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On the construction of weighing matrices using negacyclic matrices

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Abstract

We construct weighing matrices by 2-suitable negacyclic matrices, and study the conjecture by J.S. Wallis in 1972 that "For every $n \equiv 2 \pmod 4$, there exist weighing matrices $W(2n,\ w)$ constructed from two circulant / negacyclic $(0, \pm 1)$ matrices of order n for every $0 < w \leq 2n$."

1 Introduction

A *weighing matrix* $W(n, w)$ is a $(0, \pm 1)$ square matrix of order n that satisfies $WW^T = wI_n$, where W^T is the transpose of the matrix W and I_n is the identity matrix of order n and w is the weight of the matrix. When $w = n$, a weighing matrix is a Hadamard matrix. When $w = n - 1$, a weighing matrix is a conference

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matrix, or a C-matrix. Delsarte, Goethals and Seidel [3] studied the types of *weighing matrix* of weights n and $n-1$, based on circulant and negacyclic matrices which we now define.

Throughout this paper indices for matrices and sequences begin with 0. A *negacyclic shift matrix* P is a square matrix of order n, in which all entries $p_{i,j}$ are defined as follows:

$$\begin{cases} p_{i,i+1} = 1, & i = 0, 1, \ldots, n-2, \\ p_{n-1,0} = -1, & \\ p_{i,j} = 0, & \text{otherwise.} \end{cases}$$

The *negacyclic shift matrix* has the form

$$P = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

It is easily checked that $P^n = -I_n,\ (P^i)^T = P^{-i}$, for all $i,\ j$.

We denote the first row of a square matrix $A = (a_{ij})$ of order n by $(a_0, a_1, \ldots, a_{n-1})$. Then the entries of A are defined as follows:

$$a_{i,j} = \begin{cases} a_{j-i}, & \text{where } 0 \le i \le j \le n-1, \\ -a_{n+j-i}, & \text{where } 0 \le j < i \le n-1. \end{cases}$$

We call A a *negacyclic matrix*. The matrix A can also be defined as $A = \sum_{i=0}^{n-1} a_i P^i$. It is obvious that A^T is a negacyclic matrix.

Definition 1.1 *(k-suitable negacyclic matrices)* The k negacyclic matrices $A_1, \ldots, A_k$ of order n are called k-suitable negacyclic matrices if

$$A_1 A_1^T + \cdots + A_k A_k^T = wI_n, \tag{1.1}$$

for an integer w.

Recall that a circulant matrix A is one of the form $A = \sum_{i=0}^{n-1} a_i C^i$, where $(a_0, a_1, \ldots, a_{n-1})$ is the first row of the square matrix A, and C is a cyclic shift matrix defined as

$$C = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

In this paper, we are interested in the case $k = 2$.

Geramita and Seberry [5] conjectured the following.

Conjecture 1.1 *For every $n > 0$ where n is odd, there exist weighing matrices $W(2n, w)$, where $w = a^2 + b^2$, with a and b integers.*

We also study the conjecture (J. S. Wallis [10]) below.

Conjecture 1.2 *For every $n \equiv 2 \pmod 4$, there exists a weighing matrix $W(2n, w)$ constructed from two circulant or two negacyclic $(0, \pm 1)$ matrices of order n, for every w with $0 \leq w \leq 2n$.*

The rest of the paper is organized as follows. In Section 2 we give some definitions. In Section 3 we study the relationships between Golay sequences, ternary complementary pairs, and suitable sequences. In Section 4 we construct weighing matrices by 2-suitable negacyclic matrices. In Appendices A and B we list some results of weighing matrices constructed by 2-suitable negacyclic matrices.

2 Preliminaries

Definition 2.1 The weight of a sequence $a = (a_0, \ldots, a_{n-1})$, denoted by $|a|$, is the total number of non-zero elements ($a_i \neq 0$).

For two $(0, \pm 1)$ sequences a and b of length n, let s and t, respectively, be their weights. The sum $w = s + t$ is called the total weight of a and b.

Definition 2.2 The weight of a circulant matrix (or negacyclic matrix) A, denoted by $|A|$, is the weight of its first row.

Definition 2.3 The Kronecker product of two sequences $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{m-1})$ is denoted by

$$a \otimes b = (a_0b_0, a_0b_1, \ldots, a_0b_{m-1}, \ldots, a_{n-1}b_0, a_{n-1}b_1, \ldots, a_{n-1}b_{m-1}).$$

Definition 2.4 The Kronecker product of two matrices

$$A = \begin{pmatrix} a_{0,0} & a_{0,1} & \ldots & a_{0,m-1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-1,0} & a_{n-1,1} & \ldots & a_{n-1,m-1} \end{pmatrix} \text{ and } B = \begin{pmatrix} b_{0,0} & b_{0,1} & \ldots & b_{0,s-1} \\ \ldots & \ldots & \ldots & \ldots \\ b_{r-1,0} & b_{r-1,1} & \ldots & b_{r-1,s-1} \end{pmatrix}$$

is denoted by

$$A \otimes B = \begin{pmatrix} a_{0,0}B & a_{0,1}B & \ldots & a_{0,m-1}B \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-1,0}B & a_{n-1,1}B & \ldots & a_{n-1,m-1}B \end{pmatrix}.$$

Definition 2.5 A non-periodic autocorrelation function (NPAF) $N_a(j)$ of a sequence $a = (a_0, \ldots, a_{n-1})$ is defined as

$$N_a(j) = \sum_{i=0}^{n-1-j} a_i a_{i+j}, \quad j = 0, \ldots, n-1.$$

Consider $(i + j)$ mod n; then we have:

Definition 2.6 A periodic autocorrelation function (PAF) $P_a(j)$ of a sequence $a = (a_0, \ldots, a_{n-1})$ is defined as

$$P_a(j) = \sum_{i=0}^{n-1} a_i a_{i+j}, \quad j = 0, \ldots, n-1,$$

where $i + j$ represents $(i + j)$ mod n.

Golay sequences were introduced by Golay in 1949 [4].

Definition 2.7 *(Golay sequences)* Two $(1, -1)$ sequences of length n, say a and b, are called Golay sequences (or Golay complementary sequences) if $N_a(i) + N_b(i) = 0$ for $0 < i < n$.

Definition 2.8 *(Ternary complementary pairs)* Two $(0, \pm 1)$ sequences of length n, say a and b, are called ternary complementary pairs (TCPs) if $N_a(i) + N_b(i) = 0$ for $0 < i < n$.

Obviously, Golay sequences are TCPs.

Sequences of length n with zero NPAF or zero PAF can form the first rows of circulant or negacyclic matrices which can be used to construct Hadamard matrices, orthogonal designs and weighing matrices. See [3, 8] for more details. Arasu, Leung, et al. [1] have done a complete search of circulant weighing matrices of order 16.

For two sequences $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$, we define

$$\langle a, b \rangle = \sum_{i=0}^{n-1} a_i b_i.$$

For a sequence $a = (a_0, \ldots, a_{n-1})$ of length n, we define a *nega-shift operator* $f(a)$ as

$$f(a) = (-a_{n-1},\ a_0, \ldots, a_{n-2}). \tag{2.1}$$

It is easy to see that $f^n(a) = -a$. Define a shift operator $s(a)$ as

$$s(a) = (a_{n-1},\ a_0, \ldots, a_{n-2}). \tag{2.2}$$

Then $s^n(a) = a$.

Let $x = (x_0, \ldots, x_{n-1})$; then

$$\langle x, f^j(x) \rangle = N_x(j) - N_x(n-j), \quad 0 \le j < n. \tag{2.3}$$

Theorem 2.1 *If a and b form a pair of Golay sequences or TCPs then the negacycles with first rows a and b form a pair of 2-suitable negacyclic matrices.*

Proof. Note that with these definitions in (2.3), if two sequences $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$, where $a_i, b_i \in \{0, \pm 1\}$, $0 \le i < n$, are Golay sequences or TCPs, then

$$N_a(j) + N_b(j) = 0, \quad 0 < j < n.$$

We have, for $0 < j < n$,

$$\begin{aligned}\langle a, f^j(a)\rangle + \langle b, f^j(b)\rangle &= (N_a(j) - N_a(n-j)) + (N_b(j) - N_b(n-j)) \\ &= (N_a(j) + N_b(j)) - (N_a(n-j) + N_b(n-j)) \\ &= 0.\end{aligned}$$

Let a and b be the first rows of two negacyclic matrices A and B. Then A and B are 2-suitable negacyclic matrices that satisfy (1.1) with $w = 2$. The proof is now complete. □

Example 2.1 Let $a = (1, 0, 1)$, $b = (1, 0, -1)$, where a and b are Golay complement sequences with zero NPAF for $j = 1, 2$. The two negacyclic matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

that are formed by the sequences a and b, are 2-suitable negacyclic matrices.

Golay sequences and TCPs have been studied by many researchers. See [2, **?**, 7] for more details.

Definition 2.9 *(k-suitable sequences)* The k $(0, \pm 1)$ sequences of length n, say a_1, $\ldots$, a_k, are called k-suitable sequences (k-SSs) for $0 < j \le \lfloor \frac{n}{2} \rfloor$ if

$$N_{a_1}(j) + \cdots + N_{a_k}(j) = N_{a_1}(n-j) + \cdots + N_{a_k}(n-j).$$

Clearly, TCPs are 2-suitable sequences, but the converse is not always true.

Example 2.2 Let $a = (1, 1, 1)$, $b = (1, 0, 1)$; here a and b are 2-suitable sequences, but not TCPs.

Let 0_m denote a sequence of length m with all elements zero.

Corollary 2.1 *Let $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$ be TCPs of length n. The sequences a' and b' are TCPs of length $l + m + n$ where*

$$a' = (a, 0_{m+l}) \text{ or } (0_l, a, 0_m) \text{ or } (0_{m+l}, a)$$

and

$$b' = (b, 0_{m+l}) \text{ or } (0_m, b, 0_l) \text{ or } (0_{m+l}, b).$$

Proof. Since $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$ are TCPs of length n, we have for $j = 0, \ldots, n-1$,

$$N_a(j) + N_b(j) = \sum_{i=0}^{n-1-j} a_i a_{i+j} + \sum_{i=0}^{n-1-j} b_i b_{i+j} = 0.$$

Without loss of generality, let $a' = (0_l, a, 0_m)$, $b' = (b, 0_{l+m})$. Then

$$\begin{aligned}
N_{a'}(j) + N_{b'}(j) &= \sum_{i=0}^{l+m+n-1-j} a'_i a'_{i+j} + \sum_{i=0}^{l+m+n-1-j} b'_i b'_{i+j} \\
&= (\sum_{i=0}^{l-1} a'_i a'_{i+j} + \sum_{i=l}^{l+n-1} a'_i a'_{i+j} + \sum_{i=l+n}^{l+n+m-1-j} a'_i a'_{i+j}) \\
&\quad + (\sum_{i=0}^{n-1} b'_i b'_{i+j} + \sum_{i=n}^{l+m+n-1-j} b'_i b'_{i+j}) \\
&= \sum_{i=l}^{l+n-1} a'_i a'_{i+j} + \sum_{i=0}^{n-1} b'_i b'_{i+j} \\
&= \sum_{i=0}^{n-1} a_i a_{i+j} + \sum_{i=0}^{n-1} b_i b_{i+j} \\
&= N_a(j) + N_b(j) \;=\; 0.
\end{aligned}$$

The proof is now complete. □

Lemma 2.1 *Let $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$ be the first rows of negacyclic matrices A and B, respectively, i.e.,*

$$A = \sum_{i=0}^{n-1} a_i P^i, \quad B = \sum_{i=0}^{n-1} b_i P^i,$$

where P is the negacyclic shift matrix of order n. Then A and B are 2-suitable negacyclic matrices if and only if a and b are 2-suitable sequences.

Proof. We have

$$\begin{aligned}
AA^T + BB^T &= \sum_{i=0}^{n-1}(a_i^2 + b_i^2) I_n + \sum_{i=1}^{n-1}(N_a(i) + N_b(i))(P^i + P^{2n-i}) \\
&= \sum_{i=0}^{n-1}(a_i^2 + b_i^2) I_n \\
&\quad + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}(N_a(i) + N_b(i) - N_a(n-i) - N_b(n-i))(P^i + P^{2n-i}),
\end{aligned}$$

where we use the fact that $P^i + P^{2m-i} = -P^{m+i} - P^{m-i}$, for $0 < i \leq \lfloor \frac{m}{2} \rfloor$. From the above it follows that A and B are 2-suitable negacyclic matrices if and only if, for $0 < i \leq \lfloor \frac{n}{2} \rfloor$,

$$N_a(i) + N_b(i) = N_a(n-i) + N_b(n-i),$$

which means that the two sequences a and b are 2-suitable sequences. □

Corollary 2.2 *The negacyclic matrices $A_1, \ldots, A_k$ are k-suitable negacyclic matrices if and only if the first rows of $A_1, \ldots, A_k$ are k-suitable sequences.*

Proof. Using the same method as in the proof of Lemma 2.1, one can easily prove that the corollary is true. □

Lemma 2.2 *If there exist 2-suitable sequences of length n with weight w, then there exist 2-suitable sequences of length mn with weight w for all $m > 0$.*

Proof. Suppose $a = (a_0, \ldots, a_{n-1})$, $b = (b_0, \ldots, b_{n-1})$ are 2-suitable sequences of length n with weight w. Set $c = (c_0, \ldots, c_{mn-1})$, $d = (d_0, \ldots, d_{mn-1})$, where

$$c_i = \begin{cases} a_j, & i = mj, \\ 0, & \text{otherwise}, \end{cases} \qquad d_i = \begin{cases} b_j, & i = mj, \\ 0, & \text{otherwise}, \end{cases} \tag{2.4}$$

for $0 \leq j < n$, $0 \leq i < mn$. It is easy to verify that c and d are 2-suitable sequences of length mn with weight w. □

Example 2.3 The sequences $a = (1, 1, 1)$ and $b = (1, 0, 1)$ are 2-suitable sequences of length 3 with weight 5. For $m = 2$, we can construct $c = (1, 0, 1, 0, 1, 0)$ and $d = (1, 0, 0, 0, 1, 0)$ by using (2.4). Then c and d are 2-suitable sequences of length 6 with weight 5.

Proposition 2.1 *If n is odd, there does not exist a pair of 2-suitable sequences of length n with weight w if w cannot be represented by a sum of two squares.*

Proof. If there exist 2-suitable sequences of length n with weight w, we will show that w must be a sum of two squares. Set 2-suitable sequences $a = (a_0, \ldots, a_{n-1})$, $b = (b_0, \ldots, b_{n-1})$ and their corresponding negacyclic matrices A, B respectively. Let

$$\begin{array}{ll} c = (c_0, \ldots, c_{n-1}), & c_i = (-1)^i a_i, \\ d = (d_0, \ldots, d_{n-1}), & d_i = (-1)^i b_i, \end{array} \quad 0 < i < n. \tag{2.5}$$

Define two circulant matrices C and D with the first rows being c and d respectively. It is easy to see that $AA^T + BB^T = wI_n = CC^T + DD^T$. The matrix C has the same sum of each row (column), and so does the matrix D, say w_1, w_2, respectively. Let J be an n-dimensional row vector with all elements 1. We have

$$wn = J(wI_n)J^T = J(CC^T + DD^T)J^T = (w_1^2 + w_2^2)n.$$

That is,

$$w = w_1^2 + w_2^2.$$

The proof is now complete. □

3 Golay sequences, TCPs and suitable sequences

Theorem 3.1 *Suppose $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-2}, 0)$ are 2-suitable sequences, one of them being symmetric and the other skew (or symmetric). If n is even, $a_i = -a_{n-1-i}$ and $b_i = b_{n-2-i}$ for $0 \le i \le \frac{n}{2}$; if n is odd, $a_i = a_{n-1-i}$ and $b_i = -b_{n-2-i}$ for $0 \le i \le \frac{n}{2}$. Then there exists a suitable sequence of length $2n$ with weight $|a| + |b| = \sum_{i=0}^{n-1}(|a_i| + |b_i|)$.*

Proof. We write $(/a, b)$ for the sequence $(a_0, b_0, \ldots, a_{n-1}, b_{n-1})$. We will prove that $c = (/a, b)$ is the required sequence. First, for $0 < i \le \frac{n}{2}$,

$$N_c(2i) = N_a(i) + N_b(i) = N_a(n-i) + N_b(n-i) = N_c(2n-2i).$$

Then, for $2i+1$, $0 \le i < \frac{n}{2}$, we have

$$\begin{aligned} N_c(2i+1) &= \sum_{j=0}^{n-2-i} (a_j b_{i+j} + b_j a_{i+1+j}) = \sum_{j=0}^{n-2-i} (a_j b_{i+j} + b_{n-2-j} a_{i+j+1}) \\ &= \sum_{j=0}^{n-2-i} a_j b_{i+j} + \sum_{s=0}^{n-2-i} b_{s+i} a_{n-1-s} = \sum_{j=0}^{n-2-i} (a_j + a_{n-1-j}) b_{i+j} \\ &= \sum_{j=0}^{n-2-i} (a_j - a_j) b_{i+j} = 0, \end{aligned}$$

or

$$\begin{aligned} N_c(2i+1) &= \sum_{j=0}^{n-2-i} a_j b_{i+j} - \sum_{j=0}^{n-2-i} b_j a_{n-2-i-j} = \sum_{j=0}^{n-2-i} a_j b_{i+j} - \sum_{j=0}^{n-2-i} b_{n-2-j} a_{n-i-j} \\ &= \sum_{j=0}^{n-2-i} a_j b_{i+j} - \sum_{s=0}^{n-2-i} b_{s+i} a_{2+s} = 0. \end{aligned}$$

The proof is now complete. □

n	a ; b
1	1 ; 0
2	- 1 ; 1 0
3	- - - ; - 1 0
4	- - 1 1 ; - - - 0
5	1 - - - 1 ; - - 1 1 0
6	- - 1 - 1 1 ; - - 1 - - 0
7	1 - - - - - 1 ; 1 - - 1 1 - 0
8	- - - - - 1 - - ; - - 1 - 1 - - 0
9	1 1 - - - - - 1 1 ; - - 1 - 1 - 1 1 0
10	- - 1 - - - - 1 - - ; - - - 1 - 1 - - - 0
12	- - - - 1 - - 1 - - - - ; - 1 1 1 - 1 - 1 1 1 - 0
13	- - - - 1 - - - 1 - - - - ; - - 1 1 - 1 - 1 - - 1 1 0
14	- - - - - 1 - - 1 - - - - - ; - - 1 1 - 1 - 1 - 1 1 - - 0
15	- - - - - - - 1 - 1 1 - - 1 - ; - - - 1 1 - 1 0 1 - 1 1 - - -
16	- - - - 1 - - 1 1 - - 1 - - - - ; - - 1 - 1 - - - - - 1 - 1 - - 0
19	- - - - - 1 - - 1 - 1 - - 1 - - - - - ; - 1 - - - 1 1 - - 0 - - 1 1 - - - 1 -
21	- - - 1 - 1 - - 1 - - - 1 - - 1 - 1 - - - ; - - - - 1 1 - 1 1 1 0 1 1 1 - 1 1 - - - -
25	- - - 1 - - - 1 - 1 1 - 1 - 1 1 - 1 - - - 1 - - - ; - 1 1 - - 1 1 1 1 1 1 - 0 - 1 1 1 1 1 1 - - 1 1 -
27	- - - - - - 1 1 - - 1 - 1 - 1 - 1 - - 1 1 - - - - - - ; - - - 1 1 - 1 - - 1 - - - 0 - - - 1 - - 1 - 1 1 - - -

Table 1: In this table we use "-" as an abbreviation for -1.

Corollary 3.1 *There exist* 2*-suitable sequences of length* $2n$ *with weight* $2n-1$ *for* $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 19, 21, 25, 27$.

The details can be found in Table 1.

Lemma 3.1 *If there exist TCPs of length* n *with weight* w*, then there exist TCPs of length* t *for all* $t \geq n$ *with weight* w*.*

Proof. Let $a = (a_0, \ldots, a_{n-1})$ and $b = (b_0, \ldots, b_{n-1})$ be TCPs of order n with weight w. We have

$$N_a(j) + N_b(j) = 0, \ 0 < j < n.$$

Set

$$c = (c_0, \ldots, c_{t-1}) = (a_0, \ldots, a_{n-1}, 0_{t-n}),$$
$$d = (d_0, \ldots, d_{t-1}) = (b_0, \ldots, b_{n-1}, 0_{t-n}).$$

For $t \geq n$, we have

$$\begin{aligned}
N_c(j) + N_d(j) &= \sum_{i=0}^{t-1-j} c_i c_{i+j} + \sum_{i=0}^{t-1-j} d_i d_{i+j} \\
&= (\sum_{i=0}^{n-1-j} c_i c_{i+j} + \sum_{i=n-j}^{t-1-j} c_i c_{i+j}) + (\sum_{i=0}^{n-1-j} d_i d_{i+j} + \sum_{i=n-j}^{t-1-j} d_i d_{i+j}) \\
&= (\sum_{i=0}^{n-1-j} a_i a_{i+j} + \sum_{i=n-j}^{t-1-j} a_i \cdot 0) + (\sum_{i=0}^{n-1-j} b_i b_{i+j} + \sum_{i=n-j}^{t-1-j} b_i \cdot 0) \\
&= N_a(j) + N_b(j) \ = i\ 0, \quad 0 < j < n.
\end{aligned}$$

Thus c and d are TCPs of order t with weight w. □

Let $a = (a_0, \ldots, a_{n-1})$ be a sequence of length n; we denote the reverse of a by a^*, that is, $a^* = (a_{n-1}, \ldots, a_0)$.

Theorem 3.2 *Let $n \equiv 2 \pmod 4$. If there exist 2-suitable sequences of length n with weight $2n-1$, such that $a = (0, a_1, \ldots, a_{n-1})$, $b = (b_0, \ldots, b_{n-1})$, where $a_i = a_{n-i}$, $b_{i-1} = b_{n-i}$, $0 < i \leq \frac{n}{2}$, then $2n - 1 = s^2 + 2t^2$ with $st \equiv 1 \pmod 2$.*

Proof. Set four sequences of length $\frac{n}{2}$ as follows:

$$c_0 = (0, a_2, a_4, \ldots, a_{n-2}), \qquad c_1 = (a_1, a_3, \ldots, a_{n-1}),$$
$$c_2 = (b_0, b_2, b_4, \ldots, b_{n-2}), \qquad c_3 = (b_1, b_3, \ldots, b_{n-1}).$$

Then c_0, c_1, c_2 and c_3 are 4-suitable sequences of length $\frac{n}{2}$ with weight $2n-1$. Moreover, from the hypothesis of the theorem we know that c_1 is symmetric, and $c_3 = c_2^*$. Now set four sequences d_0, d_1, d_2 and d_3 of length $\frac{n}{2}$ from the sequences c_0, c_1, c_2 and c_3 by using the same method as in (2.5), such that the jth element of d_i is equal to the jth element of c_i multiplied by $(-1)^j$, $j = 0, \ldots, \frac{n}{2} - 1$, $i = 0, 1, 2, 3$. Let c_0, c_1, c_2 and c_3 be the first rows of negacyclic matrices C_0, C_1, C_2 and C_3, and d_0, d_1, d_2 and d_3 be the first rows of 4-suitable circulant matrices D_0, D_1, D_2 and D_3, respectively. From the proof of Proposition 2.1, we can easily see that

$$\sum_{i=0}^{3} C_i C_i^T = (2n-1) I_n = \sum_{i=0}^{3} D_i D_i^T.$$

Hence

$$2n - 1 = |d_0|^2 + |d_1|^2 + |d_2|^2 + |d_3|^2.$$

But $|d_0| = 0$, $|d_2|^2 = |d_3|^2$ and $|d_1||d_2| \equiv 1 \pmod{2}$. The proof is now complete. □

When $n \in \{18, 46, 58, 78, 98\}$, the weighing matrices $W(36, 35)$, $W(92, 91)$, $W(116, 115)$, $W(156, 155)$ and $W(196, 195)$ cannot be constructed by 2-suitable circulant / negacyclic matrices.

Corollary 3.2 *There exist TCPs of length* $n = 2^i 10^j 26^k$ *for* $0 \leq i, j, k$.

Proof. In the Corollary of [9], Turyn proves that binary complementary sequences of length $2^i 10^j 26^k$ exist for all i, j, $k \geq 0$. In this case, there exist two $(1, -1)$ complementary sequences of length $n = 2^i 10^j 26^k$, say a and b, that satisfy

$$N_a(0) + N_b(0) = 2n, \quad N_a(l) + N_b(l) = 0 \text{ for } 0 < l < n.$$

Thus a and b are TCPs. □

Lemma 3.2 *If there exist Golay sequences of length* $2n$*, then there exist TCPs of length* $m \leq n$ *with weight* n.

Proof. Let $a = (a_0, \ldots, a_{2n-1})$ and $b = (b_0, \ldots, b_{2n-1})$ be Golay sequences. Set

$$c = \frac{a+b}{2}, \quad d = \frac{a-b}{2}.$$

Then c and d are TCPs of length $2n$ with weight n. From the multiplication $a_i \cdot a_{2n-1-i} \cdot b_i \cdot b_{2n-1-i} = -1$ for $0 \leq i < n$, we know that $c_i = \pm d_{2n-1-i}$. That is, $c_i = 0$ or ± 1 if and only if $d_{2n-1-i} = 0$ or ± 1, respectively. Thus we can take

$$e = \frac{c+d^*}{2}, \quad f = \frac{c-d^*}{2}.$$

It is easy to verify that e and f are TCPs with weight n. Clearly, there exist TCPs of length $m \leq n$ with weight n. □

Example 3.1 $a = (-1, 1, 1, 1, 1, 1, 1, 1, -1, -1, 1)$, $b = (-1, 1, 1, 1, -1, 1, -1, 1, 1, -1)$ are Golay sequences of length 10. From Lemma 3.2, we have

$$c = \frac{a+b}{2} = (-1, 1, 1, 1, 0, 1, 0, 0, 0, 0), \quad d = \frac{a-b}{2} = (0, 0, 0, 0, 1, 0, 1, -1, -1, 1),$$

and

$$e = \frac{c+d^*}{2} = (0, 0, 0, 1, 0, 1, 0, 0, 0, 0), \quad f = \frac{c-d^*}{2} = (-1, 1, 1, 0, 0, 0, 0, 0, 0, 0).$$

From Corollary 2.1, it follows that $g = (1, 0, 1)$, $h = (-1, 1, 1)$ are TCPs of length 3 with weight 5.

Example 3.2 From Golay sequences of length 26 we can get TCPs of length 11 with weight 13:

$$g = (1, 1, 1, 0, -1, 1, 1, 0, -1, 1, -1), \quad h = (-1, 0, -1, 0, 0, 0, 1, 0, 0, 0, -1).$$

Lemma 3.3 *Let* $a = (a_0, \ldots, a_{n-1})$, $b = (b_0, \ldots, b_{n-1})$ *be TCPs such that* $|a_i| = |b_i|$ *for* $0 \le i < n$, *and* c, d *be TCPs of length* m. *Put*

$$u = c \otimes \frac{a+b}{2} + d^* \otimes \frac{a-b}{2}, \qquad v = d \otimes \frac{a+b}{2} - c^* \otimes \frac{a-b}{2}. \tag{3.1}$$

Then u *and* v *are TCPs of length* mn.

Proof. For $0 \le i < m-1$, $0 < j < n$, we have

$$\begin{aligned}
&N_u(in+j) + N_v(in+j) = \\
&\quad \{[N_c(i) + N_d(i)][N_{(a+b)}(j) + N_{(a-b}(j)] \\
&\quad\quad +[N_c(i+1) + N_d(i+1)][N_{(a+b)}(n-j) + N_{(a-b)}(n-j)]\}/4 \\
&\quad = \{[N_c(i) + N_d(i)][N_a(j) + N_b(j)] + [N_c(i+1) \\
&\quad\quad +N_d(i+1)][N_a(n-j) + N_b(n-j)]\}/2 \ = \ 0.
\end{aligned}$$

When $0 < i < m$ and $j = 0$, we have

$$N_u(in) + N_v(in) = \frac{[N_c(i) + N_d(i)][N_a(0) + N_b(0)]}{2} = 0.$$

When $i = m-1$ and $0 < j < n$, we also have

$$\begin{aligned}
N_u((m-1)n+j) + N_v((m-1)n+j) &= \frac{[N_c(m-1) + N_d(m-1)][N_c(j)+N_d(j)]}{2} \\
&= 0.
\end{aligned}$$

□

If we replace u and v in (3.1) by

$$u = \frac{a+b}{2} \otimes c + \frac{a-b}{2} \otimes d^*, \qquad v = \frac{a+b}{2} \otimes d - \frac{a-b}{2} \otimes c^*,$$

then the conclusion of Lemma 3.3 still holds.

Example 3.3 Let $a = (1, -1, -1)$, $b = (1, 0, 1)$, $c = (1, -1, -1, 1, 0, 1)$ and $d = (1, -1, -1, 1, 0, 1)$. Put

$$u = a \otimes \frac{c+d}{2} + b^* \otimes \frac{c-d}{2}, \qquad v = b \otimes \frac{c+d}{2} - a^* \otimes \frac{c-d}{2}.$$

It is easy to verify that u and v are TCPs of length 18 with weight 25.

Lemma 3.4 *Suppose* $a = (a_0, \ldots, a_{m-1})$, $b = (b_0, \ldots, b_{m-1})$ *are* 2*-suitable sequences, and* $c = (c_0, \ldots, c_{n-1})$, $d = (d_0, \ldots, d_{n-1})$ *are TCPs. If* $|a_i| = |b_i|$, $0 \le i < m$. *Set*

$$u = \frac{a+b}{2} \otimes c + \frac{a-b}{2} \otimes d^*, \qquad v = \frac{a+b}{2} \otimes d - \frac{a-b}{\otimes} c^*. \tag{3.2}$$

If $|c_j| = |d_j|$, $0 \le j < n$, *set*

$$u = a \otimes \frac{c+d}{2} + b^* \otimes \frac{c-d}{2}, \qquad v = b \otimes \frac{c+d}{2} - a^* \otimes \frac{c-d}{2}.$$

Then u *and* v *are suitable sequences.*

Proof. For the first case, when $0 \le i < m-1$, $0 < j < n$, we have

$$\begin{aligned} N_u(in+j) + N_v(in+j) &= \{[N_a)(i) + N_b(i)][N_c(j) + N_d(j)] \\ &\quad + [N_a(i+1) + N_b(i+1)][N_c(n-j) + N_d(n-j)]\}/2 \\ &= 0. \end{aligned}$$

Note that $mn - in - j = (m-1-i)n + (n-j)$, so

$$\begin{aligned} & N_u(mn-in-j) + N_v(mn-in-j) \\ &\quad = \{[N_a(m-1-i) + N_b(m-1-i)][N_c(n-j) + N_d(n-j)] \\ &\qquad + [N_a(m-i) + N_b(m-i)][N_c(j) + N_d(j)]\}/2 \\ &\quad = 0. \end{aligned}$$

That is, for $0 \le i < m-1,\ 0 < j < n$,

$$N_u(in+j) + N_v(in+j) = N_u(mn-in-j) + N_v(mn-in-j).$$

For $0 < i < m$, $j = 0$, we have

$$N_u(in) + N_v(in) = \frac{[N_a(i) + N_b(i)][N_c(0) + N_d(0)]}{2}$$

and

$$N_u((m-i)n) + N_v((m-i)n) = \frac{[N_a(m-i) + N_b(m-i)][N_c(0) + N_d(0)]}{2}.$$

Since a, b are suitable sequences, it follows that

$$N_a(i) + N_b(i) = N_a(m-i) + N_b(m-i),\ 0 < i < m.$$

That is,

$$N_u(in) + N_v(in) = N_u(mn-in) + N_v(mn-in),\ 0 < i < m.$$

From the above it follows that

$$N_u(s) + N_v(s) = N_u(mn-s) + N_v(mn-s),\ 0 < s < \lfloor \frac{mn}{2} \rfloor.$$

Consequently, u, v are 2-suitable sequences. Similarly, we can prove the second case. The proof is now complete. □

Under the assumption of Lemma 3.4, if we define two sequences u and v as in (3.1) instead of (3.2), the conclusion of the lemma may not be true (see Example 3.4 for the details).

Example 3.4 Let $c = (1, 1, -1)$, $d = (1, 0, 1)$ are TCPs of length 3, $a = b = (1, 1, -1, 0)$ are the first rows of suitable negacyclic matrices respectively. From (3.2) it follows that

$$u = (1, 1, -1, 1, 1, -1, -1, -1, 1, 0, 0, 0),\ v = (1, 0, 1, 1, 0, 1, -1, 0, -1, 0, 0, 0).$$

It easy to verify that u and v above are 2-suitable sequences of length 12 with weight 15. If we apply (3.1), then the sequences

$$u = (1, 1, -1, 0, 1, 1, -1, 0, -1, -1, 1, 0),\ v = (1, 1, -1, 0, 0, 0, 0, 0, 1, 1, -1, 0)$$

are not 2-suitable sequences.

Example 3.5 Two sequences $c = (1, -1, -1)$, $d = (-1, 0, -1)$ are TCPs of length 3 with weight 5,

$$a = b = (1, -1, -1, 1, 1, 1, 0, 1, 0, 1, -1, 0, 0)$$

are suitable sequences of length 13. By applying (3.2) and Lemma 2.1, we can construct a weighing matrix $W(78, 45)$.

In Table 2 we list minimum length TCPs for several low-valued weights.

w	TCPs of the smallest length	Remarks	Source
1	1; 0	$W(n, 1)$ exists for $n \geq 1$	
2	1;1	$W(n, 2)$ exists for $n \geq 1$	
4	1 1; 1 -	$W(n, 4)$ exists for $n \geq 2$	
5	1 - -; - 0 -	$W(n, 5)$ exists for $n \geq 3$	
8	1 - - -; - 1 - -	$W(n, 8)$ exists for $n \geq 4$	
9		does not exist	
10	1 - - - 0 -; - 1 1 - 0 -	$W(n, 10)$ exists for $n \geq 6$	
13	1 1 - 1 1 1 0 0 -; 1 1 0 0 - 1 - 0 1	$W(n, 13)$ exists for $n \geq 9$	
16	1 1 1 1 1 - - 1; 1 1 - - 1 - 1 -	$W(n, 16)$ exists for $n \geq 8$	[7]
17	1 - 1 0 - 0 0 0 1 1 1 0 1; - 0 - 0 1 1 0 - 0 1 1 - 1	$W(n, 17)$ exists for $n \geq 13$	
18		does not exist	
20	1 - 1 - - 0 0 - 0 - - 0 1 1; 1 0 1 0 0 - 1 1 - 1 1 1 0 -	$W(n, 20)$ exists for $n \geq 14$	[2]
25	1 - - - 1 1 - 1 1 1 0 1 0 0 0 1 0 1; 1 0 1 - 0 - - 0 - 1 1 - 0 0 0 1 1 -	Lemma 3.3 $W(n, 25)$ exists for $n \geq 18$	
26	1 1 1 1 - 1 1 - - 1 - 1 0 1; 1 1 1 - - 1 1 1 - 1 - - 0 -1	$W(n, 26)$ exists for $n \geq 14$	
29	1 1 1 1 - 1 - - - 1 - - - - 1 0 0 0 ; 1 1 - 1 1 - - 0 1 1 - 1 - 1 1 0 0 0	$W(n, 29)$ exists for $n \geq 18$	
32	1 1 1 1 1 1 - - - 1 1 - - 1 - 1; 1 1 1 1 - - 1 1 - 1 1 - 1 - 1 -	$W(n, 32)$ exists for $n \geq 16$	[7]

Table 2: TCPs of weight w, where $w = a^2 + b^2$, a and b are integers

4 Construction of weighing matrices from 2-suitable negacyclic matrices

Theorem 4.1 *If there exists a weighing matrix $W(n,\ w)$, then there exists a weighing matrix $W(mn,\ w)$ for any $m \geq 0$.*

Proof. It is easily checked that $I_m \otimes W(n,\ w)$ is a weighing matrix $W(mn,\ w)$. □

Corollary 4.1 *There exists a weighing matrix* $W(n,\ 1)$ *for any* $n \geq 1$.

Proof. The 1×1 matrix (1) is a weighing matrix $W(1,1)$ so by Theorem 4.1 the corollary follows. □

Theorem 4.2 *If there exist weighing matrices* $W(n,\ w_1)$ *and* $W(n,\ w_2)$*, and they are commutative, then there exists a weighing matrix* $W(2n,\ w_1+w_2)$.

Proof. Let A and B be weighing matrices of $W(n,\ w_1)$ and $W(n,\ w_2)$ respectively. Set

$$C = \begin{pmatrix} A & B \\ B^T & -A^T \end{pmatrix}. \tag{4.1}$$

Obviously, C is a weighing matrix $W(2n,\ w_1+w_2)$. □

Corollary 4.2 *If there exist Golay sequences or TCPs of length* n *and weight* w*, then there exist weighing matrices* $W(2t, 2w)$ *for all* $t \geq n$.

The proof is similar to the proof of Lemma 3.1.

Corollary 4.3 *There exists a weighing matrix of* $W(2n, 2)$ *for any* $n \geq 1$.

Proof. Since there exists a weighing matrix of $W(1,1)$, from Theorem 4.2 we know there exists a weighing matrix of $W(2,2)$. From Theorem 4.1 the corollary is true. □

Let A be a negacyclic matrix of order 4 with first row $a = (1, -, -, 0)$. It is easy to verify that $AA^T = 3I_4$. We conclude that

- There exists a weighing matrix $W(4n, 3)$ for any $n \geq 1$.
- There exists a weighing matrix $W(8n, 6)$ for any $n \geq 1$.

From the above it follows that there exists a weighing matrix $W(4, w)$ for $w = 1, 2, 3, 4$.

Theorem 4.3 *If* 2*-suitable negacyclic matrices* A *and* B *of order* n *exist, then a weighing matrix* $W(2n, w)$ *exists, where* w *is the sum of the weights of two negacyclic matrices.*

Proof. Matrices A and B are 2-suitable negacyclic matrices of order n. Let $a = (a_0, \dots, a_{n-1})$ and $b = (b_0, \dots, b_{n-1})$ be the first rows of A and B. From Lemma 2.1, we have

$$AA^T + BB^T = \sum_{i=0}^{n-1}(a_i^2 + b_i^2)I_n, \quad AB^T = BA^T.$$

By using (4.1), we can construct a matrix C. It is easy to verify that C is a weighing matrix $W(2n, w)$, where $w = |a| + |b|$. The proof is now complete. □

Lemma 4.1 *Weighing matrices $W(2n, w)$ can be constructed from 2-suitable negacyclic matrices when n is odd, $w < 2n$, $w = a^2 + b^2$, where a and b are two integers, for $n \in \{3, 5, 7, 9, 11, 13, 15\}$, except for $n = 9$ and $w = 9$.*

Proof. Tables 3 to 9 in Appendix A show the result of weighing matrices $W(2n, w)$ constructed by 2-suitable negacyclic matrices, where $n = 3, 5, 7, 9, 11, 13, 15$, $w = a^2 + b^2$, with a, b integers. No 2-suitable negacyclic matrices were found for $n = 9$ and $w = 9$ after an exhaustive search. □

Theorem 4.4 *Suppose $n > 0$ is odd. Then there exist 2-suitable negacyclic matrices of order n if and only if there exist 2-suitable circulant matrices of order n.*

Proof. Let $a = (a_0, \dots, a_{n-1})$ be the first row of a negacyclic matrix A. The i-th row of A is $f^{i-1}(a)$. Let $b = (b_0, \dots, b_{n-1})$ be the first row of a negacyclic matrix B. The i-th row of B is $f^{i-1}(b)$. Define

$$\begin{matrix} x = (x_0, \dots, x_{n-1}), \text{ where } x_i = (-1)^i a_i, \\ y = (y_0, \dots, y_{n-1}), \text{ where } y_i = (-1)^i b_i, \end{matrix} \quad 0 \le i < n.$$

Since n is odd, consider nega-shift and shift operators defined in (2.1) and (2.2). We have

$$\begin{aligned} \langle x, s^j(x)\rangle &= N_x(j) + N_x(n-j) \\ &= \sum_{i=0}^{n-1-j} (-1)^i a_i (-1)^{i+j} a_{i+j} + \sum_{i=0}^{j-1} (-1)^i a_i (-1)^{i+n-j} a_{i+n-j} \\ &= (-1)^j \Big(\sum_{i=0}^{n-1-j} a_i a_{i+j} + (-1)^{n-j} \sum_{i=0}^{j-1} a_i a_{i+n-j} \Big) \\ &= (-1)^j (N_a(j) - N_a(n-j)) = \langle (-1)^j a, f^j(a)\rangle, \ 0 < j < n. \end{aligned}$$

Similarly, we have $\langle y, s^j(y)\rangle = \langle (-1)^j b, f^j(b)\rangle$, $0 < j < n$.

Let x and y be the first rows of circulant matrices X and Y. Now A and B are 2-suitable negacyclic matrices if and only if

$$\langle a, f^i(a)\rangle + \langle b, f^i(b)\rangle = 0, \ 0 < i < n,$$

if and only if

$$\langle x, s^i(x)\rangle + \langle y, s^i(y)\rangle = 0,\ 0 < i < n,$$

if and only if X and Y are 2-suitable circulant matrices. The proof is now complete. □

Koukouvinos and Seberry [6] found, after a complete search, that the weighing matrix $W(18, 9)$ cannot be constructed by two circulant matrices of order 9. The weighing matrix $W(18, 9)$ cannot be constructed by two negacyclic matrices of order 9 either.

Theorem 4.5 *There exist weighing matrices $W(2n, w)$, $w = 1, 2, 4$, $2n \geq w$, constructed by 2-suitable negacyclic matrices.*

Proof. When $w = 1$, let $a = (1, 0_{n-1})$ and $b = (0_n)$ be the first rows of negacyclic matrices A and B. It is easy to verify that A and B are 2-suitable negacyclic matrices.

When $w = 2$, let $a = (0_i, 1, 0_{n-i-1})$, where $0 \leq i < n$, and $b = (0_j, 1, 0_{n-j-1})$, where $0 \leq j < n$ and $j \neq i$. Letting a and b be the first rows of negacyclic matrices A and B, respectively, we have $AA^T + BB^T = 2I_n$.

When $w = 4$, let $a = (1, 0_{i-1}, 1, 0_{n-i-1})$, where $0 < i < n$, and $b = (1, 0_{j-1}, -1, 0_{n-j-1})$, where $0 \leq j < n$ and $j \neq i$. Construct two negacyclic matrices A and B from a and b as their first rows; then $AA^T + BB^T = 4I_n$. The proof is now complete. □

Lemma 4.2 *When $n \equiv 2 \pmod 4$, there exist weighing matrices $W(2n, w)$, $1 \leq w \leq 2n$, constructed by two negacyclic matrices of order n, with $n \in \{6, 10, 14, 18\}$, except for $n = 18$ and $w = 35$.*

Proof. In Tables 10–13 in Appendix B, we list the first rows of 2-suitable negacyclic matrices of order n with weight w, $n = 6, 10, 14, 18$, $1 \leq w \leq 2n$. No 2-suitable negacyclic matrices were found for $n = 18$ and $w = 35$ after a complete search. □

Appendix A Weighing matrices $W(2n, w)$, n odd, $3 \leq n \leq 15$, constructed by 2-suitable negacyclic matrices

w	First rows	w	First rows
1	1 0 0 ; 0 0 0	4	1 1 0 ; 1 0 1
2	1 0 0 ; 0 1 0	5	1 1 1 ; 1 0 1

Table 3: $W(6, w)$ constructed by 2-suitable negacyclic matrices of order 3.

w	First rows	w	First rows
1	1 0 0 0 0; 0 0 0 0 0	5	1 1 0 0 1; 1 0 1 0 0
2	1 0 0 0 0; 0 1 0 0 0	8	1 1 1 0 1; 1 1 0 - 1
4	1 1 0 0 0; 1 0 0 0 1	9	1 1 1 - 1; 1 1 0 1 1

Table 4: $W(10, w)$ constructed by 2-suitable negacyclic matrices of order 5.

w	First rows	w	First rows
1	1 0 0 0 0 0 0; 0 0 0 0 0 0 0	8	1 0 0 0 - 1 -; 1 1 1 0 0 0 1
2	1 0 0 0 0 0 0; 0 1 0 0 0 0 0	9	1 0 - 1 1 1 1; 1 0 1 0 0 0 1
4	1 0 0 0 0 0 -; 1 0 0 0 0 0 1	10	1 0 - 1 1 - 0; 1 1 1 0 1 0 1
5	1 0 0 0 0 - 0; 1 1 0 0 0 0 1	13	1 - 1 1 1 1 1; 1 1 - 0 - 1 1

Table 5: $W(14, w)$ constructed by 2-suitable negacyclic matrices of order 7.

w	First rows	w	First rows
1	1 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0	10	1 0 0 0 0 0 - 1 0; 1 1 1 0 1 - 1 1 0
2	1 0 0 0 0 0 0 0 0; 0 1 0 0 0 0 0 0 0	13	1 0 0 0 - 1 - - -; 1 1 0 - 0 1 1 - 1
4	1 0 0 0 0 0 0 0 - ; 1 0 0 0 0 0 0 0 1	16	1 0 - 1 1 1 - 1 -; 1 1 1 1 - 1 1 0 1
5	1 0 0 0 0 0 0 - 0; 1 1 0 0 0 0 0 0 1	17	1 - 1 1 1 1 - - -; 1 1 - 1 0 1 - 1 1
8	1 0 0 0 0 0 0 - 0; 1 1 0 - - 1 - 0 0		

Table 6: $W(18, w)$ constructed by 2-suitable negacyclic matrices of order 9.

w	First rows	w	First rows
1	1 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0	10	1 1 1 - 1 - - 0 0 0 0; 1 0 - 0 0 0 - 0 0 0 0
2	1 0 0 0 0 0 0 0 0 0 0; 0 1 0 0 0 0 0 0 0 0 0	13	1 1 - 1 - 1 1 - - 0 0; 1 1 1 0 0 - 0 0 0 0 0
4	1 1 0 0 0 0 0 0 0 0 0; 1 - 0 0 0 0 0 0 0 0 0	16	1 1 1 1 - 1 - - - 1 0; 1 - - 0 - 1 0 - 0 0 0
5	1 1 - 0 0 0 0 0 0 0 0; 1 0 1 0 0 0 0 0 0 0 0	17	1 1 1 1 - 1 1 - 1 1 1; 1 - 1 0 1 1 0 - 0 0 0
8	1 1 1 - 1 0 - 0 0 0 0; 1 0 0 - 0 0 0 0 0 0 0	18	1 1 1 1 - 1 - 1 1 0 0; 1 1 1 - 1 1 - - 1 0 0
9	1 1 1 - 0 1 - 0 0 0 0; 1 0 1 0 0 - 0 0 0 0 0	20	1 1 1 1 1 - 1 - - 1 0; 1 1 - - 1 1 1 - 1 - 0

Table 7: $W(22, w)$ constructed by 2-suitable negacyclic matrices of order 11.

w	First rows	w	First rows
1	1 0 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0 0	13	1 - - - 1 - - 1 0 1 1 0 0; 1 0 1 0 0 0 0 - 0 0 0 0 0
2	1 0 0 0 0 0 0 0 0 0 0 0 0; 0 1 0 0 0 0 0 0 0 0 0 0 0	16	1 1 1 - - 1 - 1 1 - - - 0 ; 1 - 0 - 0 0 0 0 0 - 0 0 0
4	1 1 0 0 0 0 0 0 0 0 0 0 0; 1 - 0 0 0 0 0 0 0 0 0 0 0	17	1 1 1 - - - 1 - - 1 - 0 0 ; 1 1 - 0 0 1 0 1 0 1 0 0 0
5	1 1 - 0 0 0 0 0 0 0 0 0 0; 1 0 1 0 0 0 0 0 0 0 0 0 0	18	1 1 1 1 - 1 - 1 1 - - 0 0 ; 1 1 - - 0 0 - 1 0 1 0 0 0
8	1 1 - - 0 - 0 - 0 0 0 0 0; 1 - 0 0 0 0 0 0 0 0 0 0 0	20	- 1 1 1 0 1 0 1 - - 1 0 1; - 1 1 1 0 1 0 - 1 1 - 0 -
9	1 - - 1 1 1 0 1 0 1 - 0 0; 0 0 0 0 0 0 0 0 0 0 0 0 0	25	1 1 1 1 1 - 1 - 1 - - 1 1; 1 1 1 - - 1 - 1 1 - - - 0
10	1 - - 1 1 1 0 1 0 1 - 0 0; 1 0 0 0 0 0 0 0 0 0 0 0 0		

Table 8: $W(26, w)$ constructed by 2-suitable negacyclic matrices of order 13.

w	First rows	w	First rows
1	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	16	1 1 - 1 1 1 - 1 - - - 0 1 0 0 ; 1 0 0 1 0 0 1 0 0 0 - 0 0 0 0
2	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0; 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0	17	1 1 - 1 - 1 1 1 1 1 - - 0 0 0; 1 - - 1 0 0 0 1 0 0 0 0 0 0 0
4	1 1 0 0 0 0 0 0 0 0 0 0 0 0 0; 1 - 0 0 0 0 0 0 0 0 0 0 0 0 0	18	1 1 1 1 - 1 - - 1 1 - 0 0 0 0 ; 1 1 1 - 0 - 1 0 0 0 0 0 - 0 0
5	1 1 - 0 0 0 0 0 0 0 0 0 0 0 0; 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0	20	- 1 1 1 0 1 0 1 - - 1 0 1 0 0; - 1 1 1 0 1 0 - 1 1 - 0 - 0 0
8	1 - - - 0 0 - 0 0 0 1 0 0 0 0 ; 1 - 0 0 0 0 0 0 0 0 0 0 0 0 0	25	1 1 1 1 1 1 - 1 1 - 1 - 1 1 1 ; 1 1 1 - 0 - 1 0 - 0 1 - 0 1 0
9	1 - 1 1 0 0 - - 0 0 0 - 0 0 0 ; 1 0 0 0 0 0 - 0 0 0 0 0 0 0 0	26	1 1 - 1 1 1 1 1 - 0 - 1 - 0 1; 1 1 - 1 1 1 - - - 0 1 - 1 0 -
10	1 1 1 - - 1 - 0 0 0 0 0 0 0 0 ; 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0	29	1 1 1 1 1 1 1 - 1 - - 1 1 - 1 ; 1 - 1 1 - - - 1 1 1 - - 1 - 0
13	1 1 1 - - 1 - - 0 - 0 - 1 0 0 ; 1 0 0 0 0 0 0 - 0 0 0 0 0 0 0		

Table 9: $W(30, w)$ constructed by 2-suitable negacyclic matrices of order 15.

Appendix B Weighing matrices $W(2n, w)$, $n = 6, 10, 14, 18$, constructed by 2-suitable negacyclic matrices

w	First rows	w	First rows
1	1 0 0 0 0 0 ; 0 0 0 0 0 0	7	1 1 1 0 - 1 ; 0 0 1 0 0 1
2	1 0 0 0 0 0 ; 0 1 0 0 0 0	8	1 1 1 0 0 1 ; 1 1 0 0 - 1
3	1 0 0 0 0 0 ; 0 1 0 0 - 0	9	1 1 1 0 1 1 ; - 1 0 - 1 0
4	1 0 0 0 0 - ; 1 0 0 0 0 1	10	1 1 1 1 - 1 ; 1 1 0 - 1 0
5	1 0 0 0 - 0 ; 1 1 0 0 0 1	11	1 1 1 1 - 1 ; 1 1 1 - 0 1
6	1 1 1 0 0 1 ; 1 0 0 0 0 1	12	1 1 1 1 - 1 ; 1 1 1 - 1 1

Table 10: $W(12, w)$ constructed by 2-suitable negacyclic matrices of order 6.

w	First rows	w	First rows
1	1 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0	11	1 1 1 1 0 0 0 0 0 1; 1 1 - 1 0 0 - 0 0 1
2	1 0 0 0 0 0 0 0 0 0; 1 0 0 0 0 0 0 0 0 0	12	1 1 1 1 0 0 0 - 1 0; 0 0 0 1 - 0 - 1 1 -
3	1 0 0 0 0 1 0 0 0 0; 1 0 0 0 0 0 0 0 0 0	13	1 1 1 1 0 1 - 0 0 1; 1 0 0 - - 1 - 0 1 0
4	1 0 0 0 0 1 0 0 0 0; 1 0 0 0 0 1 0 0 0 0	14	1 1 1 1 1 - 0 0 0 1; 0 1 1 - 0 1 0 1 - 1
5	1 1 0 0 0 0 0 0 0 1; 1 0 1 0 0 0 0 0 0 0	15	1 1 1 1 1 - 0 1 0 1; 1 0 1 0 - 1 1 0 - 1
6	1 1 0 0 0 0 0 0 0 1; 1 0 1 0 0 0 - 0 0 0	16	1 1 1 1 1 0 0 - 1 1; 1 1 0 0 1 - 1 - 1 1
7	1 1 0 0 0 0 0 0 0 0; 1 1 0 0 0 0 1 - 1 0	17	1 1 1 1 0 1 - 1 1 0; 0 1 - 1 - - 1 1 - -
8	1 1 0 0 0 0 0 0 - 1; 1 1 1 0 0 0 0 0 0 1	18	1 1 1 1 1 0 - 1 1 1; 1 1 0 - 1 - 1 - - 1
9	1 1 1 1 - 1 0 0 0 1; 0 1 0 0 0 0 0 0 0 1	19	1 1 1 1 1 1 - - 1 1; 1 1 1 - 1 - 0 1 - 1
10	1 1 1 1 0 0 0 0 0 1; 0 0 1 - 0 1 - 0 0 1	20	1 1 1 1 1 1 - - 1 1; 1 1 1 - 1 - 1 1 - 1

Table 11: $W(20, w)$ constructed by 2-suitable negacyclic matrices of order 10.

w	First rows
1	1 0 0 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0 0 0
2	1 0 0 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0 0 1
3	1 0 0 0 0 0 0 0 0 0 0 0 0 0; 1 0 0 0 0 0 0 1 0 0 0 0 0 0
4	1 1 0 0 0 0 0 0 0 0 0 0 0 0; 1 0 0 0 0 0 0 0 0 0 0 0 0 1
5	1 0 0 0 0 0 0 0 0 0 0 0 0 0 ; 1 0 1 0 0 0 0 0 1 0 0 0 1 0
6	1 1 0 0 0 0 0 0 0 0 0 0 0 0; 1 0 0 1 0 0 0 0 0 0 - 0 0 1
7	1 1 0 0 0 0 0 0 0 0 0 0 0 0; 1 0 1 0 0 0 0 0 1 - 0 1 0 0
8	1 1 1 0 0 1 0 - 1 - 0 0 1 0; 0 0 0 0 0 0 0 0 0 0 0 0 0 0
9	1 1 1 0 0 0 0 0 0 0 0 0 0 0; 1 0 1 0 - 0 1 - 0 0 0 0 0 1
10	1 1 1 0 1 0 1 0 - 1 - 0 1 1; 0 0 0 0 0 0 0 0 0 0 0 0 0 0
11	0 0 1 1 1 1 - 0 0 0 0 1 - 0; 1 0 0 0 0 1 0 - 0 0 0 0 0 1
12	0 0 1 1 1 1 - 0 1 0 0 - 1 0; 1 0 0 0 1 0 0 0 - 0 0 0 0 1
13	1 - 1 1 1 1 1 - - 0 1 - - 1; 0 0 0 0 0 0 0 0 0 0 0 0 0 0
14	1 - 1 1 1 1 1 - - 0 1 - - 1; 1 0 0 0 0 0 0 0 0 0 0 0 0 0
15	1 - 1 1 1 1 1 - 0 0 1 0 0 1; 1 1 0 0 0 0 0 0 0 0 0 - 1 1
16	0 0 1 1 1 1 1 - 0 0 0 0 - 1; 1 1 0 - 0 0 1 0 0 - 1 0 - 1
17	0 0 1 1 1 1 1 - 0 0 1 0 - 1; 1 1 0 0 - 0 0 1 - 1 0 - 1 0
18	0 0 1 1 1 1 1 0 - 0 1 - 1 0; 1 - - 1 0 1 - 0 - 0 0 0 1 -
19	0 0 1 1 1 1 1 0 0 - 1 0 - 1; 1 - 1 0 0 - 1 1 - - 0 0 1 1
20	0 0 1 1 1 1 1 0 0 - 1 1 - 1; 1 - 0 0 1 - 1 - - 0 0 1 1 -
21	1 1 1 1 1 1 0 0 - 1 - - 1 1; 1 1 0 0 - 1 - 1 0 - 0 1 0 1
22	1 1 1 1 1 1 0 - - 0 1 0 - 1; 1 1 - - 1 0 1 0 - 1 0 1 - 1
23	1 1 1 1 1 1 0 - 1 - 0 - 1 1; 1 1 0 1 0 - 0 1 1 - 1 - 1 1
24	1 1 1 1 1 1 0 0 - - 1 1 - 1; 1 1 1 - 1 - 1 0 0 1 - - 1 1
25	1 1 1 1 1 0 1 - 1 1 - 1 0 1; 1 1 1 0 - 1 1 - - 1 - 1 1 1
26	1 1 1 1 1 1 0 - 1 - 1 - 1 0; 1 - - - 1 1 - 1 1 - 1 1 - -
27	1 1 1 1 1 1 1 - - 1 0 - 1 1; 1 1 1 - - - 1 - 1 - - 1 - 1
28	1 1 1 1 1 1 1 - - 1 1 - 1 1; 1 - - - 1 - 1 - - 1 - 1 - -

Table 12: $W(28, w)$ constructed by 2-suitable negacyclic matrices of order 14.

w	First rows	w	First rows
1	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	19	1 1 1 1 - - - 1 - - - 1 - - 1 - 1 0; 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0
2	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1	20	1 - 1 - - 0 0 - 0 - - 0 1 1; 1 0 1 0 0 - 1 1 - 1 1 1 0 -
3	1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0; 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0	21	1 1 1 1 1 - 1 - - 1 1 0 0 0 0 0 0 0; 1 1 - 1 - 0 0 1 - - 0 0 0 1 0 1 0 0
4	1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0; 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1	22	1 1 1 1 - 1 1 1 - - 1 - 1 1 1 - 0 0; 1 - 0 0 0 1 0 - 0 0 0 1 0 1 0 0 0 0
5	1 0 0 1 0 0 1 0 0 - 0 0 1 0 0 0 0 0; 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	23	1 1 1 1 1 - - 1 1 - 1 - 1 0 0 0 0 0 ; 1 1 1 - 1 0 - 1 0 0 0 1 0 0 0 1 1 0
6	1 0 0 1 0 0 1 0 0 - 0 0 1 0 0 0 0 0; 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	24	1 1 1 1 1 1 - - 1 - 1 1 0 0 0 0 0 0 ; 1 1 - - 0 1 - 0 1 - 1 - 0 - 0 0 1 0
7	1 0 0 1 0 0 1 0 0 - 0 0 1 0 0 0 0 0; 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0	25	1 - - - 1 1 - 1 1 1 0 1 0 0 0 1 0 1; 1 0 1 - 0 - - 0 - 1 1 - 0 0 0 1 1 -
8	1 1 1 - 0 0 0 0 0 0 0 0 0 0 0 0 0 0; 1 - 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1	26	1 1 1 1 1 1 1 - - 1 1 - 1 - 1 - 1 1; 1 1 0 - 0 0 - 0 0 - 1 0 - 0 0 1 0 0
9	1 1 1 0 0 1 - 0 0 0 0 0 0 0 - 1 0 0; 1 0 - 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	27	1 1 1 1 1 - 1 1 - - - 1 - 1 - 0 0 0 ; 1 1 - 1 0 - 1 1 0 - - - 1 0 0 1 0 0
10	1 1 1 - - 1 - 0 0 0 0 0 0 0 0 0 0 0; 1 0 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0	28	1 1 1 1 1 - 1 - - - 1 1 - - 0 0 0 0 ; 1 - 1 1 - 1 0 1 - 0 - - - 1 1 - 0 0
11	1 1 - 1 1 1 0 0 - 0 0 0 1 0 0 0 0 0; 1 - 0 0 0 0 0 0 - 0 0 0 0 0 0 0 0 0	29	1 1 1 1 - 1 - - - 1 - - - - 1 0 0 0 ; 1 1 - 1 1 - - 0 1 1 - 1 - 1 1 0 0 0
12	1 1 - - 1 0 1 0 1 1 0 0 - 0 0 0 - 0; 1 - 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	30	1 1 1 1 1 - 1 - 1 1 - - 1 1 - 0 0 0 ; 1 - 1 1 1 - 1 0 1 1 - - - - 1 0 1 0
13	1 1 - 1 1 1 0 0 - 0 0 0 0 0 0 0 0 0 1 1 0 0 - 1 - 0 1 0 0 0 0 0 0 0 0 0	31	1 1 1 1 1 1 - - 1 1 - 1 - 1 1 1 - 0 ; 1 1 1 - 0 1 - 1 0 1 1 - - 0 1 - 1 0
14	1 1 1 1 - 1 - 0 0 - 0 1 - - 0 1 0 0; 1 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0	32	1 0 1 1 1 1 1 - 1 - - 1 - - 1 1 0 -; 1 0 1 - 1 - 1 1 1 1 - - - 1 1 - 0 1
15	1 1 - 1 1 1 - 0 - 0 0 0 0 0 0 0 - 0; 1 1 0 - 0 0 - 0 0 0 1 - 0 0 0 0 0 0	33	1 - 1 - - 1 1 1 1 1 1 1 1 - - 1 - 1 ; 1 1 - 1 0 - - - 1 1 - - - 0 1 - 1 1
16	1 1 1 1 1 - - 1 0 0 0 0 0 0 0 0 0 0; 1 1 - - 1 - 1 - 0 0 0 0 0 0 0 0 0 0	34	1 - 1 - - 1 1 1 1 1 1 1 1 - - 1 - 1 ; 1 1 - 1 0 - - - 1 1 - - - 0 1 - 1 1
17	1 - 1 0 - 0 0 0 1 1 1 0 1 0 0 0 0 0 - 0 - 0 1 1 0 - 0 1 1 - 1 0 0 0 0 0	35	does not exist
18	1 1 1 1 - - - 1 - - - 1 - - 1 - 1 0 ; 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0	36	1 1 1 1 1 1 1 1 - 1 1 - - 1 - 1 1 1; 1 1 1 - 1 - - 1 - 1 - 1 1 - - 1 1 1

Table 13: $W(36, w)$ constructed by 2-suitable negacyclic matrices of order 18.

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