A note on coloring of \( \frac{3}{3} \)-power of subquartic graphs

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Abstract

For any \( k \in \mathbb{N} \), the \( k \)-subdivision of a graph \( G \) is a simple graph \( G_1^k \), which is constructed by replacing each edge of \( G \) with a path of length \( k \). In [M.N. Iradmusa, Discrete Math. 310 (10-11) (2010), 1551–1556], the \( m \)-th power of the \( n \)-subdivision of \( G \) was introduced as a fractional power of \( G \), denoted by \( G_{n,m}^2 \). F. Wang and X. Liu in [Discrete Math. Algorithms Appl. 10 (3), (2018), 1850041] showed that \( \chi(G_{3,3}^2) \leq 7 \) for any subcubic graph \( G \). In this note, we prove that the \( \frac{3}{3} \)-power of every subquartic graph admits a proper coloring with at most nine colors. We conjecture that \( \chi(G_{3}^2) \leq 2\Delta(G) + 1 \) for any graph \( G \) with maximum degree \( \Delta(G) \geq 2 \).

1 Introduction

All graphs we consider in this note are simple, finite and undirected. We mention some of the definitions that are referred to throughout this note, and for other necessary definitions and notation we refer the reader to a standard text-book [2]. Minimum degree, maximum degree and maximum size of cliques of a graph \( G \) are denoted by \( \delta(G) \), \( \Delta(G) \) and \( \omega(G) \), respectively. For each vertex \( v \in V(G) \), the neighbors of \( v \) are the vertices adjacent to \( v \) in \( G \). The neighborhood \( N_G(v) \) of \( v \) is the set of all neighbors of \( v \) in \( G \). Any vertex of degree \( k \) is called a \( k \)-vertex and any path of length \( k \) is called a \( k \)-path. Also, a path \( P : v_1, \ldots, v_k \) is a simple path if the degrees of all vertices \( v_2, \ldots, v_{k-1} \) are 2, and a cycle \( C \) is called a loop cycle if all of its vertices are 2-vertices except one that is a 3-vertex.

Let \( G \) be a graph and \( k \) be a positive integer. The \( k \)-power of \( G \), denoted by \( G^k \), is defined on the vertex set \( V(G) \) by adding edges joining any two distinct vertices \( x \) and \( y \) with distance at most \( k \). Also the \( k \)-subdivision of \( G \), denoted by \( G_1^k \), is constructed by replacing each edge \( xy \) of \( G \) with a path of length \( k \), say \( P_{xy} \). These \( k \)-paths are called superedges. We denote a vertex by \((xy)_l\) if it belongs to \( P_{xy} \) and has distance \( l \) from the vertex \( x \), where \( l \in \{0,1,2,\ldots,k\} \). Note that
Let $G$ be a graph and $m, n \in \mathbb{N}$. The graph $G_{mn}$ is defined to be the $m$-power of the $n$-subdivision of $G$. In other words $G_{mn} = (G_{1n})^m$.

We denote the set of terminal vertices of $G_{mn}$ by $V_{t}(G_{mn})$ and the set of internal vertices by $V_{i}(G_{mn})$. It is worth noting that $G_{11} = G$ and $G_{22} = T(G)$, where $T(G)$, the total graph of $G$, is the graph whose vertex set is $V(G) \cup E(G)$, in which two vertices are adjacent if and only if they are adjacent or incident in $G$ [1].

As usual, a proper $k$-coloring of $G$ is a mapping from $V(G)$ to $\{1, \ldots, k\}$, where any two adjacent vertices have distinct colors. The chromatic number of a graph $G$ is the minimum integer $k$ for which $G$ has a proper $k$-coloring, and is denoted by $\chi(G)$. By the definition of a total graph, $\chi''(G) := \chi(T(G)) = \chi(G_{2})$. In 1965, Behzad [1] conjectured that $\omega(G_{2}) = \Delta(G) + 1$ and the Total Coloring Conjecture can be reformulated as follows.

**Conjecture 1.2** For any simple graph $G$, $\chi(G_{2}) \leq \omega(G_{2}) + 1$.

There is a relation between an incidence coloring of a graph $G$ and a vertex coloring of $G_{2}$, which is one of the motivations of this note. The concept of incidence coloring was introduced by Brualdi and Massey in 1993.

**Definition 1.3** [3] Let $G = (V, E)$ be a multigraph. An incidence of $G$ is a pair $(v, e)$ where $v \in V(G)$, $e \in E(G)$ and $e$ is incident with $v$. Let $I$ be the set of incidences of $G$. An incidence graph of $G$, denoted by $I(G)$, has its vertex set $V(I(G)) = I$ such that two incidences $(v, e)$ and $(w, f)$ are adjacent in $I(G)$ if one of the following holds:

1. $v = w$,
2. $e = f$,
3. the end-vertices of $e$ or $f$ are $v$ and $w$.

**Definition 1.4** [3] An incidence coloring $\sigma$ of $G$ is a map from $I$ to the color set $C$ such that adjacent incidence pairs are assigned different colors. If $\sigma : I \to C$ is an incidence coloring with $|C| = k$, then we say that $\sigma$ is a $k$-incidence coloring of $G$. The incidence chromatic number of $G$, denoted $\chi_i(G)$, is the smallest $k$ for which there exists a $k$-incidence coloring of $G$.

In [3], it is proved that for a graph $G$ with maximum degree $\Delta$, $\chi_i(G) \leq 2\Delta$. Also, by definition of the fractional power of a graph, the incidence graph $I(G)$ is
the subgraph of \( G^{\frac{3}{2}} \) induced by the set of internal vertices. So we have \( \chi_i(G) = \chi(G^{\frac{3}{2}}[V_i(G^{\frac{3}{2}})]) \leq \chi(G^{\frac{3}{2}}) \). In addition, the partition \( \{V_i(G^{\frac{3}{2}}), V_i(G^{\frac{3}{2}})\} \) of the vertices of \( G^{\frac{3}{2}} \) implies that

\[
\chi(G^{\frac{3}{2}}) \leq \chi(G^{\frac{3}{2}}[V_i(G^{\frac{3}{2}})]) + \chi(G^{\frac{3}{2}}[V_i(G^{\frac{3}{2}})]) = \chi(G) + \chi_i(G).
\]

Also, in [6] it was proved that if \( \Delta(G) \geq 3 \), then \( \chi(G^{\frac{3}{2}}) \leq \chi(G) + \chi_i(G) - 1 \).

In this note, we are investigating the chromatic number of \( G^{\frac{3}{2}} \). When \( \Delta(G) = 1 \), one can easily show that \( \chi(G^{\frac{3}{2}}) = 4 \), and by applying the following theorem, which was proved in [5], we can prove that \( \chi(G^{\frac{3}{2}}) \leq 5 \) for any graph \( G \) with \( \Delta(G) = 2 \).

**Theorem 1.5** If \( m, n, k \in \mathbb{N} \) and \( k \geq 3 \), then

(i) \( \chi(C_m^\frac{n}{k}) = \left\{ \begin{array}{ll}
\frac{nk}{m} & \text{if } m \geq \frac{nk}{2}, \\
\left\lceil \frac{nk}{m+1} \right\rceil & \text{if } m < \frac{nk}{2},
\end{array} \right. \)

(ii) \( \chi(P_m^\frac{n}{k}) = \min\{m+1, (k-1)n+1\} \).

In [8], Wang and Liu proved that \( \chi(G^{\frac{3}{2}}) \leq 7 \) for any subcubic graph \( G \). Recall that a graph \( G \) is subcubic if \( \Delta(G) \leq 3 \). Recently, a simple proof of this result was given in [6] by using the following theorem about the 5-colorability of the incidences of any subcubic graph.

**Theorem 1.6** [7] For any subcubic graph \( G \), we have \( \chi_i(G) \leq 5 \).

The graph \( G \) is called **subquartic** if \( \Delta(G) \leq 4 \). Also any 4-regular graph is known as a **quartic** graph. The main theorem of this note is stated as follows.

**Theorem 1.7** Let \( G \) be a subquartic graph. Then \( \chi(G^{\frac{3}{2}}) \leq 9 \).

**Remark 1.8** In [3], it was conjectured that \( \chi_i(G) \leq \Delta(G) + 2 \) for every graph \( G \). This was disproved by Guiduli in [4] who showed that Paley graphs with sufficiently large maximum degree have incidence chromatic number at least \( \Delta + \Omega(\log \Delta) \). However, this conjecture seems to hold for graphs with small maximum degree. Theorem 1.6 shows that the conjecture is true for cubic graphs. It remains an open problem whether the conjecture is true for quartic graphs. But if the conjecture holds for all quartic graphs, then easily we can prove Theorem 1.7 by use of the inequality \( \chi(G^{\frac{3}{2}}) \leq \chi(G) + \chi_i(G) - 1 \). So Theorem 1.7 provides evidence that the conjecture may be true for all quartic graphs.

Considering the results for cycles, paths, subcubic and subquartic graphs, we conjecture that \( 2\Delta(G) + 1 \) colors suffice for the proper coloring of \( G^{\frac{3}{2}} \) when \( G \) is a graph with maximum degree at least two.

**Conjecture 1.9** Let \( G \) be a graph with \( \Delta(G) \geq 2 \). Then \( \chi(G^{\frac{3}{2}}) \leq 2\Delta(G) + 1 \).
The clique number of the fractional power of a graph was obtained in [5] for powers less than one. As mentioned in [8], one can easily show that $\omega(G^{\frac{3}{2}}) = \Delta(G) + 2$ when $\Delta(G) \geq 2$, and $\omega(G^{\frac{3}{2}}) = 4$ when $\Delta(G) = 1$. Therefore, if Conjecture 1.2 holds, we conclude that $\chi'(G) = \chi(G^{\frac{3}{2}}) \leq \chi(G^{\frac{3}{2}})$. So, the following conjecture seems strongly true.

**Conjecture 1.10** For any graph $G$, $\chi(G^{\frac{3}{2}}) \leq \chi(G^{\frac{3}{2}})$.

## 2 Proof of Theorem 1.7

For convenience, we need some notation and preliminaries.

Let $G$ be a graph and consider $G^{\frac{3}{2}}$. On each superedge $P_{uv}$ there are two internal vertices $(uv)_1$ and $(uv)_2$ which correspond to the incidences of the edge $uv$. We denote $(uv)_1$ and $(uv)_2$ by $(u, v)$ and $(v, u)$, respectively.

We need to establish the following lemmas before the proof of Theorem 1.7.

**Lemma 2.1** Let $P_n : v_1, v_2, \ldots, v_n$ be a path of order $n \geq 5$, $n \not\equiv 6, 7 \pmod{4}$ and the vertices $v_1, v_2, v_{n-1}$ and $v_n$ are respectively colored with the colors $a, b, c$ and $d$ from the set $C = \{1, 2, \ldots, 5\}$ that are all distinct except possibly $a$ and $d$. Then we can extend this partial coloring to a proper coloring of $P^{\frac{3}{2}}_n$ with colors from $C$.

**Proof** To extend this partial coloring to a proper coloring of $P^{\frac{3}{2}}_n$, we consider 8 cases:

1. $a = d$, $n \equiv 1 \pmod{4}$: Starting from $v_1$, we color the vertices of $P^{\frac{3}{2}}_n$ sequentially by colors $(abc)(ab)(abc) \cdots$ where $e \in C \setminus \{a, b, c\}$.

2. $a = d$, $n \equiv 2 \pmod{4}$: Starting from $v_1$, we color the vertices of $P^{\frac{3}{2}}_n$ sequentially by colors $(abcdef)(ab)(abc) \cdots$ where $e, f \in C \setminus \{a, b, c\}$ and $e \neq f$.

3. $a = d$, $n \equiv 3 \pmod{4}$: Starting from $v_1$, we color the vertices of $P^{\frac{3}{2}}_n$ sequentially by colors $(abcdef)(ab)(abc) \cdots$ where $e, f \in C \setminus \{a, b, c\}$ and $e \neq f$.

4. $a = d$, $n \equiv 0 \pmod{4}$: Starting from $v_1$, we color the vertices of $P^{\frac{3}{2}}_n$ sequentially by colors $(abc)(ab)(abc)(fbc) \cdots$ where $e, f \in C \setminus \{a, b, c\}$ and $e \neq f$.

5. $a \not= d$, $n \equiv 1 \pmod{4}$: Starting from $v_1$, we color the vertices of $P^{\frac{3}{2}}_n$ sequentially by colors $(abcd)(abcd)(abcd) \cdots$ where $e \in C \setminus \{a, b, c, d\}$.

6. $a \not= d$, $n \equiv 2 \pmod{4}$: Starting from $v_1$, we color the vertices of $P^{\frac{3}{2}}_n$ sequentially by colors $(abcd)(abcd)(abcd)(abcd) \cdots$ where $e \in C \setminus \{a, b, c, d\}$.

7. $a \not= d$, $n \equiv 3 \pmod{4}$: Starting from $v_1$, we color the vertices of $P^{\frac{3}{2}}_n$ sequentially by colors $(abcd)(ab)(abc)(fbc) \cdots$ where $e \in C \setminus \{a, b, c, d\}$.

8. $a \not= d$, $n \equiv 0 \pmod{4}$: Starting from $v_1$, we color the vertices of $P^{\frac{3}{2}}_n$ sequentially by colors $(abcd)(abcd)(abcd) \cdots$. 

Each of these colorings preserves the colors of \( v_1, v_2, v_{n-1} \) and \( v_n \) and it can be easily seen that the given coloring is a proper coloring of \( P_5^3 \) with color set \( C \).

**Lemma 2.2** Let \( G \) be a subcubic graph with \( \delta(G) \geq 2 \) and

\[
V_3 = \{ v \in V(G) \mid d_G(v) = 3 \}.
\]

Then \( \chi(G^\frac{3}{2} \setminus V_3) \leq 5 \).

**Proof** Each connected component of \( G \) is a cycle or a subcubic graph in which any 2-vertex belongs to a simple path or a loop cycle. Let \( H \) be a connected component of \( G \). If \( H \) is a cycle, then by Theorem 1.5, we can color the vertices of \( H^\frac{3}{2} \setminus V_3 = H^\frac{3}{2} \) with the colors \( C = \{1, 2, 3, 4, 5\} \). Now let \( H \) be a component of the second type in \( G \). To find a proper coloring for \( H^\frac{3}{2} \setminus V_3 \), at first we identify all 2-vertices lying in simple paths and remove all 2-vertices of the loop cycles in \( H \). Let \( H_1 \) be the resulting graph. By Theorem 1.6, \( \chi(H^\frac{3}{2}[V_i(H^\frac{3}{2})]) = \chi_i(H_1) \leq 5 \). Suppose that \( c : V_i(H^\frac{3}{2}) \rightarrow C \) is a proper coloring of \( H^\frac{3}{2}[V_i(H^\frac{3}{2})] \). Suppose the simple path \( P : v_1, v_2, \ldots, v_n \) in \( H \). The vertices \( v_2, \ldots, v_{n-1} \) of \( P \) contracted to a single vertex \( v^* \) in \( H_1 \). The subgraph of \( H^\frac{3}{2} \setminus V_3 \) induced by

\[
V_P = \{ (v_1, v_2), (v_2, v_1), (v_2, v_2), (v_2, v_3), (v_3, v_2), (v_3, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_n) \}
\]

is isomorphic to \( P_{3n-4}^3 \). Now we color the first two vertices and the last two vertices of \( V_P \) as follows:

\[
c(v_1, v_2) = c(v_1, v^*), c(v_2, v_1) = c(v^*, v_1),
\]

\[
c(v_n, v_{n-1}) = c(v_n, v^*), c(v_{n-1}, v_n) = c(v^*, v_n).
\]

Because \( 5 \leq 3n-4 \neq 6, 7 \), by Lemma 2.1 we can extend \( c \) to the other vertices of \( P^3_5 \) except \( v_1 \) and \( v_n \) and similarly, to all vertices of the other simple paths. We denote by \( c' \) this extension of \( c \).

Finally, we color the vertices of the loop cycles. Let \( L : v_1, v_2, \ldots, v_n, v_1 \) be a loop cycle of \( G \) such that \( d_G(v_1) = 3 \) and \( N_G(v_1) = \{ v_0, v_2, v_n \} \). Suppose that \( c'(v_0, v_1) = a \) and \( c'(v_1, v_0) = b \). Because \( L^\frac{3}{2} \) is isomorphic to \( C^\frac{3}{2}_n \), by Theorem 1.5, we have \( \chi(L^\frac{3}{2}) \leq 5 \). Let \( c_L \) be a proper coloring of \( L^\frac{3}{2} \) such that \( c_L(v_1) = b \), \( c_L((v_1, v_n)) = c \) and \( c_L((v_1, v_2)) = d \) such that \( \{c, d\} \subset C \setminus \{a, b\} \). Therefore, \( c_L((v_1, v_2)) \neq a = c'((v_0, v_1)) \neq c_L((v_1, v_n)) \). Because \( c_L \) is a proper coloring, \( c_L((v_1, v_2)) \neq b = c'(v_1, v_0) \neq c_L((v_n, v_1)) \). Now we delete the color of \( v_1 \) and add this coloring of \( \frac{3}{2} \)-power of \( L \) to \( c' \). By repeating this method for each loop cycle, \( c' \) gives rise to a proper coloring of \( G^\frac{3}{2} \setminus V_3 \).

**Proof of Theorem 1.7.** Since each subquartic graph is a subgraph of a quartic graph, we only prove the theorem for quartic graphs. Let \( G \) be a quartic graph. If \( G \) is a complete graph, then \( G = K_5 \). In Figure 1, a proper 7-coloring of \( K_5^\frac{3}{2} \) is shown. Some edges of \( K_5^\frac{3}{2} \) were removed in the figure for simplicity. Now, suppose that \( G \)
is not a complete graph and $M_1 = \{e_1, \ldots, e_k\}$ is a maximum matching. Since $M_1$ is a maximum matching of $G$, $A = V(G) \setminus V(M_1)$ is an independent set of $G$ and so $N_G(v) \subseteq V(M_1)$ for each $v \in A$.

Consider the bipartite subgraph $H$ with bipartition $(A, V(M_1))$ that contains all edges between $A$ and $V(M_1)$. Since $d_H(v) = 4 > d_H(u)$ for any vertex $v \in A$ and any vertex $u \in V(M_1)$, by Hall’s Theorem, $H$ has a matching, named $M_2$, which covers all the vertices of $A$. Without loss of generality, suppose that $M_2 = \{f_1, \ldots, f_k\}$. Note that none of the edges in $M_1$ is adjacent to two edges of $M_2$; otherwise, $M_1$ is not a maximum matching of $G$.

Consider the subgraph $F$ of $G$ induced by $M_1 \cup M_2$ and let $B_1 = V(F^2) \setminus (V(M_1) \cap V(M_2))$ and $B_2 = V(G^3) \setminus B_1$. Now consider the partition $\{B_1, B_2\}$ of $V(G^3)$. We prove that $\chi(G^3[B_1]) \leq 4$ and $\chi(G^3[B_2]) \leq 5$ and then we conclude that $\chi(G^3) \leq \chi(G^3[B_1]) + \chi(G^3[B_2]) \leq 9$.

Since $G$ is neither a complete graph nor an odd cycle, by Brooks’ Theorem, $\chi(G) \leq 4$. In addition, the subgraph of $G^3$ induced by the $t$-vertices is isomorphic to $G$. Therefore, there exists a proper coloring $c$ of the $t$-vertices of $G^3[B_1]$ with colors in $\{1, 2, 3, 4\}$. Now, we color the $i$-vertices of $G^3[B_1]$ with colors in $\{1, 2, 3, 4\}$. For any edge $e = uv \in M_1$ that is not adjacent with any edge of $M_2$, color $i$-vertices $(u, v)$ and $(v, u)$ with two different colors of $\{1, 2, 3, 4\} \setminus \{c(u), c(v)\}$. Now, suppose that $e = uv \in M_1$ is adjacent with an edge $f = vw \in M_2$. If $c(u) \neq c(w)$, then color the $i$-vertices $(u, v)$ and $(w, v)$ with a same color from $\{1, 2, 3, 4\} \setminus \{c(u), c(w)\}$. Also, assign colors $c(u)$ and $c(w)$ to $i$-vertices $(v, w)$ and $(v, u)$, respectively. If $c(u) = c(w)$, color the $i$-vertices $(u, v)$ and $(w, v)$ with a same color from $\{1, 2, 3, 4\} \setminus \{c(u)\}$ and then assign two different colors in $\{1, 2, 3, 4\} \setminus \{c(u), c((u, v))\}$ to the $i$-vertices $(v, w)$ and $(v, u)$. Therefore $\chi(G^3[B_1]) \leq 4$.

To find a proper 5-coloring of $G^3[B_2]$, we apply Lemma 2.2. Let $G_1$ be the spanning subgraph of $G$ with edge set $E(G_1) = E(G) \setminus (M_1 \cup M_2)$ and $V_3$ be the set of 3-vertices of $G_1$. Because $d_{G_1}(v) = 2$ for any vertex $v \in V(M_1) \cap V(M_2)$ and $d_{G_1}(v) = 3$ for any vertex $v \notin V(M_1) \cap V(M_2)$, we have $\delta(G_1) \geq 2$, $\Delta(G_1) = 3$ and $G^3_1 \setminus V_3 = G^3[B_2]$. Therefore, by Lemma 2.2, we have $\chi(G^3[B_2]) = \chi(G^3_1 \setminus V_3) \leq 5$, which completes the proof. □
Problem 1 We did not find a subquartic graph $G$ with $\chi(G^3) = 9$. Therefore, it remains an open problem as to whether the upper bound in Theorem 1.7 can be decreased to 8 colors.

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References


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