are uniquely determined by the equalities $d_{ij} = \frac{b_i + b_j}{2}$ for $1 \leq i < j \leq 3$ we conclude that $\{b_1, b_2, b_3\} = -\{a_1, a_2, a_3\}$. This is a contradiction, as we assume that every two of $v_0, v_1, v_2, v_3, v_4, -(v_0 + v_1 + v_2 + v_3 + v_4)$ are linearly dependent. □

**Lemma 4.2.** *Let $v_0, \ldots, v_4$ be 5 vectors in $\mathbb{R}^2$ no two of which are linearly dependent. Assume that for $i = 0, 1, 2$ the following is true: There exists $\alpha_i$ such that $v_i + v_{i+1} = -\alpha_i(v_{i+2} + v_{i+4})$ (in particular the two vectors $v_i + v_{i+1}$ and $v_{i+2} + v_{i+4}$ are linearly dependent). Assume moreover that $\alpha_1 = (1 - \alpha_0)(1 - \alpha_2)$. Then there exists $\alpha_3$ and $\alpha_4$ such that $v_i + v_{i+1} = -\alpha_i(v_{i+2} + v_{i+4})$ for $i = 3$ and $i = 4$. Moreover, for every $i = 0, 1, 2, 3, 4$ we have $\alpha_i = (1 - \alpha_{i-1})(1 - \alpha_{i+1})$. The summation of indices is done modulo 5.*

**Proof.** Let $W$ be the vector space $W = \{(a_0, \ldots, a_4) \mid \sum_{i=0}^{4} a_i v_i = 0\}$. Because $v_0, \ldots, v_4$ span $\mathbb{R}^2$, the dimension of $W$ is equal to 3.

By our assumption, the vectors $w_0 = (1, 1, \alpha_0, 0, \alpha_0)$, $w_1 = (\alpha_1, 1, 1, \alpha_1, 0)$, and $w_2 = (0, \alpha_2, 1, 1, \alpha_2)$ are in $W$.

We claim that $w_0, w_1$, and $w_2$ are linearly independent. This is regardless of the assumption that $\alpha_1 = (1 - \alpha_0)(1 - \alpha_2)$. To see this, observe that $w_0$ and $w_2$ are clearly linearly independent. Assume to the contrary that $w_1$ is equal to a linear combination of $w_0$ and $w_2$, that is,

$$w_1 = aw_0 + bw_2. \tag{4}$$

By considering the first coordinate of equality (4), we get $a = \alpha_1$. By considering the fourth coordinate of equality (4), we get $b = \alpha_1$. By considering the fifth coordinate of equality (4), we get $\alpha_1(\alpha_0 + \alpha_2) = 0$. Clearly, $\alpha_1 \neq 0$ (or else $w_1 = 0$, which is not the case) and therefore $\alpha_0 = -\alpha_2$. By considering the second coordinate of equality (4), we get $\alpha_1(1 + \alpha_2) = 1$. By considering the third coordinate of equality (4), we get $\alpha_1(1 + \alpha_0) = 1$. This is possible only if $\alpha_0 = \alpha_2 = 0$. However, this is impossible as we assume $v_0 + v_1 = \alpha_0(v_2 + v_4)$. If $\alpha_0 = 0$, then $v_0 + v_1 = 0$, contrary to our assumption that every two vectors in $V$ are linearly independent.

Having shown that $w_0, w_1, w_2$ are linearly independent, we conclude that they form a basis for the space $W$.

We will now show that that there is a unique $\alpha_3$ such that the vector $(\alpha_3, 0, \alpha_3, 1, 1)$ is a linear combination of $w_0, w_1, w_2$. We will also show that $\alpha_3$ is given by $\alpha_2 = (1-\alpha_1)(1-\alpha_3)$.

We would like to find $a_0, a_1$, and $a_2$, and $\alpha_3$ such that $a_0w_0 + a_1w_1 + a_2w_2 = (\alpha_3, 0, \alpha_3, 1, 1)$. It is easy to uniquely find $a_0, a_1$, and $a_2$ that will satisfy the equality in the second, fourth, and fifth coordinates of this equality (that are independent of $\alpha_3$). A direct calculation shows that

$$\begin{aligned} a_0 &= \frac{1-\alpha_2-\alpha_1\alpha_2}{\alpha_0+\alpha_1\alpha_2(1-\alpha_0)} \\ a_1 &= \frac{\alpha_2(1-\alpha_0)-1}{\alpha_0+\alpha_1\alpha_2(1-\alpha_0)} \\ a_2 &= \frac{\alpha_0+\alpha_1}{\alpha_0+\alpha_1\alpha_2(1-\alpha_0)}. \end{aligned}$$

Then indeed, $a_0w_0+a_1w_1+a_2w_2 = (\alpha_3, 0, \alpha_3, 1, 1)$ for $\alpha_3 = \frac{1-\alpha_1-\alpha_2-\alpha_0\alpha_1\alpha_2}{\alpha_0+\alpha_1\alpha_2(1-\alpha_0)}$. Under our assumption that $\alpha_1 = (1-\alpha_0)(1-\alpha_2)$, we get $\alpha_3 = \frac{\alpha_0-\alpha_0\alpha_2}{\alpha_0+\alpha_2-\alpha_0\alpha_2}$. Because the vector $(\alpha_3, 0, \alpha_3, 1, 1)$ is in $W$ we conclude that $v_3 + v_4 = -\alpha_3(v_0 + v_2)$. It is easy to verify that $\alpha_2 = (1-\alpha_1)(1-\alpha_3)$. We can now apply the same argument for the vectors $v'_0 = v_1, v'_1 = v_2, v'_2 = v_3, v'_3 = v_4$, and $v'_4 = v_0$ with $\alpha'_0 = \alpha_1, \alpha'_1 = \alpha_2$, and $\alpha'_2 = \alpha_3$ and conclude that there is $\alpha'_3$ such that $v'_3 + v'_4 = -\alpha'_3(v'_0 + v'_2)$. If we define $\alpha_4 = \alpha'_3$ we get $v_4 + v_0 = -\alpha_4(v_1 + v_3)$. Moreover, $\alpha_3 = \alpha'_2 = (1-\alpha'_1)(1-\alpha'_3) = (1-\alpha_2)(1-\alpha_4)$. A direct calculation shows that we also have $\alpha_4 = (1-\alpha_0)(1-\alpha_3)$ and $\alpha_0 = (1-\alpha_4)(1-\alpha_1)$. (The last two equalities follow also by two repeated application of the same argument for the vectors $v_2, v_3, v_4, v_0, v_1$ and for $v_3, v_4, v_0, v_1, v_2$.) □

Although we will not use this fact, it is not difficult to check that also the following converse of Lemma 4.2 is true.

**Lemma 4.3.** *Let $v_0, \ldots, v_4$ be 5 vectors in $\mathbb{R}^2$ no two of which are linearly dependent. Assume that for $i = 0, 1, 2, 3, 4$ the following is true: There exists $\alpha_i$ such that $v_i + v_{i+1} = -\alpha_i(v_{i+2} + v_{i+4})$ (this is equivalent to saying that the two vectors $v_i + v_{i+1}$ and $v_{i+2} + v_{i+4}$ are linearly dependent). Then necessarily $\alpha_1 = (1-\alpha_0)(1-\alpha_2)$.*

**Proof.** We start exactly as in the proof of Lemma 4.2. Let $W$ be the vector space $W = \{(a_0, \ldots, a_4) \mid \sum_{i=0}^{4} a_i v_i = 0\}$. Because $v_0, \ldots, v_4$ span $\mathbb{R}^2$, the dimension of $W$ is equal to 3.

By our assumption, the vectors $w_0 = (1, 1, \alpha_0, 0, \alpha_0)$, $w_1 = (\alpha_1, 1, 1, \alpha_1, 0)$, and $w_2 = (0, \alpha_2, 1, 1, \alpha_2)$ are in $W$.

Recall that the fact that $w_0, w_1$, and $w_2$ are linearly independent was part of the proof of Lemma 4.2 and this part did not rely on any relation between $\alpha_0, \alpha_1$, and $\alpha_2$.

Because $w_0, w_1$, and $w_2$ form a basis for $W$, then $w_3$ is equal to a linear combination of $w_0, w_1$, and $w_2$. As in the proof of Lemma 4.2, we find $w_3 = a_0w_0+a_1w_1+a_2w_2$, where

$$a_0 = \frac{1-\alpha_2-\alpha_1\alpha_2}{\alpha_0+\alpha_1\alpha_2(1-\alpha_0)}$$
$$a_1 = \frac{\alpha_2(1-\alpha_0)-1}{\alpha_0+\alpha_1\alpha_2(1-\alpha_0)}$$
$$a_2 = \frac{\alpha_0+\alpha_1}{\alpha_0+\alpha_1\alpha_2(1-\alpha_0)}.$$

We now consider the first and third coordinate of $w_3$ and observe that they must be equal (in fact they are both equal to $\alpha_3$). From the equality $w_3 = a_0w_0 + a_1w_1 + a_2w_2$ it now follows that $a_0 + \alpha_1 a_1 = a_0\alpha_0 + a_1 + a_2$. Plugging in the expressions for $a_0, a_1$, and $a_2$ in terms of $\alpha_0, \alpha_1$, and $\alpha_2$, we get the relation $\alpha_1 = (1-\alpha_0)(1-\alpha_2)$, as desired. □

The following result, which we discovered in the course of proving Lemma 4.2, is stated here, although it is not used in this paper:

**Lemma 4.4.** *Assume $a_0, \ldots, a_{n-1}$ are $n$ real numbers different from 0 that satisfy $a_i = (1-a_{i-1})(1-a_{i+1})$ for every $i$ (summation of indices is modulo $n$). Then $n$ must be divisible by 5, unless $a_0 = a_1 = a_2 = \ldots = a_{n-1} = \frac{3\pm\sqrt{5}}{2}$.*

We remark that as we will see in the proof, there are infinitely many (two degrees of freedom) distinct sequences $a_0, \ldots, a_{n-1}$ that satisfy the conditions of Lemma 4.4. Lemma 4.4 was used in one of the problems in the Grossman Math Olympiad in Israel 2020.

**Proof.** Let $x = a_0$ and $y = a_2$. Then $a_1 = (1-a_0)(1-a_2) = (1-x)(1-y)$. We may assume that both $x$ and $y$ are different from 1 and from 0 because we assume that $a_i \neq 0$ for every $i$.

We know that for every $i$ we have $a_i = (1-a_{i-1})(1-a_{i+1})$. From here we conclude that for every $i$ we have

$$a_{i+1} = \frac{1-a_i-a_{i-1}}{1-a_{i-1}}. \tag{5}$$

We can now find $a_3$ in terms of $x$ and $y$ using (5): $a_3 = \frac{1-a_2-a_1}{1-a_1} = \frac{x-xy}{x+y-xy}$. In the same way $a_4 = \frac{1-a_3-a_2}{1-a_2}$. After substituting the expressions of $a_3$ and $a_2$ in terms of $x$ and $y$ and simplifying, we get $a_4 = \frac{y-xy}{x+y-xy}$. Now moving on to $a_5$ we get $a_5 = \frac{1-a_4-a_3}{1-a_3}$. After substituting $a_3$ and $a_4$ and simplifying, we get $a_5 = x$.

We can continue and check that $a_6 = (1-x)(1-y)$ and $a_7 = y$ but this follows already from our calculations above using the symmetry between $x$ and $y$. We conclude that the sequence $a_0, a_1, \ldots, a_{n-1}$ must be periodic with period equal to 5, that is, $a_{i+5} = a_i$ for every $i$.

If $n$ is not divisible by 5, then we must have $a_0 = a_1 = \ldots = a_{n-1}$, because we know that $a_i = a_{i+5}$ for every $i$. If we denote this common value by $x$, we see that we must have $x = (1-x)^2$. This equation has only two solutions $\frac{3\pm\sqrt{5}}{2}$. □

We shall now continue with the analysis of the case $n = 6$ in Theorem 1.1. Combining Lemma 4.2 and Claim 4.1, we can get a method for generating a counterexample to the case $n = 6$ in Theorem 1.1. Indeed, assume we can find $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 = (1-\alpha_0)(1-\alpha_2)$ and vectors $v_0, v_1, v_2, v_3, v_4 \in \mathbb{R}^2$, such that every two of $v_0, v_1, v_2, v_3, v_4, \sum_{i=0}^{4} v_i$ are linearly independent and such that for $i = 0, 1, 2$ we have $v_i + v_{i+1} = -\alpha_i(v_{i+2} + v_{i+4})$. Then by Lemma 4.2, $v_0, \ldots, v_4$ satisfy the conditions of Claim 4.1 and therefore $v_0, v_1, v_2, v_3, v_4$ together with $-\sum_{i=0}^{4} v_i$ form a counterexample of size six to Theorem 1.1.

We will now follow this recipe and prove that such a construction does exist. Let $\epsilon > 0$ be a very small positive number to be described later. One can take $\epsilon = \frac{1}{1000}$. We take $v_1 = (1, 1)$ and $v_2 = (1, -1)$. We take $v_4 = (-1-\epsilon, \epsilon)$. It remains to choose $v_0$ and $v_3$. Denote by $A_1$ the midpoint of the segment connecting $v_4$ to $v_1$, that is $A_1 = \frac{1}{2}(v_4 + v_1) = (-\frac{\epsilon}{2}, \frac{1+\epsilon}{2})$. Let $m_1$ be the line through $O$ and $A_1$ and let $\ell_1$ be the line parallel to $m_1$ below $m_1$ whose distance from $m_1$ is equal to the distance of $v_2$ from $m_1$. In order that $v_2 + v_3$ and $v_1 + v_4$ will be linearly dependent, $v_3$ must lie on $\ell_1$. The line $\ell_1$ has slope equal to $-\frac{1+\epsilon}{\epsilon}$ and it intersects that $x$-axis at $(-1 + \frac{\epsilon}{1+\epsilon}, 0)$. It is equal to the line $y = -\frac{1+\epsilon}{\epsilon}(x + 1 - \frac{\epsilon}{1+\epsilon})$.

Similarly, let $A_2 = \frac{1}{2}(v_4 + v_2) = (-\frac{\epsilon}{2}, \frac{-1+\epsilon}{2})$. Let $m_2$ be the line through $O$ and $A_2$. Let $\ell_2$ be the line parallel to $m_2$ above $m_2$, whose distance from $m_2$ is equal to the distance of $v_1$ from $m_2$. We observe that $v_0$ must lie on $\ell_2$. The line $\ell_2$ has a slope equal to $\frac{1-\epsilon}{\epsilon}$ and it intersects the $x$-axis at $-1+\frac{\epsilon}{1-\epsilon}$. It is the line $y = \frac{1-\epsilon}{\epsilon}(x+1-\frac{\epsilon}{1-\epsilon})$.

We will choose $v_0$ and $v_3$ in the following way. We will choose a number $h > 0$ in a way that will be specified shortly and take $v_0$ to the the point on $\ell_2$ with $y$ coordinate that is equal to $h$. That is, $v_0 = ((h - \frac{1-2\epsilon}{\epsilon})\frac{\epsilon}{1-\epsilon}, h)$. Then we take $v_3$ to the the point on $\ell_1$ with $y$ coordinate that is equal to $-h$. That is, $v_3 = ((h - \frac{1}{\epsilon})\frac{\epsilon}{1+\epsilon}, -h)$. By choosing $v_0$ and $v_3$ in this way we guarantee that $v_1 + v_2 = -\alpha_1(v_0 + v_3)$ for some $\alpha_1 \in \mathbb{R}$. Let $\alpha_0$ be such that $v_0 + v_1 = -\alpha_0(v_4 + v_2)$. Let $\alpha_2$ be such that $v_2 + v_3 = -\alpha_2(v_4 + v_1)$. It remains to show that we can choose $h$ such that $\alpha_1 = (1-\alpha_0)(1-\alpha_2)$.

Notice that $v_1 + v_2 = (2, 0)$ and $v_0 + v_3 = (h\frac{2\epsilon}{1-\epsilon^2} - \frac{2-2\epsilon-2\epsilon^2}{1-\epsilon^2}, 0)$. Therefore, $\alpha_1 = \frac{1-\epsilon^2}{-h\epsilon+1-\epsilon-\epsilon^2}$.

What about $(1-\alpha_0)(1-\alpha_2)$? We have $v_0 + v_1 = ((h - \frac{1-2\epsilon}{\epsilon})\frac{\epsilon}{1-\epsilon} + 1, h + 1)$ and $v_4 + v_2 = (-\epsilon, -1+\epsilon)$. We know already that $v_0 + v_1 = -\alpha_0(v_4 + v_2)$. Therefore, $\alpha_0 = \frac{h+1}{1-\epsilon}$.

Similarly, $v_2 + v_3 = ((h - \frac{1}{\epsilon})\frac{\epsilon}{1+\epsilon} + 1, -h-1)$ and $v_1 + v_4 = (-\epsilon, 1+\epsilon)$. Therefore, because $v_2 + v_3 = -\alpha_2(v_1 + v_4)$, we have $\alpha_2 = \frac{h+1}{1+\epsilon}$. We get $(1-\alpha_0)(1-\alpha_2) = \frac{h^2-\epsilon^2}{1-\epsilon^2}$.

We want to find $h$ such that $\alpha_1 = (1-\alpha_0)(1-\alpha_2)$. Substituting the expressions for $\alpha_0, \alpha_1$, and $\alpha_2$, we would like the following equality to hold:

$$\frac{1-\epsilon^2}{-h\epsilon+1-\epsilon-\epsilon^2} = \frac{h^2-\epsilon^2}{1-\epsilon^2}.$$

It is not difficult to solve this, after observing that $h = -1$ gives equality. We

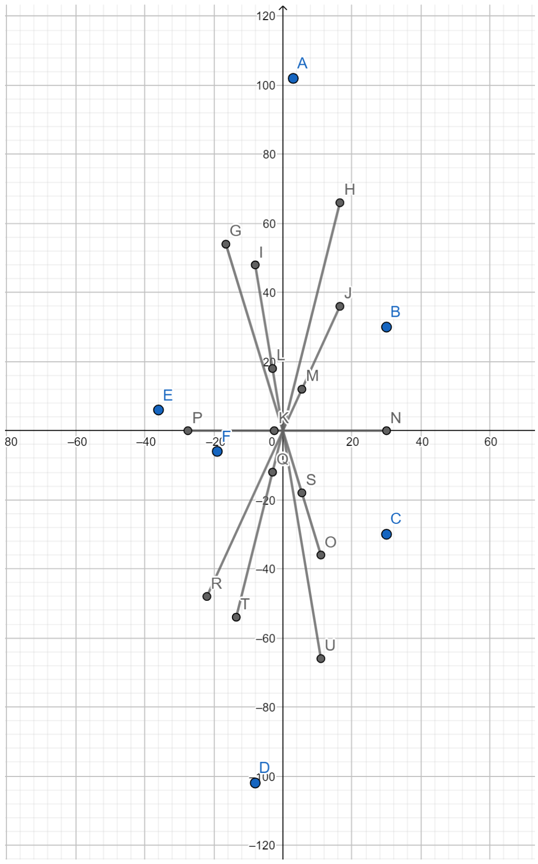


Figure 2: A counterexample to the case $n = 6$ in Theorem 1.1.

obtain the equation

$$h^2\epsilon - h(1-\epsilon^2) - (\epsilon^3 + \epsilon^2 - 1) = 0.$$

We get $h = \frac{1}{2\epsilon}(1 - \epsilon^2 + \sqrt{5\epsilon^4 + 4\epsilon^3 - 2\epsilon^2 - 4\epsilon + 1})$. Keeping in mind that the function $\sqrt{1+x} = 1 + \frac{1}{2}x + o(x)$, we observe that $h = \frac{1}{\epsilon} - 1 + o(1)$. This yields $v_0 = ((h - \frac{1-2\epsilon}{\epsilon})\frac{\epsilon}{1-\epsilon}, h) = (\epsilon + o(\epsilon), \frac{1}{\epsilon} - 1 + o(1))$. Similarly, $v_3 = ((h - \frac{1}{\epsilon})\frac{\epsilon}{1+\epsilon}, -h) = (-\epsilon + o(\epsilon), -\frac{1}{\epsilon} + 1 + o(1))$.

In particular, we see that when $\epsilon$ is very small every two of the vectors $v_0, v_1, v_2, v_3, v_4$ are linearly independent. What about $\sum_{i=0}^{4} v_i$?

We have $\sum_{i=0}^{4} v_i = (h\frac{2\epsilon}{1-\epsilon^2} + 1 - \epsilon - \frac{2-3\epsilon}{1-\epsilon^2}, \epsilon)$. Hence, $\sum_{i=0}^{4} v_i = (1 + o(1), \epsilon)$. Therefore, the only vector among $v_0, v_1, v_2, v_3, v_4$ that may be a scalar multiple of $\sum_{i=0}^{4} v_i$ is the vector $v_4$. However, in such a case, by comparing the $y$-coordinate, we must have $v_4 = \sum_{i=0}^{4} v_i$, or in other words, $v_0 + v_1 + v_2 + v_3 = 0$. But this is not the case because $\alpha_0 \neq 1$.

This completes the proof. We can take a specific value of $\epsilon$ to get a concrete counterexample for the case $n = 6$ in Theorem 1.1. Taking $\epsilon = \frac{1}{5}$ yields a particularly nice example (in the sense that all the vectors are rational): $h = \frac{17}{5}$ and $v_0 = (\frac{1}{10}, \frac{17}{5})$, $v_1 = (1, 1)$, $v_2 = (1, -1)$, $v_3 = (-\frac{4}{15}, -\frac{17}{5})$, and $v_4 = (-\frac{6}{5}, \frac{1}{5})$. Finally, $v_5 = -(v_0 + \ldots + v_4) = (-\frac{19}{30}, -\frac{1}{5})$. One can directly check that for $V = \{v_0, v_1, \ldots, v_5\}$ the set

$S(V)$ contains at most five vectors no two of which are proportional. Moreover, for every distinct $i, j, k$ the vectors $v_i + v_j$ and $v_i + v_k$ are linearly independent. Figure 2 contains the points $v_0, v_1, v_2, v_3, v_4$, and $v_5$, multiplied by a factor of 30, drawn (using the platform of Geogebra) as $A, B, C, D, E$, and $F$, respectively. The gray points correspond to all the possible midpoints of segments connecting a pair of the points $A, B, C, D, E$, and $F$. These precisely correspond to $\frac{1}{2}(v_i + v_j)$. One can see that all the gray points lie on a union of 5 gray lines through the origin. This shows that $S(V)$ contains at most five vectors no two of which are proportional.

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