A Generalization of Fan-Type Conditions for Hamiltonian and Hamiltonian-Connected Graphs*

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Abstract

In this paper, we prove if G is a 2-connected graph of order n and $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of nonadjacent vertices u, v of G with $1 \leq |N(u) \cap N(v)| \leq \alpha - 2$, then either G is Hamiltonian or else G belongs to one of a family of exceptional graphs. We give a similar sufficient condition for Hamiltonian-connected graphs.

§1. Introduction

We consider only finite undirected graphs without loops or multiple edges. For notation and terminology not defined here we refer to [3].

For a graph $G = (V, E)$, let $N(v)$ be the set of vertices adjacent to the vertex v in G and $d(v)$ be the degree of v in G. We denote by α and $d_G(u, v)$ the independence number of G and the distance between vertices u, v in G, respectively.

Geng-Hua Fan [4] established the following result.

Theorem 1 *If G is a 2-connected graph of order n and $\max\{d(u), d(v)\} \geq \frac{n}{2}$ for each pair of vertices u, v with $d(u, v) = 2$ in G, then G is Hamiltonian.*

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More recently, Guantao Chen [5] and Song Zeng Min [6] independently generalized Fan's theorem as follows.

Theorem 2 *If G is a 2-connected graph of order n and* $\max\{d(u), d(v)\} \geq \frac{n}{2}$ *for each pair of nonadjacent vertices u, v with* $1 \leq |N(u) \cap N(v)| \leq \alpha - 1$, *then G is Hamiltonian.*

Since $d(u,v) = 2$ if and only if $|N(u) \cap N(v)| \geq 1$ for a pair of nonadjacent vertices u, v of G, Theorem 1 is an immediate consequence of Theorem 2. In fact, there are many examples showing that Theorem 2 is stronger than Fan's Theorem (see [5]).

In this paper, we shall prove the following two theorems.

Theorem 3 *Let G be a 2-connected graph of order n. If* $\max\{d(u), d(v)\} \geq \frac{n}{2}$ *for every pair of nonadjacent vertices u, v with* $1 \leq |N(u) \cap N(v)| \leq \alpha - 2$, *then either G is Hamiltonian or G is a spanning subgraph of the nonhamiltonian graph* $(\bigcup_{i=1}^{\alpha} K_{n_i}) \vee K_{\alpha-1}$.

Theorem 4 *Let G be a 3-connected graph of order n. If* $\max\{d(u), d(v)\} \geq \frac{n+1}{2}$ *for every pair of nonadjacent vertices u, v with* $1 \leq |N(u) \cap N(v)| \leq \alpha - 1$, *then either G is Hamiltonian-connected or G is a spanning subgraph of the nonhamiltonian-connected graph* $(\bigcup_{i=1}^{\alpha} K_{n_i}) \vee K_{\alpha}$.

We also obtain A. Benhocine and A.P. Wojda's a result [1] below as a special case of Theorem 4.

Theorem 5 *([1]) If G is a 3-connected graph of order n and* $\max\{d(u), d(v)\} \geq \frac{n+1}{2}$ *for every pair of nonadjacent vertices u, v with $d(u,v) = 2$ in G, then G is Hamiltonian-connected.*

Remark 1. There are many Hamiltonian graphs showing that Theorem 3 is stronger than Theorem 1 and 2. We construct one of these by taking three vertex disjoint graphs, a complete graph K_{m-1}, a complete bipartite graph $K_{2,m}$ and a graph $K_{n-2m+1} \setminus \{e\}$ obtained by deleting an edge of K_{n-2m-1}, so that two vertices belonging to the same part of the bipartition of $K_{2,m}$ are joined to all vertices of $K_{n-2m-1} \setminus \{e\}$ and each vertex of another part of $K_{2,m}$ is joined to all vertices in K_{m-1}, where $m \geq 2$ and $n \geq 4(m+1)$. This graph is shown in Figure 1.

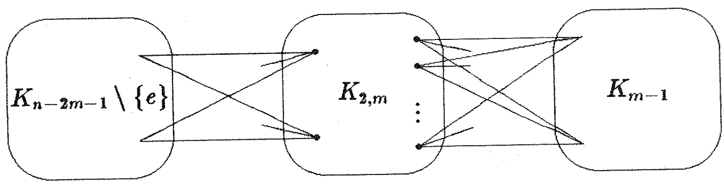

Figure 1

Another example is depicted in Figure 2.

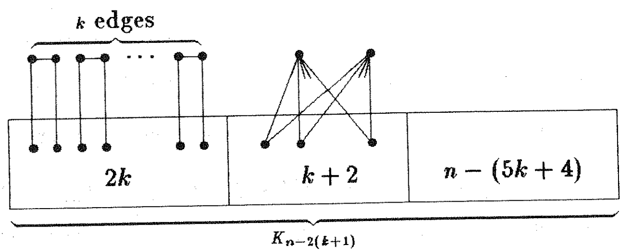

Figure 2

It is easy to check that the two graphs as shown in Figures 1 and 2 satisfy the condition of Theorem 3, but not that of Theorem 1 and 2 when $m < \frac{n-2}{2}$.

Remark 2. There are Hamiltonian-connected graphs which satisfy the condition of Theorem 4 but do not satisfy the condition of Theorem 5. One of these is depicted in Figure 3, where m and n are two positive integers with $3 \leq m \leq \frac{n-3}{2}$ and $n \geq 4(m+1)$. Therefore, Theorem 4 is stronger than Theorem 5.

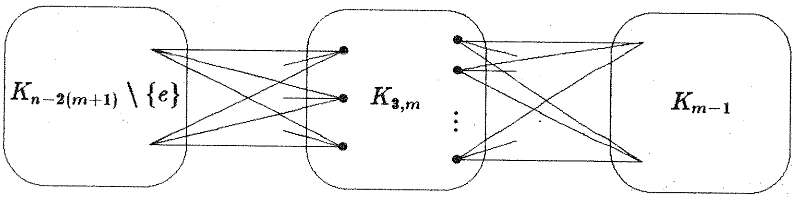

Figure 3

§2. The proof of Theorems

Before proving Theorems 3 and 4, we give a number of definitions.

Let G be a graph. For F and H subgraphs or vertex subsets of G, let $G[F]$ denote the subgraph of G induced by the vertices of G and $N_F(H)$ denote the set of neighbours of vertices of H that belong to F. For X a path or a cycle of G, let $\overrightarrow{X}$ denote the set X with a given orientation. If $u, v \in V(X)$, then $u \overrightarrow{X} v$ denotes the subpath of X on $\overrightarrow{X}$ from u to v. The same vertices, in reverse order, are given by $v \overleftarrow{X} u$. For $S \subseteq V(X)$, we use S^+ (resp. S^-) to denote the successors (resp. predecessors) of vertices of S on $\overrightarrow{X}$. Let uHv denote a u–v path in which all internal vertices belong to H.

The proof of Theorem 3. Suppose that $G = (V, E)$ is a nonhamiltonian graph satisfying the hypothesis of Theorem 3. Let $B = \{v \in V(G) \mid d(v) \geq \frac{n}{2}\}$ and $E' = E(G) \cup \{uv \notin E(G) \mid u, v \in B\}$. Consider the graph $G' = (V(G), E')$. By the Bondy and Chvátal Closure Theorem [2], G' in nonhamiltonian. Clearly, B is contained in some cycle of G'. Let C be a maximal cycle containing B in G' and let H be a component of $G' - V(C)$. Let $v_1, v_2, \cdots, v_k$ be the elements of $N_C(H)$ occurring on $\vec{C}$ in consecutive order and let $x_i \in N(v_i) \cap V(H)$ for $i = 1, 2, \cdots, k$. Since G is 2-connected, we have $k \geq 2$. Note that, for any $i, j \in \{1, 2, \cdots, k\}$ with $i \neq j$, the path

$$v_i^+ \vec{C} v_j H v_i \overleftarrow{C} v_j^+$$

contains at least one vertex of H and contains C. Since C is a maximal cycle, we must have $v_i^+ v_j^+ \notin E(G')$. Thus it follows from the definition of $N_C^+(H)$ that

(2.1) For any i with $1 \leq i \leq k$, $\{x_i\} \cup N_C^+(H)$ is an independent set.

Also, by the construction of G', we conclude that there exists at most one vertex in $N_C^+(H)$ belonging to B. Without loss of generality, we assume that $d(v_i^+) < \frac{n}{2}$ for $i = 1, 2, \cdots, k-1$ and so $v_i v_i^+ \in E(G)$ by the definitions of G, B and G'. Hence $d(x_i, v_i^+) = 2$ for every $i \neq k$. Since $\max\{d(x_i), d(v_i^+)\} < \frac{n}{2}$, by the assumption of Theorem we have

(2.2) $$|N(x_i) \cap N(v_i^+)| \geq \alpha - 1, \quad i = 1, 2, \cdots, k-1.$$

Now, we see by (2.1) that $\alpha \geq k+1$. Note that $N(x_i) \cap N(v_i^+) \subseteq \{v_1, v_2, \cdots, v_k\}$. Hence, the following two statements hold by (2.2).

(2.3) $$\alpha = k + 1$$

(2.4) $$N(x_i) \cap N(v_i^+) = \{v_1, v_2, \cdots, v_k\}, \; i = 1, 2, \cdots, k-1.$$

For $i \neq j$, let $R = v_i^+ \vec{C} v_j^+$ and $S = V(C) - R$. Then we conclude that

(2.5) $$N_R^-(v_i^+) \cap N_R(v_j^+) = \emptyset.$$

To prove (2.5) suppose $v \in N_R^-(v_i^+) \cap N_R(v_j^+)$. By (2.1), $v \neq v_i^+$ and $v \neq v_j$. Hence we see that

$$v_i H v_j \overleftarrow{C} v^+ v_i^+ \vec{C} v v_j^+ \vec{C} v_i$$

is a cycle longer than C. This contradiction shows (2.5).

An analogous argument proves that the statement given below also holds.

(2.6) $$N_S^+(v_i^+) \cap N_S(v_j^+) = \emptyset.$$

Next, we shall give a characterization of G by showing three statements. Set $V_0 = V(H)$ and $V_i = v_i^+ \vec{C} v_{i+1}^-$ for $i = 1, 2, \cdots, k$, (indices taken modulo k).

(2.7) For each $i = 0, 1, \cdots, k$, $G[V_i]$ is complete.

Applying (2.1) and (2.3), the conclusion follows for $i = 0$. If $|V_i| \leq 2$ for $i \neq 0$, then we are done. So assume that $|V_i| \geq 3$ for some $i \neq 0$. If $i \neq k$, then by (2.4), $v_i^+ v_{i+1} \in E(G')$ and hence (2.5) implies $v_{i+1}^- \notin N(v_j^+)$ for every $j \neq i$ with $1 \leq j \leq k$. Thus, by using (2.1) and (2.3), we must have $v_i^+ v_{i+1}^- \in E(G')$ for otherwise $\{x_i, v_{i+1}^-\} \cup N_C^+(H)$ would be an independent set of cardinality $k+2$ since $v_i^{++} \neq v_{i+1}^-$. This contradicts (2.3). Continuing the process if $|V_i| > 3$, we shall eventually obtain

$$V_i \setminus \{v_i^+\} \subset N(v_i^+) .$$

Now, if $G[V_i]$ is not complete, then there must exist two vertices $u, v \in V_i \setminus \{v_i^+\}$ so that $uv \notin E(G')$. Clearly, $u^+, v^+ \notin \{u, v\}$. Since $u^+, v^+ \in N(v_i^+)$, using (2.5) we have

$$(N(u) \cup N(v)) \cap (N_C^+(H) \setminus \{v_i^+\} = \emptyset.$$

Therefore, we see that $\{u, v, x_1\} \cup (N_C^+(H) \setminus \{v_i^+\})$ is an independent set of cardinality $k+2$ contradicting (2.3). Thus, $G[V_i]$ is a complete subgraph for $i \neq k$. Now, the proof only for the case $i = k$ remains. If one of $d(v_k^+)$ and $d(v_1^-)$ is less than $\frac{n}{2}$, then, by an argument analogous to one above, we have finished. So assume that $d(v_k^+) \geq \frac{n}{2}$ and $d(v_1^-) \geq \frac{n}{2}$ and so $v_1^- v_k^+ \in E(G')$. Applying (2.5) we obtain

$$N(v_1^{--}) \cap \{x_i, v_1^+, v_2^+, \cdots, v_{k-1}^+\} = \emptyset$$

and thus $v_1^{--}v_k^+ \in E(G')$ if $|V_k| > 3$ since otherwise $\{x_i, v_1^{--}\} \cup N_C^+(H)$ would be an independent set of cardinality $k+2$, which contradicts to (2.3). Continuing the same process, we further obtain

$$V_k \setminus \{v_k^+\} \subset N(v_k^+).$$

By symmetry, we also have that

$$V_k \setminus \{v_1^-\} \subset N(v_1^-).$$

Again, by using the arguments of $i \neq k$, it follows that $G[V_k]$ is complete. Therefore, (2.7) is verified.

(2.8) For any $i, j \in \{0, 1, \cdots, k\}$ with $i \neq j$, $N(v_i) \cap V_j = \emptyset$.

Otherwise, assume there exists two vertices $u \in V_i$ and $v \in V_j$ with $i \neq j$ such that $uv \in E(G')$. By the assumption, we have $i, j \neq 0$. Without loss of generality, we suppose that $i \neq k$. It is easy to see that

$$C' = \begin{cases} uv \overleftarrow{C} v_j^+ v^+ \overrightarrow{C} v_i H v_j \overrightarrow{C} u^- v_i^+ \overrightarrow{C} u & \text{if } v \neq v_{j+1}^- \\ uv \overleftarrow{C} v_{i+1} H v_{j+1} \overrightarrow{C} u^- v_{i+1}^- \overleftarrow{C} u & \text{if } v = v_{j+1}^- \end{cases}$$

is a cycle longer than C. This contradiction shows (2.8).

(2.9) $$V(G) = V(C) \cup V(H).$$

Suppose the contrary. Let H' be another component of $G' - V(C)$ and y be a vertex of H'. Then we can conclude that $N(y) \cap N_C^+(H) \neq \emptyset$ and $N(y) \cap N_C^-(H) \neq \emptyset$ for otherwise $\{x_1, y\} \cup N_C^+(H)$ or $\{x_1, y\} \cup N_C^-(H)$ would be an independent set of cardinality $k+2$ contradicting (2.3). So we may assume, without loss of generality, that $yv_1^+ \in E(G')$ and $yv_i^- \in E(G')$. If either $|V_1| \geq 2$ or $|V_i| \geq 2$, then by (2.1) and (2.8), we see that either $\{v_1^{++}, x_1, y\} \cup (N_C^+(H) \setminus \{v_1^+\})$ or $\{v_i^{--}, x_1, y\} \cup (N_C^-(H) \setminus \{v_1^-\})$ would be an independent set of cardinality $k+2$, which contradicts (2.3).

Thus we must have that $|V_1| = |V_i| = 1$. Since G' is 2-connected, we may assume that $i \neq 2$. Hence, the cycle

$$v_1 H v_2 \overrightarrow{C} v_i^- y v_1^+ v_i \overrightarrow{C} v_1$$

is longer than C. We again obtain a contradiction and thus statement (2.9) holds.

Combining the statements (2.7) through (2.9), it is easily seen that G' is a spanning subgraph, and therefore G is also one, of the nonhamiltonian graph $(\bigcup_{i=1}^{\alpha} K_{n_i}) \vee K_{\alpha-1}$. The proof of Theorem 3 is completed. ■

The proof of Theorem 4 Suppose that G is not a Hamiltonian-connected graph satisfying the condition of Theorem 4. Set $B = \{v \in V(G) \mid d(v) \geq \frac{n+1}{2}\}$ and $E' = \{uv \mid u, v \in B \text{ and } uv \notin E(G)\}$. Consider the graph $G' = G + E$. Since, by the Bondy and Chvátal Closure Theorem [2], G is Hamiltonian-connected if and only if G' is Hamiltonian-connected, G' is also not Hamiltonian-connected. Thus there exists a pair of vertices u, v of G' such that no Hamiltonian u–v path in G' exists. Clearly G' contains a u–v path through B. Let P be a maximal u–v path containing B in G' and H be a component of $G - V(P)$. Let $v_1, v_2, \cdots, v_k$ be the elements of $N_P(H)$, and assume they appear on $\overrightarrow{C}$ in consecutive order. Let $x_i \in N(v_i) \cap V(H)$ for each i with $1 \leq i \leq k$. Since G' is 3-connected, we have $k \geq 3$. Moreover, we establish the following statements.

(3.1)

For any $x \in V(H)$, both $\{x\} \cup N_P^+(H)$ and $\{x\} \cup N_P^-(H)$ are independent sets.

Since P is a maximal u–v path, $N(x) \cap N_P^+(H) = \emptyset$. If there exists $v_i^+, v_j^+ \in N_P^+(H)$ such that $v_i^+, v_j^+ \in E(G')$, then we see that

$$u \overrightarrow{P} v_i H v_j \overleftarrow{P} v_i^+ v_j^+ \overrightarrow{P} v$$

is a path longer than P, a contradiction. Thus $\{x\} \cup N_P^+(H)$ is an independent set and $\{x\} \cup N_P^-(H)$ is also one by symmetry.

From (3.1) and the construction of G', it follows that

(3.2) There exists at most one $h \in \{1, 2, \cdots, k\}$ such that $d(v_h^+) \geq \frac{n+1}{2}$.

(3.3) For each $i \neq h$, $|N_P(x_i)| = \alpha = k$ and $v_1 = u$ and $v_k = v$

By (3.2), $d(v_i^+) < \frac{n+1}{2}$ and hence $d_G(x_i, v_i^+) = 2$. Since $\max\{d(x_i), d(v_i^+)\} < \frac{n+1}{2}$, it follows from the hypothesis of the Theorem that

$$|N(x_i) \cap N(v_i^+)| \geq \alpha.$$

Note that $N(x_i) \cap N(v_i^+) \subseteq \{v_1, v_2, \cdots, v_k\}$. We must then have $\alpha = k$ and $N_P(x_i) = \{v_1, v_2, \cdots, v_k\}$. From this fact and (3.1), we deduce that $v_1 = u$ and $v_k = v$. Thus, (3.3) is proved.

By using an analogous argument to that of Theorem 3, we can show that the statements (3.4) through (3.8) as given below hold.

(3.4) For any $i \neq h$, $N(x_i) \cap N(v_i^+) = \{v_1, v_2, \cdots, v_k\}$.

(3.5) Let $R = v_i^+ \overrightarrow{P} v_j^-$ and $S = V(P) \setminus R$. Then

$$N_R^-(v_i^+) \cap N_R(v_j^+) = \emptyset \text{ and } N_S^+(v_i^+) \cap N_S(v_j^+) = \emptyset.$$

Put $V_0 = V(H)$ and $V_i = v_i^+ \overrightarrow{P} v_{i+1}^-$ for $i = 1, 2, \cdots, k-1$. Then we have

(3.6) For each $i = 0, 1, \cdots, k-1$, $G[V_i]$ is complete.

(3.7) For $i, j = 0, 1, \cdots, k-1$ with $i \neq j$, $N(v_i) \cap V_j = \emptyset$.

(3.8) $$V(G) = V(P) \cup V(H).$$

Now, by combining statements (3.6), (3.7) and (3.8), we easily see that G' is a spanning subgraph of the graph $(\bigcup_{i=1}^{\alpha} K_{n_i}) \vee K_\alpha$ and so G is also one. Obviously, the graph $(\bigcup_{i=1}^{\alpha} K_{n_i}) \vee K_\alpha$ is not Hamiltonian-connected. Thus the proof of Theorem 4 is completed. ■

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