AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **80(1)** (2021), Pages 30–47

# On optimal orientations of complete tripartite graphs

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**Abstract**

Given a connected and bridgeless graph $G$, let $\mathscr{D}(G)$ be the family of strong orientations of $G$. The orientation number of $G$ is defined to be $\bar{d}(G) := \min\{d(D) \mid D \in \mathscr{D}(G)\}$, where $d(D)$ is the diameter of the digraph $D$. In this paper, we focus on the orientation number of complete tripartite graphs. We prove a conjecture raised by Rajasekaran and Sampathkumar. Specifically, for $q \geq p \geq 3$, if $\bar{d}(K(2,p,q)) = 2$, then $q \leq \binom{p}{\lfloor p/2 \rfloor}$. We also present some sufficient conditions on $p$ and $q$ for $\bar{d}(K(p,p,q)) = 2$.

## 1 Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. In this paper, we consider only graphs $G$ having no loops or parallel edges. For any vertices $v, x \in V(G)$, the *distance* from $v$ to $x$, $d_G(v,x)$, is defined as the length of a shortest path from $v$ to $x$. For $v \in V(G)$, its *eccentricity* $e_G(v)$ is defined as $e_G(v) := \max\{d_G(v,x) \mid x \in V(G)\}$. The *diameter* of $G$, denoted by $d(G)$, is defined as $d(G) := \max\{e_G(v) \mid v \in V(G)\}$. The above notions are defined similarly for a digraph $D$ with vertex set $V(D)$ and arc set $A(D)$. Furthermore, a vertex $x$ is said to be *reachable* from another vertex $v$ if $d_D(v,x) < \infty$. For $u, v \in V(D)$, we write $u \to v$ if $(u,v) \in A(D)$. For $V \subseteq V(D) - \{u\}$, $u \to V$ (respectively, $V \to u$) means $u \to v$ (respectively, $v \to u$) for every $v \in V$. The *outset* and *inset* of a vertex $v \in V(D)$ are defined to be $O_D(v) := \{x \in V(D) \mid v \to x\}$ and $I_D(v) := \{y \in V(D) \mid y \to v\}$ respectively. If there is no ambiguity, we shall omit the subscript for the above notation.

An *orientation* $D$ of a graph $G$ is a digraph obtained from $G$ by assigning a direction to every edge $e \in E(G)$. An orientation $D$ of $G$ is said to be *strong* if every two vertices in $V(D)$ are mutually reachable. An edge $e \in E(G)$ is a *bridge* if $G - e$

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ISSN: 2202-3518 

is disconnected. Robbins' One-way Street Theorem [14] states that for a connected graph $G$, $G$ has a strong orientation if and only if $G$ is bridgeless. Given a connected and bridgeless graph $G$, let $\mathscr{D}(G)$ be the family of strong orientations of $G$. The *orientation number* of $G$ is defined as

$$\bar{d}(G) := \min\{d(D) \mid D \in \mathscr{D}(G)\}.$$

The general problem of finding the orientation number of a connected and bridgeless graph is very difficult. Moreover, Chvátal and Thomassen [3] proved that it is NP-hard to determine whether a graph admits an orientation of diameter 2. Hence it is natural to focus on special classes of graphs. The orientation number was evaluated for various classes of graphs, such as complete graphs [1, 9, 11] and complete bipartite graphs [4, 16]. For general results on orientations of graphs and digraphs, we refer the reader to a survey by Koh and Tay [8].

In this paper, we focus on the orientation number of some complete tripartite graphs. To put our results in context, let us introduce some notation and recall some closely related classical results.

Given any positive integers, $n, p_1, p_2, \ldots, p_n$, let $K_n$ denote the complete graph of order $n$ and $K(p_1, p_2, \ldots, p_n)$ denote the complete $n$-partite graph having $p_i$ vertices in the $i$th partite set for $i = 1, 2, \ldots, n$, where $p_1 \leq p_2 \leq \ldots \leq p_n$. The $n$ partite sets are denoted by $V_i$, $i = 1, 2, \ldots, n$, i.e., $|V_i| = p_i$ for $i = 1, 2, \ldots, n$. Furthermore, $i_j$ denotes the $j$th vertex in $V_i$ for $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, p_i$. Thus, $K_n \cong K(p_1, p_2, \ldots, p_n)$, where $p_1 = p_2 = \ldots = p_n = 1$. For any real number $x$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$ while $\lceil x \rceil$ denotes the smallest integer greater than or equal to $x$.

The orientation number for a complete bipartite graph was determined independently by Šoltés [16] and Gutin [4].

**Theorem 1.1.** *(Soltés [16] and Gutin [4])*
*For $q \geq p \geq 2$,*

$$\bar{d}(K(p,q)) = \begin{cases} 3, & \text{if } q \leq \binom{p}{\lfloor p/2 \rfloor}, \\ 4, & \text{if } q > \binom{p}{\lfloor p/2 \rfloor}. \end{cases}$$

Of interest, Gutin ingeniously made use of a celebrated result in combinatorics, Sperner's Lemma, in his proof. Two sets $T$ and $S$ are *independent* if $T \nsubseteq S$ and $S \nsubseteq T$. If $T$ and $S$ are independent, we may say that $S$ is independent of $T$ or $T$ is independent of $S$.

**Lemma 1.2.** *(Sperner [15])*
*Let $n$ be a positive integer and let $\mathscr{C}$ be a collection of subsets of $\mathbb{N}_n = \{1, 2, \ldots, n\}$ such that $S$ and $T$ are independent for any two distinct sets $S$ and $T$ in $\mathscr{C}$. Then $|\mathscr{C}| \leq \binom{n}{\lfloor n/2 \rfloor}$ with equality holding if and only if all members in $\mathscr{C}$ have the same size, $\lfloor \frac{n}{2} \rfloor$ or $\lceil \frac{n}{2} \rceil$.*

Plesnik [12], Gutin [5], and Koh and Tan [6] independently proved that the orientation number of a complete multipartite graph is 2 or 3. Some sufficient and necessary conditions were also established in the same papers. However, a complete characterisation remains elusive.

**Theorem 1.3.** *(Plesnik [12], Gutin [5] and Koh and Tan [6])*
*For all positive integers $n \geq 3$ and $p_1, p_2, \ldots, p_n$, $2 \leq \bar{d}(K(p_1, p_2, \ldots, p_n)) \leq 3$.*

**Theorem 1.4.** *(Gutin [5] and Koh and Tan [6])*
*For all integers $n \geq 3$ and $p \geq 2$, $\bar{d}(K(\overbrace{p, p, \ldots, p}^{n})) = 2$.*

**Theorem 1.5.** *(Koh and Tan [6])*
*Let $n \geq 3$ and $p_1, p_2, \ldots, p_n$ be positive integers. Denote $h = \sum_{k=1}^{n} p_i$. If*

$$p_i > \binom{h - p_i}{\lfloor (h - p_i)/2 \rfloor}$$

*for some $i = 1, 2, \ldots, n$, then $\bar{d}(K(p_1, p_2, \ldots, p_n)) = 3$.*

Next, we state some existing results on complete tripartite graphs, most of which were established by Rajasekaran and Sampathkumar.

**Theorem 1.6.** *(Rajasekaran and Sampathkumar [13])*
*For $q \geq p \geq 2$, $\bar{d}(K(1, p, q)) = 3$.*

**Theorem 1.7.** *(Koh and Tan [7])*
*For $q \geq p \geq 2$, if $q \leq \binom{p}{\lfloor p/2 \rfloor}$, then $\bar{d}(K(2, p, q)) = 2$.*

**Theorem 1.8.** *(Rajasekaran and Sampathkumar [13])*
*For $q \geq 3$, $\bar{d}(K(2, 2, q)) = 3$.*

**Theorem 1.9.** *(Rajasekaran and Sampathkumar [13])*
*For $q \geq 4$, $\bar{d}(K(2, 3, q)) = 3$.*

**Theorem 1.10.** *(Rajasekaran and Sampathkumar [13])*
*For $p \geq 4$, $4 \leq q \leq 2p$, $\bar{d}(K(p, p, q)) = 2$.*

In this paper, we prove a conjecture raised by Rajasekaran and Sampathkumar (see Section 2) and present some sufficient conditions on $p$ and $q$ for $\bar{d}(K(p, p, q)) = 2$ (see Section 3).

## 2 A conjecture on $K(2, p, q)$

Based on Theorems 1.8 and 1.9 and an unpublished paper "The orientation number of the complete tripartite graph $K(2, 4, p)$", Rajasekaran and Sampathkumar conjectured that the converse of Theorem 1.7 holds for complete tripartite graphs $K(2, p, q)$, $q \geq p \geq 5$. Ng [10] showed for $q \geq p \geq 2$, $\bar{d}(K(1, 1, p, q)) = 2$ implies $q \leq \binom{p}{\lfloor p/2 \rfloor}$. Since an orientation $D$ of $K(2, p, q)$, where $d(D) = 2$, is a spanning subdigraph of an orientation of $K(1, 1, p, q)$, the conjecture follows from Ng's result. In this section, we provide a different and shorter proof of the conjecture. We start with some observations which will be used in our proof later.

**Lemma 2.1.** *Let $G = K(p_1, p_2, \ldots, p_n)$, $n \geq 3$, and $D$ be an orientation of $G$. Suppose there exist vertices $i_s$ and $j_t$ for some $i, j, s$ and $t$, where $i \neq j$, $1 \leq i, j \leq n$, $1 \leq s \leq p_i$ and $1 \leq t \leq p_j$, such that $O(i_s) \cap (V(D) - V_j) = O(j_t) \cap (V(D) - V_i)$. Then, $d(D) \geq 3$.*

*Proof*: Without loss of generality, we assume $j_t \to i_s$. It follows that $d_D(i_s, j_t) > 2$ and $d(D) \geq 3$. □

**Lemma 2.2.** *Let $D$ be an orientation of a graph $G$. Let $\tilde{D}$ be the orientation of $G$ such that $(u, v) \in A(\tilde{D})$ if and only if $(v, u) \in A(D)$. Then, $d(\tilde{D}) = d(D)$.*

*Proof*: Suppose not. Then there exist some vertices $u, v \in V(\tilde{D})$ such that $d_{\tilde{D}}(u, v) > d(D)$. Since $d_D(v, u) = d_{\tilde{D}}(u, v)$, it follows that $d_D(v, u) > d(D)$, yielding a contradiction. □

**Theorem 2.3.** *For any integers $q \geq p \geq 3$, if $\bar{d}(K(2, p, q)) = 2$, then $q \leq \binom{p}{\lfloor p/2 \rfloor}$.*

*Proof*: Since $\bar{d}(K(2, p, q)) = 2$, there exists an orientation $D$ of $K(2, p, q)$ such that $d(D) = 2$.

Case 1. $V_1 \to V_2$.

It follows from $d_D(3_i, 1_j) \leq 2$, for every $i = 1, 2, \ldots, q$, and $j = 1, 2$, that $V_3 \to V_1$. Also, since $d_D(2_i, 3_j) \leq 2$ for every $i = 1, 2, \ldots, p$, and $j = 1, 2, \ldots, q$, we have $V_2 \to V_3$. However, $d_D(3_i, 3_j) \geq 3$ for any $1 \leq i, j \leq q$, $i \neq j$, which contradicts $d(D) = 2$.

Similarly, from Lemma 2.2, we cannot have $V_2 \to V_1$.

Case 2. $1_i \to V_2 \to 1_{3-i}$ for exactly one of $i = 1, 2$.

Without loss of generality, we may assume that $1_1 \to V_2 \to 1_2$. It follows from $d_D(1_2, 3_i) \leq 2$ and $d_D(3_i, 1_1) \leq 2$ for every $i = 1, 2, \ldots, q$ that $1_2 \to V_3 \to 1_1$. Now, for any $i \neq j$, $1 \leq i, j \leq q$, $d_D(3_i, 3_j) \leq 2$ and thus, $O(3_i) \cap V_2$ and $O(3_j) \cap V_2$ are independent. By Sperner's Lemma, $q \leq \binom{p}{\lfloor p/2 \rfloor}$.

Case 3. $1_i \to V_2$ for exactly one of $i = 1, 2$.

Without loss of generality, let $i = 1$. Furthermore, we assume that $\emptyset \neq O(1_2) \cap V_2 \subset V_2$ in view of Cases 1 and 2. Hence, let $|O(1_2) \cap V_2| = k$, where $0 < k < p$. Since $d_D(u, 3_j) \leq 2$ for every $u \in O(1_2) \cap V_2$ and every $j = 1, 2, \ldots, q$, it follows that $O(1_2) \cap V_2 \to V_3$. It also follows from $d_D(3_j, 1_1) \leq 2$ for every $j = 1, 2, \ldots, q$, that $V_3 \to 1_1$.

Partition $V_3$ into $L_1$ and $L_2$ such that $L_1 := \{v \in V_3 \mid v \to 1_2\}$ and $L_2 := \{v \in V_3 \mid 1_2 \to v\}$. Note that $L_1 \to V_1$. Since $d_D(2_j, v) \leq 2$ for all $j = 1, 2, \ldots, p$, and $v \in L_1$, we have $V_2 \to L_1$. Thus, $|L_1| \leq 1$, otherwise if $u, v \in L_1$, then $d_D(u, v) \geq 3$. Also, $|L_2| \leq \binom{p-k}{\lfloor (p-k)/2 \rfloor}$. Otherwise, by Sperner's Lemma, there exist $3_i, 3_j \in L_2$ such that $O(3_i) \cap V_2 \subseteq O(3_j) \cap V_2$ for some $i \neq j$ and $1 \leq i, j \leq q$, which implies $d_D(3_i, 3_j) > 2$. Hence, $q = |V_3| = |L_1| + |L_2| \leq 1 + \binom{p-k}{\lfloor (p-k)/2 \rfloor} \leq 1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor} \leq \binom{p}{\lfloor p/2 \rfloor}$.

Similarly, the case where $V_2 \to 1_i$ for exactly one of $i = 1, 2$, follows from Lemma 2.2.

Case 4. $\emptyset \neq O(1_i) \cap V_2 \subset V_2$ for $i = 1, 2$.

Partition $V_2$ into the sets $K_A := \{v \in V_2 \mid A \to v \to (V_1 - A)\}$, where $A \subseteq V_1$. Similarly, partition $V_3$ into the sets $L_A := \{v \in V_3 \mid A \to v \to (V_1 - A)\}$, where $A \subseteq V_1$.

Since $d_D(u, 2_j) \leq 2$ for any $u \in V_3$ and $j = 1, 2, \ldots, p$, it follows that $L_\emptyset \to K_\emptyset$, $L_{\{1_1\}} \to K_{\{1_1\}} \cup K_\emptyset$, $L_{\{1_2\}} \to K_{\{1_2\}} \cup K_\emptyset$ and $L_{V_1} \to V_2$. Similarly, since $d_D(u, 3_j) \leq 2$ for any $u \in V_2$ and $j = 1, 2, \ldots, q$, it follows that $K_\emptyset \to L_\emptyset$, $K_{\{1_1\}} \to L_{\{1_1\}} \cup L_\emptyset$, $K_{\{1_2\}} \to L_{\{1_2\}} \cup L_\emptyset$ and $K_{V_1} \to V_3$.

Invoking Sperner's Lemma on each $L_A$, $A \subseteq V_1$, we have $|L_\emptyset| \leq 1$, $|L_{\{1_1\}}| \leq \binom{|K_{\{1_2\}}|}{\lfloor |K_{\{1_2\}}|/2 \rfloor}$, $|L_{\{1_2\}}| \leq \binom{|K_{\{1_1\}}|}{\lfloor |K_{\{1_1\}}|/2 \rfloor}$ and $|L_{V_1}| \leq 1$. Otherwise, there would exist $3_i, 3_j \in L_A$ such that $O(3_i) \cap V_2 \subseteq O(3_j) \cap V_2$ for some $i \neq j$ and $1 \leq i, j \leq q$, implying $d_D(3_i, 3_j) > 2$.

Subcase 4.1. $|K_{V_1}| = 0$.

For $i = 1, 2$, $K_{\{1_i\}} \neq \emptyset$, since $O(1_i) \cap V_2 \neq \emptyset$ by assumption. From Lemma 2.1 it follows that $L_{\{1_1\}} = L_{\{1_2\}} = \emptyset$. So, $q = |V_3| = |L_\emptyset| + |L_{V_1}| \leq 1 + 1 < \binom{p}{\lfloor p/2 \rfloor}$.

Subcase 4.2. $|K_{V_1}| > 0$.

Then $L_{V_1} = \emptyset$ by Lemma 2.1. Recall that $|K_\emptyset| + |K_{\{1_1\}}| + |K_{\{1_2\}}| + |K_{V_1}| = p$. By Lemma 2.1, for each $i = 1, 2$, if $K_{\{1_i\}} \neq \emptyset$, then $L_{\{1_i\}} = \emptyset$. Hence, if $K_{\{1_1\}} \neq \emptyset$ and $K_{\{1_2\}} \neq \emptyset$, then $q = |V_3| = |L_\emptyset| \leq 1$. If $K_{\{1_1\}} = \emptyset$ and $K_{\{1_2\}} \neq \emptyset$ , then $q = |L_\emptyset| + |L_{\{1_1\}}| \leq 1 + \binom{|K_{\{1_2\}}|}{\lfloor |K_{\{1_2\}}|/2 \rfloor} \leq 1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor}$. By symmetry, if $K_{\{1_1\}} \neq \emptyset$ and $K_{\{1_2\}} = \emptyset$, it also follows that $q \leq 1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor}$. Lastly, if $K_{\{1_1\}} = K_{\{1_2\}} = \emptyset$, it follows that $q = |L_\emptyset| + |L_{\{1_1\}}| + |L_{\{1_2\}}| \leq 1 + 1 + 1$. Therefore, $q \leq \max\Big\{1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor}, 3\Big\} \leq \binom{p}{\lfloor p/2 \rfloor}$. $\square$

**Corollary 2.4.** *For any integers* $p \geq 4$ *and* $1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor} < q \leq \binom{p}{\lfloor p/2 \rfloor}$*, let* $D$ *be an optimal orientation of* $K(2, p, q)$*, where* $d(D) = 2$*. Then,*
*(i)* $1_i \to V_2 \to 1_{3-i} \to V_3 \to 1_i$ *for exactly one of* $i = 1, 2$*;*
*(ii)* $\{O(3_i) \cap V_2 \mid i = 1, 2, \ldots, q\}$ *is a family of independent subsets of* $V_2$*.*
*In particular, there are at most two optimal orientations* $D$ *(up to isomorphism) in the case where* $q = \binom{p}{\lfloor p/2 \rfloor}$*.*

*Proof*: Case 1 of the proof of Theorem 2.3 shows that it is impossible for $V_1 \to V_2$ or $V_2 \to V_1$. Since $q > 1 + \binom{p-1}{\lfloor (p-1)/2 \rfloor}$ and $p \geq 4$, Cases 3 and 4 are also impossible. This leaves us with the result of Case 2, i.e., $1_i \to V_2 \to 1_{3-i} \to V_3 \to 1_i$ for exactly one of $i = 1, 2$.

Now, for any $i, j$ where $i \neq j$ and $1 \leq i, j \leq q$, $3_i, 3_j \in V_3$, $d(3_i, 3_j) = 2$ if and only if $O(3_i) \cap V_2 \not\subseteq O(3_j) \cap V_2$. Thus, (ii) follows.

Furthermore, if $q = \binom{p}{\lfloor p/2 \rfloor}$, then $|O(3_i) \cap V_2| = \lfloor \frac{p}{2} \rfloor$ or $\lceil \frac{p}{2} \rceil$ by Sperner's Lemma. Thus, there are at most two optimal orientations (up to isomorphism) $D$. $\square$

Theorem 2.3 completes the characterizaion of complete tripartite graphs $K(2, p, q)$ with $\bar{d}(K(2, p, q)) = 2$. Together with Theorems 1.7 and 1.8, we have the following theorem. Interestingly, this characterisation has the same bounds for $q$ as the general bipartite graph $K(p, q)$. (See Theorem 1.1.)

**Theorem 2.5.** *For any integers $q \geq p \geq 2$, $\bar{d}(K(2, p, q)) = 2$ if and only if $q \leq \binom{p}{\lfloor p/2 \rfloor}$.*

## 3 Sufficient conditions for $\bar{d}(K(p, p, q)) = 2$

In this section, we provide some sufficient conditions on $p$ and $q$ so that $\bar{d}(K(p, p, q)) = 2$. Our result (see Theorem 3.11) improves significantly the upper bound $2p$ of $q$ given in Theorem 1.10, especially when $p$ increases. We begin by solving a combinatorics problem, which will be of assistance later.

**Definition 3.1.** Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$, i.e., $1 < k, d < p$. Denote a solution $(x_1, x_2, \ldots, x_{2d})^*$ if $(x_1, x_2, \ldots, x_{2d})$ satisfies

$$\left.\begin{array}{l} x_1 + x_2 + \ldots + x_{2d} = p, \text{ and} \\ 1 \leq x_i \leq k-1, \text{ for } i = 1, 2, \ldots, 2d. \end{array}\right\} \tag{1}$$

Define $\Phi^*(p, d) := \sum_{(x_1, x_2, \ldots, x_{2d})^*} \binom{k}{x_1}\binom{k}{x_2} \cdots \binom{k}{x_{2d}}$.

**Definition 3.2.** Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. For any non-negative integers $i, j$, define $[i, j]$ to be the set of solutions $(x_1, x_2, \ldots, x_{2d})$ satisfying

$$\begin{array}{l} x_1 + x_2 + \ldots + x_{2d} = p, \\ x_{s_m} = 0, \text{ for } m = 1, 2, \ldots, i, \text{ where } \{s_1, s_2, \ldots, s_i\} \subseteq \{1, 2, \ldots, 2d\}, \\ x_{t_n} = k, \text{ for } n = 1, 2, \ldots, j, \text{ where } \{t_1, t_2, \ldots, t_j\} \subseteq \{1, 2, \ldots, 2d\}, \text{ and} \\ 1 \leq x_r \leq k-1, \text{ for } r \in \{1, 2, \ldots, 2d\} - (\{s_1, s_2, \ldots, s_i\} \cup \{t_1, t_2, \ldots, t_j\}). \end{array}$$

Furthermore, we denote $\Phi(p, d, [i, j]) := \sum_{(x_1, x_2, \ldots, x_{2d}) \in [i, j]} \binom{k}{x_1}\binom{k}{x_2} \cdots \binom{k}{x_{2d}}$.

**Remark 3.3.** The following may be verified easily.
(a) If $p$ is even, then $\Phi^*(p, \frac{p}{2}) = 2^p$.
(b) For each $[i, j]$ defined above, $0 \leq i, j \leq d$.
(c) $\Phi(p, d, [i, j]) \geq 0$ for $0 \leq i, j \leq d$.
(d) $\Phi(p, d, [d, d]) = \binom{2d}{d}$.
(e) $\Phi(p, d, [i, d]) = \Phi(p, d, [d, i]) = 0$ for $0 \leq i \leq d-1$.

In the proof of Lemma 3.5, we will make use of the following combinatorial identities which we quote without proof. (See [2] for details.)

**Lemma 3.4.** *For non-negative integers $x_i, n_i, n, k, r$, $n \geq 1$, $r \leq k \leq n$ and $x_i \leq n_i$ for $i = 1, 2 \ldots, r$,*
*(a)* $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$;
*(b)* $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n} = 0$;
*(c)* $\sum\limits_{x_1+x_2+\ldots+x_r=p} \binom{n_1}{x_1}\binom{n_2}{x_2}\ldots\binom{n_r}{x_r} = \binom{n_1+n_2+\ldots+n_r}{p}$. *(Generalised Vandermonde's identity)*

**Lemma 3.5.** *Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. Then*

$$\Phi(p, d, [i, j]) = \sum_{s=i}^{d}\sum_{t=j}^{d}\left[(-1)^{(s-i)+(t-j)}\binom{2d}{s, t, 2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}\binom{s}{i}\binom{t}{j}\right].$$

*Proof*: Let $\mu, \lambda$ be any two integers such that $i \leq \mu \leq d$ and $j \leq \lambda \leq d$. We proceed using a double counting method. Suppose $\alpha = \binom{k}{\bar{x}_1}\binom{k}{\bar{x}_2}\ldots\binom{k}{\bar{x}_{2d}}$, where $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{2d})$ is an element of $[\mu, \lambda]$. We shall show that each $\alpha$ contributes the same count to both sides of the equality.

Case 1. $\mu = i$ and $\lambda = j$.

On the left side, $\alpha$ is counted exactly once. The expression

$$\binom{2d}{s, t, 2d-(s+t)}\binom{k}{0}^s\binom{k}{k}^t\binom{(2d-(s+t))k}{(d-t)k}$$

represents choosing $s$ and $t$ groups from all $2d$ groups of $k$ elements to select 0 and $k$ elements, respectively, from each group, after which $(d-t)k$ elements are selected from the remaining $(2d-(s+t))k$ elements to form a total of $p = dk$ selected elements.

It follows that, on the right, $\alpha$ is counted exactly once in the first term

$$\begin{aligned}&(-1)^{(i-i)+(j-j)}\binom{2d}{i, j, 2d-(i+j)}\binom{(2d-(i+j))k}{(d-j)k}\binom{i}{i}\binom{j}{j}\\ =\ &\binom{2d}{i, j, 2d-(i+j)}\binom{k}{0}^i\binom{k}{k}^j\binom{(2d-(i+j))k}{(d-j)k}\end{aligned}$$

and contributes a zero count in the subsequent terms

$$\binom{2d}{s, t, 2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k} = \binom{2d}{s, t, 2d-(s+t)}\binom{k}{0}^s\binom{k}{k}^t\binom{(2d-(s+t))k}{(d-t)k}$$

if $s > i$ or $t > j$. Thus, $\alpha$ is counted once on each side.

By definition of $\alpha$, $\alpha$ is counted by the term, $\Phi(p, d, [i, j])$, on the left if and only if $[\mu, \lambda] = [i, j]$. Therefore, $\alpha$ has a zero count on the left side for the following three cases. It suffices to show that $\alpha$ contributes to a count of zero on the right in each of the following cases as well.

Case 2. $\mu = i$ and $\lambda > j$.

Similar to above, on the right, $\alpha$ is counted

$$\begin{array}{ccc} \binom{\lambda}{j} & \text{times in} & \binom{2d}{i, j, 2d-(i+j)}\binom{(2d-(i+j))k}{(d-j)k}, \\ \binom{\lambda}{j+1} & \text{times in} & \binom{2d}{i, j+1, 2d-(i+j+1)}\binom{(2d-(i+j+1))k}{(d-(j+1))k}, \\ & \vdots & \\ \binom{\lambda}{\lambda} & \text{times in} & \binom{2d}{i, \lambda, 2d-(i+\lambda)}\binom{(2d-(i+\lambda))k}{(d-\lambda)k} \end{array}$$

and none in the subsequent terms $\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}$ if $s > i$ or $t > \lambda$. So, $\alpha$ has a total count of $\sum_{s=i}^{i}\sum_{t=j}^{\lambda}[(-1)^{(s-i)+(t-j)}\binom{\lambda}{t}\binom{s}{i}\binom{t}{j}] = (-1)^{(i-i)}\binom{i}{i}\sum_{t=j}^{\lambda}(-1)^{(t-j)}\binom{\lambda}{t}\binom{t}{j}$ $= \sum_{t=j}^{\lambda}(-1)^{(t-j)}\binom{\lambda}{j}\binom{\lambda-j}{t-j} = \binom{\lambda}{j}\sum_{t=j}^{\lambda}(-1)^{(t-j)}\binom{\lambda-j}{t-j} = \binom{\lambda}{j}(0) = 0$, where Lemma 3.4(a) and (b) were invoked in the second and fourth equalities respectively. Thus, $\alpha$ has a zero count on each side.

Case 3. $\mu > i$ and $\lambda = j$.

This case is similar to Case 2.

Case 4. $\mu > i$ and $\lambda > j$.

On the right, $\alpha$ is counted $\binom{\mu}{s}\binom{\lambda}{t}$ times in the term $\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}$, $i \leq s \leq \mu$ and $j \leq t \leq \lambda$ and 0 times if $\mu < s \leq d$ or $\lambda < t \leq d$. In other words, on the right, $\alpha$ is counted

$$\begin{aligned}
& \sum_{s=i}^{\mu}\sum_{t=j}^{\lambda}[(-1)^{(s-i)+(t-j)}\binom{\mu}{s}\binom{\lambda}{t}\binom{s}{i}\binom{t}{j} \\
= & \sum_{s=i}^{\mu}\left\{(-1)^{(s-i)}\binom{\mu}{s}\binom{s}{i}\sum_{t=j}^{\lambda}[(-1)^{(t-j)}\binom{\lambda}{t}\binom{t}{j}]\right\} \\
= & \sum_{s=i}^{\mu}\left\{(-1)^{(s-i)}\binom{\mu}{s}\binom{s}{i}\sum_{t=j}^{\lambda}[(-1)^{(t-j)}\binom{\lambda}{j}\binom{\lambda-j}{t-j}]\right\} \\
= & \sum_{s=i}^{\mu}\left\{(-1)^{(s-i)}\binom{\mu}{s}\binom{s}{i}\binom{\lambda}{j}\sum_{t=j}^{\lambda}[(-1)^{(t-j)}\binom{\lambda-j}{t-j}]\right\} \\
= & \sum_{s=i}^{\mu}\left\{(-1)^{(s-i)}\binom{\mu}{s}\binom{s}{i}\binom{\lambda}{j}(0)\right\} \\
= & \ 0
\end{aligned}$$

times, where Lemma 3.4(a) and (b) were invoked in the second and fourth equalities above respectively. Thus, $\alpha$ contributes a count of zero on each side. $\square$

**Corollary 3.6.** *Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. Then*
*(i)* $\Phi^*(p,d) = \sum_{s=0}^{d}\sum_{t=0}^{d}[(-1)^{(s+t)}\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}]$;
*(ii)* $\binom{2p}{p} = \sum_{i=0}^{d}\sum_{j=0}^{d}\sum_{s=i}^{d}\sum_{t=j}^{d}[(-1)^{(s-i)+(t-j)}\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}\binom{s}{i}\binom{t}{j}]$;
*(iii)* $\Phi(p,d,[i,j]) = \Phi(p,d,[j,i])$ *for* $0 \leq i, j \leq d$.

*Proof*:
(i) This follows from the fact that $\Phi^*(p,d) = \Phi(p,d,[0,0])$.
(ii) By generalised Vandermonde's identity, $\binom{2p}{p} = \sum_{i=0}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j])$.
(iii) Since $\binom{2d}{s,t,2d-(s+t)} = \binom{2d}{t,s,2d-(s+t)}$ and $\binom{(2d-(s+t))k}{(d-t)k} = \binom{(2d-(s+t))k}{(d-s)k}$, it follows that
$\Phi(p,d,[j,i]) = \sum_{s=j}^{d}\sum_{t=i}^{d}[(-1)^{(s-j)+(t-i)}\binom{2d}{s,t,2d-(s+t)}\binom{(2d-(s+t))k}{(d-t)k}\binom{s}{j}\binom{t}{i}]$
$= \sum_{t=i}^{d}\sum_{s=j}^{d}[(-1)^{(t-i)+(s-j)}\binom{2d}{t,s,2d-(s+t)}\binom{(2d-(s+t))k}{(d-s)k}\binom{t}{i}\binom{s}{j}] = \Phi(p,d,[i,j])$. □

Now, we shall construct an orientation $F$ of $K(p,p,q)$, which resembles the definition of $\Phi^*(p,d)$ (see (1)). We divide each of $V_1$ and $V_2$ into $d$ groups of size $k$. Then orientate $F$ such that for all $1 \leq i \leq q$, $|O(3_i)| = p$ and $O(3_i)$ contains some but not all vertices of each group. This distinctive property will aid in ensuring $d(F) = 2$.

**Proposition 3.7.** *Suppose $p \geq 4$ is an integer such that $p = kd$ for some non-trivial divisors $k, d \in \mathbb{Z}^+$. Then $\bar{d}(K(p,p,q)) = 2$ if $2k+2 \leq q \leq \max_d\{\Phi^*(p,d)\} + 2$, where the maximum is taken over all positive divisors $d$ of $p$ satisfying $1 < d < p$.*

*Proof*: Partition $V_1 \cup V_2$ into $X_1, X_2, \ldots, X_{2d}$ where

$$X_s = \{1_j \mid j \equiv s \pmod d\} \text{ and}$$
$$X_{d+s} = \{2_{(s-1)k+1}, 2_{(s-1)k+2}, \ldots, 2_{(s-1)k+k}\}$$

for $s = 1, 2, \ldots, d$. Observe that $|X_r| = k$ for all $r = 1, 2, \ldots, 2d$. First, we define an orientation $F$ for $K(p,p,2k+2)$ as follows.
(I) $(V_2 - X_{d+s}) \to X_s \to X_{d+s} \to (V_1 - X_s)$ for $s = 1, 2, \ldots, d$.
(II) $V_1 \to 3_{2k+1} \to V_2 \to 3_{2k+2} \to V_1$.
(III) For $t = 1, 2, \ldots, k$,

$$\begin{aligned}(a)\ &\{2_k, 2_{2k}, \ldots, 2_{dk}\} \cup (V_1 - \{1_{(t-1)d+1}, 1_{(t-1)d+2}, \ldots, 1_{(t-1)d+d}\}) \to 3_t \to \\ &\{1_{(t-1)d+1}, 1_{(t-1)d+2}, \ldots, 1_{(t-1)d+d}\} \cup (V_2 - \{2_k, 2_{2k}, \ldots, 2_{dk}\}), \text{ and} \\ (b)\ &\{1_1, 1_2, \ldots, 1_d\} \cup (V_2 - \{2_t, 2_{t+k}, \ldots, 2_{t+(d-1)k}\}) \to 3_{t+k} \to \\ &\{2_t, 2_{t+k}, \ldots, 2_{t+(d-1)k}\} \cup (V_1 - \{1_1, 1_2, \ldots, 1_d\}).\end{aligned}$$

Now, consider the case where $q > 2k+2$. Let $x_i = |O(3_j) \cap X_i|$ for some $j$, where $2k+2 < j \leq q$, and $i = 1, 2, \ldots, 2d$. So, for each solution $(x_1, x_2, \ldots, x_{2d})^*$

of (1), there are $\binom{k}{x_1}\binom{k}{x_2}\ldots\binom{k}{x_{2d}}$ ways to choose $p$ vertices (as the outset of a vertex $3_j$), where $x_i$ vertices are selected from the set $X_i$, satisfying $1 \leq x_i \leq k-1$, for $i = 1, 2, \ldots, 2d$ and $x_1 + x_2 + \ldots + x_{2d} = p$. Summing over all possible solutions $(x_1, x_2, \ldots, x_{2d})^*$, there is a total of $\Phi^*(p,d) := \sum_{(x_1,x_2,\ldots,x_{2d})^*} \binom{k}{x_1}\binom{k}{x_2}\ldots\binom{k}{x_{2d}}$ of such combinations of $p$ vertices of $V_1 \cup V_2$. Denote this set of combinations as $\Psi$.

Note from (III) that the $2k$ outsets of $3_1, 3_2, \ldots, 3_{2k}$ are elements of $\Psi$. That leaves $|\Psi| - 2k = \Phi^*(p,d) - 2k$ combinations of $p$ vertices of $V_1 \cup V_2$. Note however that $O(3_{2k+1})$ and $O(3_{3k+2})$ from (II) are not elements of $\Psi$. Hence, for $2k+2 < j \leq q \leq \max_d\{\Phi^*(p,d)\} + 2$, we extend the definition of the above orientation so that the outsets of vertices $3_{2k+3}, 3_{2k+4}, \ldots, 3_q$ are these remaining elements of $\Psi$. (See Figure 1 for $F$ when $d = 3$ and $k = 2$.)

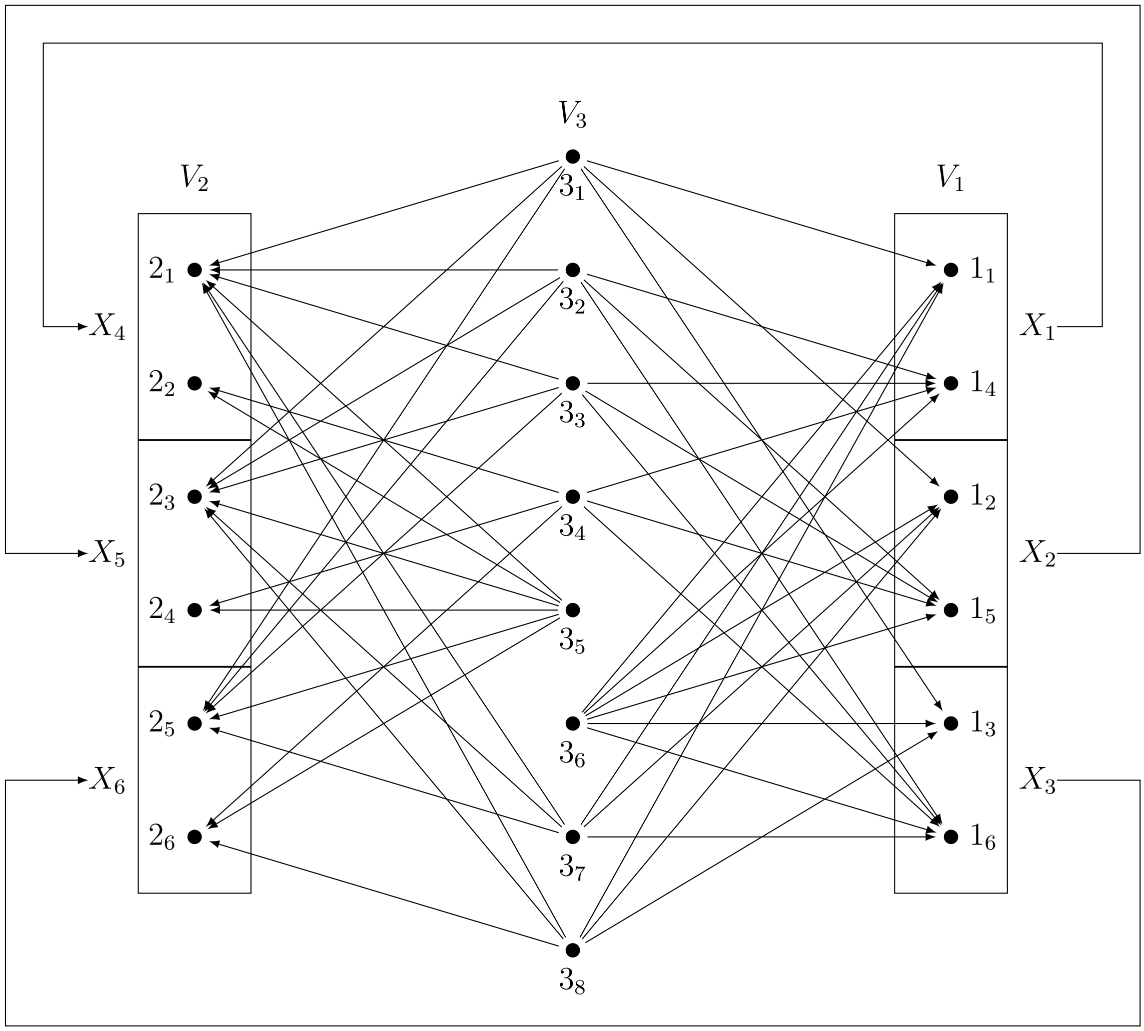


Figure 1: Orientation $F$ for $d = 3$, and $k = 2$.
For clarity, only the arcs from (1) $V_1$ to $V_2$ and (2) $V_3$ to $V_1$ and $V_2$ are shown.

Claim: For all $u, v \in V(K(p,p,q))$, $d_F(u,v) \leq 2$.

Case 1. $u = 1_a, v = 1_b$, $a \neq b$.

Since $1 \leq a, b \leq p = kd$, let $a = (\alpha_1 - 1)d + \alpha_2$ and $b = (\beta_1 - 1)d + \beta_2$ for some $\alpha_i, \beta_i$, $i = 1, 2$, satisfying $1 \leq \alpha_1, \beta_1 \leq k$ and $1 \leq \alpha_2, \beta_2 \leq d$. If $\alpha_2 = \beta_2$, then $\alpha_1 \neq \beta_1$. Note that $1_a$ and $1_b$ are in the same $X_i$ and by (III)(a), $1_a \to 3_{\beta_1} \to 1_b$. If $\alpha_2 \neq \beta_2$, then $1_a$ and $1_b$ are in different $X_i$'s and by (I), $1_a \to X_{d+\alpha_2} \to 1_b$.

Case 2. $u = 2_a, v = 2_b$, $a \neq b$.

Since $1 \leq a, b \leq p = kd$, let $a = (\alpha_1 - 1)k + \alpha_2$ and $b = (\beta_1 - 1)k + \beta_2$ for some $\alpha_i, \beta_i$, $i = 1, 2$, satisfying $1 \leq \alpha_1, \beta_1 \leq d$ and $1 \leq \alpha_2, \beta_2 \leq k$. If $\alpha_1 = \beta_1$, then $\alpha_2 \neq \beta_2$. Note that $2_a$ and $2_b$ are in the same $X_i$ and by (III)(b), $2_a \to 3_{\beta_2+k} \to 2_b$. If $\alpha_1 \neq \beta_1$, then $2_a$ and $2_b$ are in different $X_i$'s and by (I), $2_a \to X_{\beta_1} \to 2_b$.

Case 3. $u = 1_a, v = 2_b$.

By (II), $1_a \to 3_{2k+1} \to 2_b$.

Case 4. $u = 2_a, v = 1_b$.

By (II), $2_a \to 3_{2k+2} \to 1_b$.

Case 5. $u = 1_a, v = 3_b$.

If $b = 2k+1$, then $V_1 \to 3_{2k+1}$ by (II). Suppose $b \neq 2k+1$ and $1_a \in X_{i^*}$ for some $i^* = 1, 2, \ldots, d$. Then $1_a \to X_{d+i^*}$ by (I). Since for each $3_b$, $I(3_b) \cap X_{d+i} \neq \emptyset$ for each $i = 1, 2, \ldots, d$, by (II) and (III), let $w \in I(3_b) \cap X_{d+i^*}$. It follows that $1_a \to w \to 3_b$.

Case 6. $u = 2_a, v = 3_b$.

If $b = 2k+2$, then $V_2 \to 3_{2k+2}$ by (II). Suppose $b \neq 2k+2$ and $2_a \in X_{d+i^*}$ for some $i^* = 1, 2, \ldots, d$. Then $2_a \to V_1 - X_{i^*}$ by (I). Since for each $3_b$, $I(3_b) \cap X_i \neq \emptyset$ for each $i = 1, 2, \ldots, d$, by (II) and (III), let $w \in I(3_b) \cap X_j$ for some $j = 1, 2, \ldots, d$ and $j \neq i^*$. It follows that $2_a \to w \to 3_b$.

Case 7. $u = 3_a, v = 1_b$.

If $a = 2k+2$, then $3_{2k+2} \to V_1$ by (II). Suppose $a \neq 2k+2$ and $1_b \in X_{i^*}$ for some $i^* = 1, 2, \ldots, d$. Then $X_{d+j} \to 1_b$ for all $j = 1, 2, \ldots, d$ and $j \neq i^*$ by (I). Since for each $3_a$, $O(3_a) \cap X_{d+i} \neq \emptyset$ for each $i = 1, 2, \ldots, d$, by (II) and (III), let $w \in O(3_a) \cap X_{d+j}$ for some $j = 1, 2, \ldots, d$, and $j \neq i^*$. It follows that $3_a \to w \to 1_b$.

Case 8. $u = 3_a, v = 2_b$.

If $a = 2k+1$, then $3_{2k+1} \to V_2$ by (II). Suppose $a \neq 2k+1$ and $2_b \in X_{d+i^*}$ for some $i^* = 1, 2, \ldots, d$. Then $X_{i^*} \to 2_b$ by (I). Since for each $3_a$, $O(3_a) \cap X_i \neq \emptyset$ for each $i = 1, 2, \ldots, d$, by (II) and (III), let $w \in O(3_a) \cap X_{i^*}$. It follows that $3_a \to w \to 2_b$.

Case 9. $u = 3_a, v = 3_b$.

Subcase 9a. $a \neq 2k+1, 2k+2$ and $b \neq 2k+1, 2k+2$.

Observe from (III) that $|O(3_x) \cap (V_1 \cup V_2)| = p$ for $x = a, b$. Furthermore,

$O(3_a) \cap (V_1 \cup V_2) \nsubseteq O(3_b) \cap (V_1 \cup V_2)$ if $b \neq a$. Thus, there exists a vertex $w \in V_1 \cup V_2$ such that $3_a \to w \to 3_b$.

Subcase 9b. $a = 2k+1$ and $b \neq 2k+1, 2k+2$.

Note that $3_{2k+1} \to V_2$ by (II), and $I(3_b) \cap X_{d+i} \neq \emptyset$ for every $i = 1, 2, \ldots, d$, imply the existence of $w \in I(3_b) \cap V_2$. Hence, $3_a \to w \to 3_b$.

Subcase 9c. $a = 2k+2$ and $b \neq 2k+1, 2k+2$.

Note that $3_{2k+2} \to V_1$ by (II), and $I(3_b) \cap X_i \neq \emptyset$ for every $i = 1, 2, \ldots, d$, imply the existence of $w \in I(3_b) \cap V_1$. Hence, $3_a \to w \to 3_b$.

Subcase 9d. $a \neq 2k+1, 2k+2$ and $b = 2k+1$.

Note that $V_1 \to 3_{2k+1}$ by (II), and $O(3_a) \cap X_i \neq \emptyset$ for every $i = 1, 2, \ldots, d$, imply the existence of $w \in O(3_a) \cap V_1$. Hence, $3_a \to w \to 3_b$.

Subcase 9e. $a \neq 2k+1, 2k+2$ and $b = 2k+2$.

Note that $V_2 \to 3_{2k+2}$ by (II), and $O(3_a) \cap X_{d+i} \neq \emptyset$ for every $i = 1, 2, \ldots, d$, imply the existence of $w \in O(3_a) \cap V_2$. Hence, $3_a \to w \to 3_b$.

Subcase 9f. $\{a, b\} = \{2k+1, 2k+2\}$.

By (II), $3_{2k+1} \to V_2 \to 3_{2k+2} \to V_1 \to 3_{2k+1}$. □

Since $p$ may have different factorisations, the natural question to ask is which non-trivial divisor(s) $d$ of $p$ gives the best bound. Verification, using Maple [17], for all non-trivial divisors $d$ of each composite integer $p \leq 100$ shows that $\max_d\{\Phi^*(p,d)\} = \Phi^*(p, d_0)$ with $d_0$ being the smallest non-trivial divisor of each $p$. Therefore, if $p$ is even, we define

$$\begin{aligned}\Phi_{even}(p) &:= \Phi^*(p,2)\\ &= \sum_{s=0}^{2}\sum_{t=0}^{2}\left[(-1)^{(s+t)}\binom{4}{s,t,4-(s+t)}\binom{(4-(s+t))\frac{p}{2}}{(2-t)\frac{p}{2}}\right]\\ &= \binom{2p}{p} - 8\binom{\frac{3p}{2}}{p} + 12\binom{p}{\frac{p}{2}} - 6.\end{aligned}$$

Furthermore, we wish to extend Definition 3.1 and Proposition 3.7 for prime numbers and $d = 2$ seems to be the best candidate. Hence, we have the following analogue, $\Phi_{odd}(p)$, for odd integers $p \geq 5$, which also provides a better bound than $\Phi(p, d_0)$ in cases where $p$ is odd and composite.

**Definition 3.8.** Suppose $p \geq 5$ is an odd integer. Denote a solution $(x_1, x_2, x_3, x_4)^{**}$ if $(x_1, x_2, x_3, x_4)$ satisfies

$$\begin{aligned}&x_1 + x_2 + x_3 + x_4 = p,\\ &1 \leq x_i \leq \lfloor\frac{p}{2}\rfloor, \text{ for } i = 1, 2, \text{ and}\\ &1 \leq x_i \leq \lfloor\frac{p}{2}\rfloor - 1, \text{ for } i = 3, 4.\end{aligned}$$

Define $\Phi_{odd}(p) := \sum_{(x_1,x_2,x_3,x_4)^{**}} \binom{\lfloor\frac{p}{2}\rfloor+1}{x_1}\binom{\lfloor\frac{p}{2}\rfloor+1}{x_2}\binom{\lfloor\frac{p}{2}\rfloor}{x_3}\binom{\lfloor\frac{p}{2}\rfloor}{x_4}$.

The following expression for $\Phi_{odd}(p)$ can be derived by exhausting all cases and is provided without proof.

**Lemma 3.9.** *If $p \geq 5$ is an odd integer, then $\Phi_{odd}(p) = \binom{2p}{p} - 4\binom{3x+2}{x+1} - 4\binom{3x+1}{x} + 2\binom{2x+2}{x+1} + 8\binom{2x+1}{x} + 2\binom{2x}{x} - 4$, where $x = \lfloor\frac{p}{2}\rfloor$.*

We shall now prove that $\Phi_{even}(p)$ and $\Phi_{odd}(p)$ are both greater than $\max\limits_{3\leq d<p}\{\Phi^*(p,d)\}$ for each $p \geq 4$.

**Proposition 3.10.** *Suppose $p \geq 4$ is a composite integer and $d$ is a divisor of $p$, where $3 \leq d < p$.*

$$\max_{3\leq d<p}\{\Phi^*(p,d)\} < \begin{cases} \Phi_{even}(p), & \text{if } p \text{ is even,} \\ \Phi_{odd}(p), & \text{if } p \text{ is odd.} \end{cases}$$

*Proof*: Case 1. $p$ is even.

For any even integer $p \geq 14$ and any divisor $3 \leq d < p$ of $p$, we have

$$\begin{aligned} \binom{2p-\frac{p}{d}}{p} - 8\binom{\frac{3p}{2}}{p} + 12\binom{p}{\frac{p}{2}} - 6 &\geq \binom{2p-\frac{p}{d}}{p} - 8\binom{\frac{3p}{2}}{p} \\ &\geq \binom{\frac{5p}{3}}{p} - 8\binom{\frac{3p}{2}}{p} \\ &> 0. \end{aligned} \tag{2}$$

The first inequality above is due to the fact that $12\binom{p}{\frac{p}{2}} \geq 6$, while the second inequality follows as $d \geq 3$ and $f(z) := \binom{z}{p}$ is an increasing function for $z \geq p$. Since $f(z)$ is also strictly convex for $z \geq p$ and $\binom{\frac{5(13)}{3}}{13} - 8\binom{\frac{3(13)}{2}}{13} > 0$, the last inequality above follows for all $p \geq 13$.

Now, for each even integer $p \leq 12$, we have verified, using Maple, that $\Phi^*(p,d) < \Phi_{even}(p)$ for all divisors $3 \leq d < p$ of $p$. (See Table 1.) Let $p \geq 14$ be an even integer. Note that $\sum\limits_{i=1}^{d}\sum\limits_{j=0}^{d}\Phi(p,d,[i,j]) \geq \binom{k}{0}\binom{(2d-1)k}{p} = \binom{2p-\frac{p}{d}}{p}$ as the expression $\binom{k}{0}\binom{(2d-1)k}{p}$ counts the number of ways such that none is selected from a (fixed) group of $k$ elements and $p$ elements are selected from the remaining $2d-1$ groups of $k$ elements. Also, recall that $\binom{2p}{p} = \sum\limits_{i=0}^{d}\sum\limits_{j=0}^{d}\Phi(p,d,[i,j]) = \Phi(p,d,[0,0]) + \sum\limits_{i=1}^{d}\sum\limits_{j=0}^{d}\Phi(p,d,[i,j]) + \sum\limits_{j=1}^{d}\Phi(p,d,[0,j])$ by generalised Vandermonde's identity. It follows for each even integer $p \geq 14$ and each divisor $3 \leq d < p$ of $p$ that

$$\binom{2p}{p} - \Phi^*(p,d) = \binom{2p}{p} - \Phi(p,d,[0,0])$$

$$\begin{aligned}
&= \sum_{i=1}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j]) + \sum_{j=1}^{d}\Phi(p,d,[0,j]) \\
&\geq \binom{2p-\frac{p}{d}}{p} \\
&> 8\binom{\frac{3p}{2}}{p} - 12\binom{p}{\frac{p}{2}} + 6 \\
&= \binom{2p}{p} - \Phi_{even}(p),
\end{aligned}$$

where the last inequality is due to (2).

Case 2. $p$ is odd and composite.

Denote $x := \lfloor \frac{p}{2} \rfloor$. For any composite and odd integer $p \geq 17$ and any divisor $3 \leq d < p$ of $p$, we have

$$\begin{aligned}
\binom{2p-\frac{p}{d}}{p} - 4\binom{3x+2}{x+1} - 4\binom{3x+1}{x} &= \binom{2p-\frac{p}{d}}{p} - 4\binom{3x+2}{2x+1} - 4\binom{3x+1}{2x+1} \\
&\geq \binom{2p-\frac{p}{3}}{p} - 8\binom{3x+2}{2x+1} \\
&\geq \binom{\frac{10x+5}{3}}{2x+1} - 8\binom{3x+2}{2x+1} \\
&> 0. \qquad (3)
\end{aligned}$$

The first inequality above is due to the assumption that $d \geq 3$ and $f(z)$ is an increasing function for $z \geq p$. Since $f(z)$ is also strictly convex for $z \geq p$ and $\binom{\frac{10(8)+5}{3}}{2(8)+1} - 8\binom{3(8)+2}{2(8)+1} > 0$, the last inequality above follows for all $x \geq 8$.

For each composite and odd integer $p \leq 15$, we have verified, using Maple, that $\Phi^*(p,d) < \Phi_{odd}(p)$ for all divisors $3 \leq d < p$ of $p$. (See Table 1.) Now, consider any composite and odd integer $p \geq 17$. As in Case 1, $\sum_{i=1}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j]) \geq \binom{2p-\frac{p}{d}}{p}$. It follows for each composite and odd integer $p \geq 17$ and each divisor $3 \leq d < p$ of $p$ that

$$\begin{aligned}
&\binom{2p}{p} - \Phi^*(p,d) \\
&= \binom{2p}{p} - \Phi(p,d,[0,0]) \\
&= \sum_{i=1}^{d}\sum_{j=0}^{d}\Phi(p,d,[i,j]) + \sum_{j=1}^{d}\Phi(p,d,[0,j]) \\
&\geq \binom{2p-\frac{p}{d}}{p} \\
&> 4\binom{3x+2}{x+1} + 4\binom{3x+1}{x}
\end{aligned}$$

$$\begin{aligned} &\geq \; 4\binom{3x+2}{x+1}+4\binom{3x+1}{x}-2\binom{2x+2}{x+1}-8\binom{2x+1}{x}-2\binom{2x}{x}+4 \\ &= \; \binom{2p}{p}-\Phi_{odd}(p), \end{aligned}$$

where the second last inequality follows from (3). $\square$

In a way similar to Proposition 3.7, we can derive a sufficient condition for $\bar{d}(K(p,p,q)) = 2$ using $\Phi_{odd}(p)$ when $p$ is odd. For clarity, we summarise the results in the next theorem.

**Theorem 3.11.** *Suppose $p \geq 4$ is an integer. Then*

$$\bar{d}(K(p,p,q)) = 2 \text{ if } \begin{cases} p+2 \leq q \leq \Phi_{even}(p)+2 \text{ and } p \text{ is even}, \\ p+3 \leq q \leq \Phi_{odd}(p)+2 \text{ and } p \text{ is odd}, \end{cases}$$

*where $\Phi_{even}(p) = \binom{2p}{p} - 8\binom{\frac{3p}{2}}{p} + 12\binom{p}{\frac{p}{2}} - 6$ and $\Phi_{odd}(p) = \binom{2p}{p} - 4\binom{3x+2}{x+1} - 4\binom{3x+1}{x} + 2\binom{2x+2}{x+1} + 8\binom{2x+1}{x} + 2\binom{2x}{x} - 4$, $x = \lfloor \frac{p}{2} \rfloor$.*

**Corollary 3.12.** *Suppose $n \geq 2$ and $p_i$ are positive integers for $i = 1, 2, \ldots, n$ such that $p_1 + p_2 + \ldots + p_r = p_{r+1} + p_{r+2} + \ldots + p_n = p \geq 4$ for some integers $r$ and $p$. Let $G = K(p_1, p_2, \ldots, p_n, q)$. Then*

$$\bar{d}(G) = 2 \text{ if } \begin{cases} p+2 \leq q \leq \Phi_{even}(p)+2 \text{ and } p \text{ is even}, \\ p+3 \leq q \leq \Phi_{odd}(p)+2 \text{ and } p \text{ is odd}. \end{cases}$$

*Proof*: Note that $G$ is a supergraph of $K(p,p,q)$ and $\bar{d}(K(p,p,q)) = 2$ by Theorem 3.11. So, there exists an orientation $D$ for $K(p,p,q)$, where $d(D) = 2$. Partition $V(G)$ into three parts $\bigcup_{i=1}^{r} V_i$, $\bigcup_{i=r+1}^{n} V_i$ and $V_{n+1}$, and define an orientation $F$ for $G$ such that $D$ is a subdigraph of $F$ and edges not in $D$ are oriented arbitrarily. It follows that $d(F) = 2$. $\square$

| $p$ | $d$ | $\begin{cases} \Phi_{even}(p) - \Phi^*(p,d), & \text{if } p \text{ is even}, \\ \Phi_{odd}(p) - \Phi^*(p,d), & \text{if } p \text{ is odd}. \end{cases}$ |
|---|---|---|
| 4 | 2 | 16-16=0 |
| 6 | 2 | 486-486=0 |
| 6 | 3 | 486-64=422 |
| 8 | 2 | 9,744-9,744=0 |
| 8 | 4 | 9,744-256=9,488 |
| 9 | 3 | 39,400-14,580=24,820 |
| 10 | 2 | 163,750-163,750=0 |
| 10 | 5 | 163,750-1,024=162,726 |
| 12 | 2 | 2,566,726-2,566,726=0 |
| 12 | 3 | 2,566,726-1,580,096=986,630 |
| 12 | 4 | 2,566,726-459,270=2,107,456 |
| 12 | 6 | 2,566,726-4,096=2,562,630 |
| 14 | 2 | 39,227,538-39,227,538=0 |
| 14 | 7 | 39,227,538-16,384=39,211,154 |
| 15 | 3 | 152,558,168-121,562,500=30,995,668 |
| 15 | 5 | 152,558,168-14,880,348=137,677,820 |
| 16 | 2 | 595,351,056-595,351,056=0 |
| 16 | 4 | 595,351,056-269,992,192=325,358,864 |
| 16 | 8 | 595,351,056-65,536=595,285,520 |
| 18 | 2 | 9,038,224,134-9,038,224,134=0 |
| 18 | 3 | 9,038,224,134-8,120,234,620=917,989,514 |
| 18 | 6 | 9,038,224,134-491,051,484=8,547,172,650 |
| 18 | 9 | 9,038,224,134-262,144=9,037,961,990 |
| 20 | 2 | 137,608,385,766-137,608,385,766=0 |
| 20 | 4 | 137,608,385,766-95,227,343,750=42,381,042,016 |
| 20 | 5 | 137,608,385,766-47,519,843,328=90,088,542,438 |
| 20 | 10 | 137,608,385,766-1,048,576=137,607,337,190 |

Table 1: Comparison of $\Phi^*(p,d)$ with $\Phi_{even}(p)$ and $\Phi_{odd}(p)$ for $4 \leq p \leq 20$.

As a concluding remark, let us mention that for $r \geq 3$ and $p \geq 2$, Koh and Tan [6] defined the function $f(r,p)$ to be the greatest integer such that $\bar{d}(K(\overbrace{p,p,\ldots,p}^{r},q)) = 2$ for all $q$ with $1 \leq q \leq f(r,p)$. They posed the problem of determining $f(r,p)$, which looks very difficult. By Theorems 1.10 and 3.11, it follows that

$$f(2,p) \geq \begin{cases} \Phi_{even}(p) + 2, & \text{if } p \text{ is even}, \\ \Phi_{odd}(p) + 2, & \text{if } p \text{ is odd}, \end{cases}$$

for $p \geq 4$.

## Acknowledgements

We would like to express our gratitude to Dr Toh Pee Choon for the helpful discussions. The first author would like to thank the National Institute of Education, Nanyang Technological University of Singapore, for the generous support of the Nanyang Technological University Research Scholarship.

We would also like to thank the editors and referees for their helpful comments.

## References

[1] F. Boesch and R. Tindell, Robbin's Theorem for Mixed Multigraphs, *Amer. Math. Monthly* 87 (1980), 716–719.

[2] C. C. Chen and K. M. Koh, *Principles and Techniques in Combinatorics*, World Scientific (1992), 69–74.

[3] V. Chvátal and C. Thomassen, Distances in orientations of graphs, *J. Combin. Theory Ser. B* 24 (1978), 61–75.

[4] G. Gutin, $m$-sources in complete multipartite digraphs, *Vestsi Acad. Navuk BSSR Ser. Fiz-Mat.* 5 (1989), 101–106 (in Russian).

[5] G. Gutin, Minimizing and maximizing the diameter in orientations of graphs, *Graphs Combin.* 10 (1994), 225–230.

[6] K. M. Koh and B. P. Tan, The diameter of an orientation of a complete multipartite graph, *Discrete Math.* 149 (1996), 131–139.

[7] K. M. Koh and B. P. Tan, The minimum diameter of orientations of complete multipartite graphs, *Graphs Combin.* 12 (1996), 333–339.

[8] K. M. Koh and E. G. Tay, Optimal orientations of graphs and digraphs: A survey, *Graphs Combin.* 18 (2002), 745–756.

[9] S. B. Maurer, The king chicken theorems, *Math. Mag.* 53 (1980), 67–80.

[10] K. L. Ng, *A new direction in the study of the orientation number of a graph*, Ph.D. Thesis, National University of Singapore, Department of Mathematics, (2004), 15–23.

[11] J. Plesnik, Diametrically critical tournaments, *Časopis pro pěstovánái matematiky* 100 (1975), 361–370.

[12] J. Plesnik, Remarks on diameters of orientations of graphs, *Acta Math. Univ. Comenian. (N.S.)* 46/47 (1985), 225–236.

[13] G. Rajasekaran and R. Sampathkumar, Optimal orientations of some complete tripartite graphs, *Filomat* 29 (2015), 1681–1687.

[14] H. E. Robbins, A Theorem on Graphs, with an Application to a Problem of Traffic Control, *Amer. Math. Monthly* 46 (1939), 281–283.

[15] E. Sperner, Ein satz über Untermengen einer endlichen Menge, *Math. Z.* 27 (1928), 544–548.

[16] L. Šoltés, Orientations of graphs minimizing the radius or the diameter, *Math. Slovaca* 36 (1986), 289–296.

[17] Maple (2017), Maplesoft, a division of Waterloo Maple Inc., Waterloo, Ontario.

(Received 31 Dec 2018; revised 4 Feb 2021)