

Formulas in connection with parameters
related to convexity of paths on three vertices:
caterpillars and unit interval graphs

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Abstract

We present formulas to compute the P_3 -interval number, the P_3 -hull number and the percolation time for a caterpillar, in terms of certain sequences associated with it. In addition, we find a connection between the percolation time of a unit interval graph and a parameter involving the diameter of a unit interval graph related to it. Finally, we present a hereditary graph class, defined by forbidden induced subgraphs, such that its percolation time is equal to one.

1 Introduction

The convexity generated by paths of length two has been widely studied in the specialized literature. From an algorithmic perspective, there are several results in connection with different parameters related to convexity in graphs, in particular tied to P_3 -convexity. Centeno et al. proved that it is NP-complete to decide whether the P_3 -hull number of a graph is at most k [8]. Recently, two articles, considering this parameter, have been published [5, 10]. Another parameter in connection with convexity is the interval number of a graph that, under the P_3 -convexity, agrees

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with the 2-domination number [9]. Complexity aspects of the P_3 -hull number and the P_3 -interval number were investigated for graphs of bounded degree and planar graphs [18, 19]. An interesting problem, considered by many researchers, is determining the percolation time of a graph. Interesting enough is the problem of deciding whether the percolation time is at least k . This decision problem can be solved in polynomial time if $k = 4$. Nevertheless, the problem becomes NP-complete if $k = 5$ [16]. Parameters related to P_3 -convexity, not considered in this article, but extensively studied are: the Helly number [6], and the Carathéodory number [2, 11]. Besides, the problem of partitioning a graph into P_3 -convex sets have been considered in [7, 14]. For results related to other parameters or path convexities we refer the reader to [1, 4, 12, 13, 15, 20] and the references therein.

In this article we study the existence of formulas to compute the P_3 -interval number, the P_3 -hull number and the percolation time for caterpillars, which are precisely those acyclic connected interval graphs. In the case of a unit interval graph, we find a formula for the percolation time in terms of the diameter of a certain unit interval graph related to it.

All graphs considered in this article are finite, undirected, without loops, and without multiple edges. All graph-theoretic concepts and definitions not given here can be found in [21]. Let G be a graph. Let us denote by $V(G)$ and $E(G)$ its vertex set and edge set, respectively. Given u and v in $V(G)$, we say that u is *adjacent* to v if $uv \in E(G)$. The *neighborhood* of a vertex u , denoted $N_G(u)$, is the set $\{v \in V(G) : uv \in E(G)\}$, and $N_G[v]$ stands for the set $N_G(v) \cup \{v\}$. If X is a finite set, $|X|$ denotes its cardinality. The *degree* of a vertex u is the cardinality of its neighborhood (i.e., $|N_G(u)|$) and it is denoted by $d_G(u)$. The *length* of a path is its number of edges. The *distance* between two vertices u and v of G , denoted $d_G(u, v)$, is the minimum length among all paths having u and v as their endpoints. The *diameter* of G , when it is connected, denoted $\text{diam}(G)$, is the maximum $d(u, v)$ among all pairs $u, v \in V(G)$; a path P having $\text{diam}(G)$ edges is called a *diameter path*. A *pendant vertex* is a vertex of degree 1. A *support vertex* is a vertex adjacent to a pendant vertex. A *cut vertex* of G is a vertex v such that the number of connected components of $G - v$ is greater than the number of connected components of G . An *independent set* of G is a set of pairwise nonadjacent vertices. The maximum cardinality of an independent set of G is denoted by $\alpha(G)$. A *tree* is a connected acyclic graph. A *leaf* of a tree is a pendant vertex of it. A *caterpillar* is a tree such that the removal of all its pendant vertices turns out to be a path.

A *convexity* of a graph G is a pair $(V(G), \mathcal{C})$ where \mathcal{C} is a family of subsets of $V(G)$ satisfying all the following conditions: $\emptyset \in \mathcal{C}$, $V(G) \in \mathcal{C}$, and \mathcal{C} is closed under intersections; i.e., $V_1 \cap V_2 \in \mathcal{C}$ for each $V_1, V_2 \in \mathcal{C}$. Each set of the family \mathcal{C} is called a \mathcal{C} -*convex set*. Let \mathcal{P} be a set of paths in G and let $S \subseteq V(G)$. If u and v are two vertices of G , then the \mathcal{P} -*interval* of u and v , denoted $I_{\mathcal{P}}[u, v]$, is the set of all vertices lying in some path $P \in \mathcal{P}$ having u and v as endpoints. Let $I_{\mathcal{P}}[S] = \bigcup_{u, v \in S} I_{\mathcal{P}}[u, v]$. Let \mathcal{C} be the family of all sets S of vertices of G such that, for each path $P \in \mathcal{P}$ whose endpoints belong to S , every vertex of P also belongs to S ; i.e., \mathcal{C} consists of those subsets S of $V(G)$ such that $I_{\mathcal{P}}[S] = S$. It is easy to show that $(V(G), \mathcal{C})$ is a

convexity of G and \mathcal{C} is called the *path convexity generated by \mathcal{P}* . The P_3 -convexity is the path convexity generated by the set of all paths of length two. Equivalently, a P_3 -convex set is a set $S \subseteq V(G)$ such that for each vertex $v \in V(G) - S$, v has at most one neighbor in S . The P_3 -hull set of a set $R \subseteq V(G)$ is the minimum P_3 -convex set of G containing R . A P_3 -hull set of G is a set of vertices whose P_3 -hull set is $V(G)$, and the minimum cardinality of a P_3 -hull set of G , denoted $h(G)$, is the P_3 -hull number. If S is a P_3 -hull set of G , we also say that S *percolates* G . In order to decide whether a set S percolates G , we may build a sequence S_0, S_1, S_2, \dots in which $S_0 = S$ and S_{i+1} is obtained from S_i by adding those vertices of G having at most one neighbor in $\bigcup_{j=1}^{i-1} S_j$ and at least two neighbors in $\bigcup_{j=1}^i S_j$. We use $I_{P_3}^k[S]$ to denote S_k . If there exists some t such that $S_t = V(G)$, then S percolates G and we define $\tau_S(G)$ as the minimum t such that $S_t = V(G)$. We also define the *percolation time of G* as $\tau(G) = \max\{\tau_S(G) : S \text{ percolates } G\}$. Let G be a graph and let $v \in V(G)$. We say that S *percolates* v if $v \in I_{P_3}^k[S]$ for some integer k , and $\tau_S(v)$ stands for the minimum k with this property. If $A \subseteq V(G)$ and S percolates every vertex $v \in A$, $\tau_S(A)$ stands for the maximum $\tau_S(w)$ among all vertices $w \in A$. When the context is clear enough, we will use $\tau(v)$ and $\tau(A)$ for short. A set $S \subseteq V(G)$ is called a P_3 -interval set if for each vertex $v \in V(G)$, $v \in S$ or v has at least two neighbors in S . The minimum cardinality of a P_3 -interval set of G , denoted $g(G)$, is called the P_3 -interval number of G . Notice that, by definition, a P_3 -interval set is also a P_3 -hull set and thus $h(G) \leq g(G)$ for every graph G . In this article we will deal with P_3 -convexity only. So, from now on, in some cases, we will omit the ‘ P_3 -’ prefix and \mathcal{P} will denote the set of paths on three vertices.

A *unit interval graph* is a graph such that there exists a one-to-one assignment between its vertex set and a family of closed intervals of unit length in the real line, such that two different vertices are adjacent if and only if their corresponding intervals intersect. Equivalently, G is a unit interval graph if there exists a linear order, called a *unit interval order*, of its vertices so that the closed neighborhood of each vertex is an interval under that order; i.e, a sequence of consecutive vertices.

This article is organized as follows. In Section 2 we deal with the interval number and the hull number of a caterpillar. More precisely, in Subsection 2.1, we associate a sequence of positive integers to every caterpillar, called a basic sequence, used along the rest of the section: in Subsections 2.2 and 2.3 we present a closed formula for the P_3 -interval number and the P_3 -hull number of a caterpillar, respectively, in terms of its associated basic sequence. Section 3 is devoted to giving formulas for the percolation time for caterpillars and unit interval graphs. In Subsection 3.1 we study the percolation time of a caterpillar and in Subsection 3.2 we summarize some known results of unit interval graphs and then we deal with the problem of finding the percolation time of a unit interval graph. Finally, in Section 4 we present a hereditary graph class for which the percolation time of any graph in the class is 1.

2 Computing the interval number and hull number of a caterpillar

2.1 Basic sequence of a caterpillar

A *dominating set* of a graph G is a set S of vertices of G such that every vertex $v \in V(G) \setminus S$ has at least one neighbor on S . A *dominating path* in a graph G is a path P whose vertex set is a dominating set of G . Let T be a caterpillar. By definition, T has some dominating path P . Let $P = v_1, \dots, v_k$ be chosen so that its endpoints are leaves of T . For each vertex w of T , we define the *reduced degree of w in T* by $\tilde{d}_T(w) = \min\{d_T(w), 4\}$. The sequence $\tilde{d}_T(v_1), \tilde{d}_T(v_2), \dots, \tilde{d}_T(v_k)$, denoted by $s(T)$, is called a *reduced degree sequence of T* . Since a dominating path of T having pendant endpoints is unique up to the choice of its endpoints, the reduced degree sequence of T is well-defined. Notice also that this sequence is unique up to reversing the order of the vertices of P . By construction, the first and the last term of $s(T)$ are equal to 1 (i.e., $\tilde{d}_T(v_1) = \tilde{d}_T(v_k) = 1$).

Let α be a sequence. The *length* of α , denoted $|\alpha|$, is the number of terms of α . For each $i \in \{1, \dots, |\alpha|\}$, we denote by α_i the i -th term of α . Let α^1 and α^2 be two sequences of lengths n_1 and n_2 , respectively. The *concatenation of α^1 and α^2* , denoted by $\alpha^1\alpha^2$, is the sequence β of length $n_1 + n_2$ characterized by $\beta_i = \alpha_i^1$ for each $i \in \{1, 2, \dots, n_1\}$ and $\beta_{n_1+i} = \alpha_i^2$ for each $i \in \{1, 2, \dots, n_2\}$. If either α^1 or α^2 is the empty sequence, then $\alpha^1\alpha^2$ is either α^2 or α^1 , respectively. Let Γ be the family of (possibly empty) sequences γ such that all the terms of γ are equal to 4. A *basic sequence* λ is a sequence satisfying exactly one of the following conditions:

- $\lambda = 1$ (type I), or
- $\lambda \in \{21, 22, 23, 24\}$ (where each single element of the set should be thought of as a sequence) or $\lambda \in \{x\gamma y, x \in \{3, 4\}, y \in \{1, 2\} \text{ and } \gamma \in \Gamma\}$ (type II), or
- $\lambda \in \{x\gamma 3y, x \in \{3, 4\}, y \in \{1, 2, 3, 4\} \text{ and } \gamma \in \Gamma\}$ (type III).

We denote by Λ the family of all basic sequences. A *prefix* of a sequence α is a sequence β such that $\alpha = \beta\eta$ for some (possibly empty) sequence η . If η is nonempty, then β is a *proper prefix of α* . Notice that no basic sequence is a proper prefix of another basic sequence.

Lemma 2.1. *Let σ be a finite sequence whose terms belong to $\{1, 2, 3, 4\}$ and whose last term is 1. Then, there exists some integer $p \geq 0$ and some $\lambda^0, \lambda^1, \dots, \lambda^p \in \Lambda$ such that $\sigma = \lambda^0\lambda^1 \dots \lambda^p$. Moreover, the integer p and the sequences $\lambda^0, \lambda^1, \dots, \lambda^p$ are uniquely determined.*

Proof. We prove the lemma by induction on the length of σ . Since the last term of σ is equal to 1, it is possible to define a prefix λ^0 of σ as follows:

- (i) if $\sigma_1 = 1$, let $\lambda^0 = 1$ (type I);

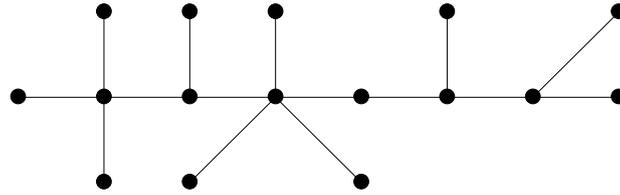


Figure 1: The two possible reduced degree sequences of the depicted caterpillar T is 14342331 and 13324341. Notice that the second sequence is the first one traversed backward. They are the concatenation of four basic sequences 1, 434, 23, 31, and 1, 332, 434, 1, respectively. Thus, $p(T) = 3$.

- (ii) if $\sigma_1 = 2$, let $\lambda^0 = \sigma_1\sigma_2 = 2\sigma_2$ (type II);
- (iii) if $\sigma_1 \in \{3, 4\}$ and j is the smallest integer greater than 1 such that $\sigma_j \neq 4$, then:
 - (a) if $\sigma_j \in \{1, 2\}$, let $\lambda^0 = \sigma_1\sigma_2 \cdots \sigma_j = \sigma_1\gamma\sigma_j$ for some $\gamma \in \Gamma$ (type II);
 - (b) if $\sigma_j = 3$, let $\lambda^0 = \sigma_1\sigma_2 \cdots \sigma_j\sigma_{j+1} = \sigma_1\gamma 3\sigma_{j+1}$ for some $\gamma \in \Gamma$ (type III).

By construction, $\lambda^0 \in \Lambda$. If $\sigma = \lambda^0$, the lemma holds trivially. Otherwise, $\sigma = \lambda^0\sigma'$ for some sequence σ' whose last term is 1 and, by the induction hypothesis, there is some $p \geq 1$ and some $\lambda^1, \dots, \lambda^p \in \Lambda$ such that $\sigma' = \lambda^1 \cdots \lambda^p$; thus $\sigma = \lambda^0\lambda^1 \cdots \lambda^p$. The uniqueness of p and $\lambda^0, \lambda^1, \dots, \lambda^p$ follows immediately by induction from the fact that no element of Λ is a proper prefix of another element of Λ . \square

From now on, all caterpillars considered in this article have at least two vertices. The following result is an immediate consequence of Lemma 2.1.

Lemma 2.2. *If T is a caterpillar and $s(T)$ is a reduced degree sequence of it, then there exists some integer $p \geq 1$ and some $\lambda^1, \lambda^2, \dots, \lambda^p \in \Lambda$ such that $s(T) = 1\lambda^1\lambda^2 \cdots \lambda^p$. Moreover, the integer p and the sequences $\lambda^1, \lambda^2, \dots, \lambda^p$ are uniquely determined.*

Now, we are ready to introduce a parameter $p(T)$ defined for each caterpillar T and a reduced degree sequence $s(T)$. The *basic sequence number* of a caterpillar T , denoted by $p(T)$, is the only positive integer p such that $s(T) = 1\lambda^1\lambda^2 \cdots \lambda^p$, where $\lambda^i \in \Lambda$ for each $i \in \{1, 2, \dots, p\}$. An example is shown in Figure 1.

2.2 P_3 -interval number

We use $\ell(T)$ (respectively $\mathcal{L}(T)$) to denote the number of leaves (respectively the set of leaves) of a tree T . Recall that an interval set of a graph G is a set S of vertices such that every vertex outside S has at least two neighbors in S and $g(G)$ stands for the size of a minimum interval set in G .

Theorem 2.1. *If T is a caterpillar having at least two vertices, then*

$$g(T) = p(T) + \ell(T) - 1.$$

Proof. Let $P = v_1, v_2, \dots, v_k$ be a dominating path of T where v_1 and v_k are leaves of T . Let $\sigma = s(T) = \tilde{d}_T(v_1)\tilde{d}_T(v_2)\cdots\tilde{d}_T(v_k)$. By Lemma 2.2, $\sigma = 1\lambda^1\cdots\lambda^p$ for some $\lambda^1, \dots, \lambda^p \in \Lambda$, where $p = p(T)$. Let $t_0 = 1$ and let $t_j = |1\lambda^1\cdots\lambda^j| = 1 + |\lambda^1| + \cdots + |\lambda^j|$ for each $j \in \{1, 2, \dots, p\}$. Thus $1 = t_0 < t_1 < \cdots < t_p = k$. Let S be the set $\{v_{t_1}, v_{t_2}, \dots, v_{t_{p-1}}\} \cup \mathcal{L}(T)$. Since none of $v_{t_1}, v_{t_2}, \dots, v_{t_{p-1}}$ is a leaf of T , $|S| = p(T) + \ell(T) - 1$.

We claim that S is an interval set of T . Let v be a vertex in $V(T) \setminus S$. Since $\mathcal{L}(T) \subseteq S$, $v \in V(P)$. Thus $v = v_h$ for some $h \in \{1, 2, \dots, k\}$. Moreover, since $v_{t_0}, v_{t_1}, \dots, v_{t_p}$ are vertices of S , it follows that $h \neq t_j$ for each $j \in \{0, 1, \dots, p\}$. Let $j \in \{1, \dots, p\}$ such that $t_{j-1} < h < t_j$ and let i such that $h = t_{j-1} + i$. Notice that, $\tilde{d}_T(v_h) = \sigma_h = \lambda_i^j$ for some $1 \leq i < |\lambda^j|$, because $\sigma = 1\lambda^1 \dots \lambda^p$. Since $h \notin \{t_0, t_p\}$, $\lambda_i^j \neq 1$.

Suppose first that $\lambda_i^j = 2$. Hence $\lambda^j \in \{21, 22, 23, 24\}$ and $i = 1$, because $\lambda^j \in \Lambda$. Consequently $t_{j-1} = h - 1$ and $t_j = h + 1$. Therefore, v_h is adjacent to $v_{t_{j-1}}$ and v_{t_j} . Suppose now that $\lambda_i^j = 3$. Hence $\lambda^j \in \Lambda$ and $\lambda_1^j \in \{3, 4\}$. Besides, $i = 1$ or $i = |\lambda^j| - 1$. Notice that $t_{j-1} = h - 1$, $t_j = h + 1$, and $d_T(v_h) \geq \tilde{d}_T(v_h) = \lambda_i^j = 3$. On the one hand, if $i = 1$, then v_h is adjacent to $v_{t_{j-1}}$ and some leaf of T . On the other hand, if $i = |\lambda^j| - 1$, then v_h is adjacent to v_{t_j} and some leaf of T . Therefore, v is adjacent to at least two vertices in S . It remains to consider the case in which $\lambda_i^j = 4$. In this case, $\tilde{d}_T(v_h) = \lambda_i^j = 4$. Hence v_h is adjacent to at least two leaves of T . Therefore v_h has at least two neighbors in S . This completes the proof of our claim.

Now we must prove that S is an interval set of T of minimum cardinality. Consider any interval set S' of T and let $j \in \{1, 2, \dots, p - 1\}$. We claim that $v_h \in S'$ for some $h \in \{t_{j-1} + 1, t_{j-1} + 2, \dots, t_j\}$. Suppose first that λ^j has some term equal to 2. Since $\lambda^j \in \Lambda$, either the first or the last term of λ^j is equal to 2. On the one hand, if the first term of λ^j is equal to 2, then $\tilde{d}_T(v_{t_{j-1}+1}) = 2$ and thus $v_h \in S'$ for $h = t_{j-1} + 1$ or $h = t_j$. On the other hand, if the last term of λ^j is equal to 2, then $\tilde{d}_T(v_{t_j}) = 2$ and thus $v_h \in S'$ for $h = t_j - 1$ or $h = t_j$. In both cases the claim holds. We can assume now, without loss of generality, that λ^j has no term equal to 2. Since $j \neq p$, no term of λ^j is equal to 1. Hence each term of λ is equal to 3 or to 4. Notice that $t_j - 2 \geq t_{j-1} + 1$, because $|\lambda^j| \geq 3$. Besides, since $\lambda^j \in \Lambda$, $\tilde{d}_T(v_{t_{j-1}}) = 3$ and thus $v_h \in S'$ for some $h \in \{t_j - 2, t_j - 1, t_j\}$. Thus S' contains at least $p - 1$ nonpendant vertices of P . Since S' contains all leaves of T , $|S'| \geq p + \ell(T) - 1 = |S|$. We have already proved that S is an interval set of T of minimum cardinality. Therefore $g(T) = p(T) + \ell(T) - 1$. □

Remark 2.1. It follows from Theorem 2.1 that $p(T)$ does not depend on the choice among the two possible linear orders of the vertices of a dominating path of T (see Fig. 1).

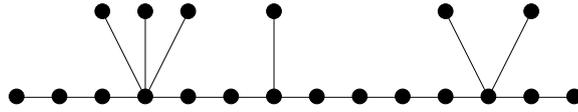


Figure 2: In this caterpillar T we have $s'(T) = 1224222222421$ and $\ell(T) = 8$, and $Z(s'(T)) = 4$. Consequently, $h(T) = 12$.

2.3 P_3 -hull number

Notice that the pendant vertices belong to every hull set of any connected graph.

Lemma 2.3. *If G is a tree, then there is a minimum hull set having no vertex of degree 3 adjacent to a pendant vertex of G .*

Proof. Arguing towards a contradiction, suppose that every minimum hull set of some tree G has at least one support vertex of degree 3. Let S be a minimum hull set of G having the minimum possible number s of support vertices of degree 3 in G . Assume that v_1 is a support vertex of degree 3 in S , where $N_G(v_1) = \{a, b, c\}$ and a is a pendant vertex of G . If b were also a pendant vertex of G , then $b \in S$ and v_1 would have two pendant vertices, a and b , as neighbors. Hence $S \subseteq I[S - \{v_1\}]$ which implies that $S - \{v_1\}$ is a hull set of G , contradicting that S is a minimum hull set of G . Hence b is not a pendant vertex of G . By symmetry, c is not a pendant vertex of G . Besides, b and c do not belong to S . If $b \in S$, since $a \in S$, then $S \subseteq I_P[S - v_1]$ and thus $S - \{v_1\}$ would not be a minimum hull set.

Let $v_2 = c$. Notice that $S_1 = (S - \{v_1\}) \cup \{v_2\}$ is also a hull set of T . This implies that, if v_2 is not a support vertex of degree 3, then S_1 is a minimum hull set with $s - 1$ support vertices of degree 3, contradicting that s is the minimum number of support vertices of degree 3 for a hull set of G . Following this construction, we obtain a path $P = v_1, v_2, \dots, v_k$ in G such that, for every $i \in \{1, 2, \dots, k\}$, v_i is a support vertex of degree 3 and $v_k \in S$, where k is the maximum positive integer having this property for some hull set S of G . Hence, v_k has two neighbors, a pendant vertex w_1 and, since G is a tree, a vertex w_2 different from those in $\{v_1, \dots, v_{k-1}\}$ such that w_2 is not a support vertex of degree 3. Otherwise, $P' = v_1, \dots, v_k, w_2$ for $S' = (S - \{v_k\}) \cup \{w_2\}$ would be a path longer than P with the same property as P . On the other hand, w_2 is not a pendant vertex because otherwise v_k would be adjacent to two pendant vertices and thus $S - \{v_k\}$ would be a minimum hull set having fewer vertices than s support vertices of degree 3. Hence $(S - \{v_k\}) \cup \{w_2\}$ is a minimum hull set having $s - 1$ support vertices of degree 3, a contradiction. The contradiction arose from supposing that G does not have a minimum hull without support vertices of degree 3. \square

Remark 2.2. Let G be a graph and let S be a hull set.

- If u and v are adjacent vertices of degree 2 of G , then $u \in S$ or $v \in S$.

- Let u be a cut vertex of degree 2 of G such that $u \notin S$ and whose neighbors are v and w . If G_v and G_w are the connected components of $G - u$ such that $v \in V(G_v)$ and $w \in V(G_w)$, then $V(G_v) \cap S$ and $V(G_w) \cap S$ are hull sets of G_v and G_w , respectively.

Let s be a sequence of positive integers. We use $Z(s)$ to denote the summation $\sum \lfloor \frac{z_i}{2} \rfloor$ taken over all the lengths z_i in s of the maximal consecutive subsequences containing only the integer 2.

Theorem 2.2. *Let T be a caterpillar such that $|V(T)| \geq 2$. If $s'(T)$ is the sequence obtained by removing from $s(T)$ those terms corresponding to the integer 3 (i.e., $s'(T)$ is the subsequence of whose terms belong to $\{1, 2, 4\}$), then*

$$h(T) = \ell(T) + Z(s'(T)).$$

Proof. Let $P = v_1, v_2, \dots, v_k$ be a maximum dominated path of T . Let $s(T) = \tilde{d}_T(v_1), \tilde{d}_T(v_2), \dots, \tilde{d}_T(v_k)$. Recall that $\tilde{d}_T(v_1) = \tilde{d}_T(v_k) = 1$. Let $B_j = \{v_{j_1}, v_{j_2}, \dots, v_{j_{z_j}}\}$ be a maximal set of vertices in P satisfying the following conditions:

- $\tilde{d}_T(v_{j_i}) = 2$, for every $1 \leq i \leq z_j$, and
- either $j_{i+1} = j_i + 1$ for every $1 \leq i \leq z_j - 1$ or $\tilde{d}_T(v_h) = 3$ for every $j_i < h < j_{i+1}$.

Let J be the number of those maximal subsequences in $s(T)$ and S be a minimum hull set of T having no vertex of degree 3. Lemma 2.3 guarantees the existence of that minimum hull set. We know that $\mathcal{L}(T) \subset S$. Every vertex v such that $\tilde{d}_T(v) = 4$ does not belong to S because it is adjacent to at least two vertices in $\mathcal{L}(T)$ and thus $v \in I_{\mathcal{P}}[S]$ for every hull set S . We consider the subsets $A_i = \{v_{j_{2i-1}}, v_{j_{2i}}\}$ of B_j , for every $1 \leq i \leq z_j/2$ if z_j is even and the subsets $A_i = \{v_{j_{2i-1}}, v_{j_{2i}}\}$ of B_j , for every $1 \leq i \leq (z_j - 3)/2$ and $A_{\lfloor \frac{z_j}{2} \rfloor} = \{v_{j_{z_j-2}}, v_{j_{z_j-1}}, v_{j_{z_j}}\}$ if z_j is odd and greater than one. In virtue of the first statement of Remark 2.2, $|S \cap A_i| \geq 1$ for all $1 \leq i \leq \lfloor \frac{z_j}{2} \rfloor$. Hence $|S \cap B_j| \geq \lfloor \frac{z_j}{2} \rfloor$. This lower bound trivially holds even when $z_j = 1$. Therefore, $h(T) \geq \ell(T) + Z(s'(T))$.

It remains to prove that there exists a hull set S such that $|S| = \ell(T) + Z(s'(T))$. For every $1 \leq j \leq J$, let $C_j = \{v_{j_2}, v_{j_4}, \dots, v_{j_{h_j}}\} \subset B_j$ where $v_{j_{h_j}} = v_{j_{z_j-1}}$ if z_j is odd and $v_{j_{h_j}} = v_{j_{z_j}}$ if z_j is even. Let $S = \mathcal{L}(T) \cup \left(\bigcup_{j=1}^J C_j\right)$. It is not hard to prove that S is a hull set.

We have proved that the cardinality of the minimum hull set is equal to $\mathcal{L}(T)$ plus $\sum_{j=1}^J \lfloor \frac{z_j}{2} \rfloor$. Therefore, $h(T) = \ell(T) + Z(s'(T))$. □

For an example of Theorem 2.2, see Fig. 2

3 Percolation time

In this section we present two results on percolation time of a graph. We consider two graph classes, namely, caterpillar graphs and unit interval graphs. For definitions in connection with percolation time we refer to the reader to Section 1.

3.1 Percolation time of a caterpillar

In [3] it was proved that if T is a tree, then $\tau(T)$ can be computed in linear time. In this subsection we give a simple closed formula for $\tau(T)$ when T is a caterpillar, in terms of certain sequences associated to T .

Let T be a caterpillar and let $s(T)$ be a reduced degree sequence of T . Let $\ell(i)$ be defined recursively for every $1 \leq i \leq |s(T)|$ as follows: $\ell(1) = 1$ and

$$\ell(i) = \begin{cases} \ell(i-1) + 1 & \text{if either } \tilde{d}(v_i) \neq 3, \text{ or both } \tilde{d}(v_i) = 3 \text{ and } \tilde{d}(v_{i-1}) \neq 3, \\ \ell(i-1) & \text{otherwise.} \end{cases}$$

Let us denote, for every $i \in \{1, \dots, |s(T)|\}$, by $n(i)$ (respectively $m(i)$) the minimum (respectively maximum) integer j such that $\ell(j) = \ell(i)$. Consider the graph of Figure 1, in this example $s = 14342331$, $\ell = 12345667$, $n = 12345668$ and $m = 12345778$. We define the sequence $f(T)$, called *percolation sequence of T* , as follows. Let i be an integer such that $1 \leq i \leq |s(T)|$.

- $f_i(T) = 0$, whenever $\tilde{d}(v_i) = 1$.
- $f_i(T) = 1$, whenever $\tilde{d}(v_i) = 2$ and $\tilde{d}(v_{i-1}), \tilde{d}(v_{i+1}) \in \{1, 2\}$.
- $f_i(T) = f_{i+1}(T) + 1$, whenever $\tilde{d}(v_i) = 2$, $\tilde{d}(v_{i-1}) \in \{1, 2\}$ and $\tilde{d}(v_{i+1}) \notin \{1, 2\}$.
- $f_i(T) = f_{i-1}(T) + 1$, whenever $\tilde{d}(v_i) = 2$, $\tilde{d}(v_{i-1}) \notin \{1, 2\}$ and $\tilde{d}(v_{i+1}) \in \{1, 2\}$.
- $f_i(T) = \max\{f_{i-1}(T), f_{i+1}(T)\} + 1$, whenever $\tilde{d}(v_i) = 2$, $\tilde{d}(v_{i-1}), \tilde{d}(v_{i+1}) \notin \{1, 2\}$.
- $f_i(T) = \min\{i - n(i), m(i) - i\} + 1$, whenever $\tilde{d}(v_i) = 3$ and $\tilde{d}(v_{n(i)-1}) = \tilde{d}(v_{m(i)+1}) = 1$.
- $f_i(T) = \max\{i - n(i), m(i) - i\} + 1$, whenever $\tilde{d}(v_i) = 3$ and $\tilde{d}(v_{n(i)-1}) = \tilde{d}(v_{m(i)+1}) = 2$.
- $f_i(T) = i - n(i) + 1$, whenever $\tilde{d}(v_i) = 3$, $\tilde{d}(v_{n(i)-1}) = 1$ and $\tilde{d}(v_{m(i)+1}) = 2$.
- $f_i(T) = m(i) - i + 1$, whenever $\tilde{d}(v_i) = 3$, $\tilde{d}(v_{n(i)-1}) = 2$ and $\tilde{d}(v_{m(i)+1}) = 1$.
- $f_i(T) = \min\{i - n(i) + 1, m(i) - i + 2\}$, whenever $\tilde{d}(v_i) = 3$, $\tilde{d}(v_{n(i)-1}) = 1$ and $\tilde{d}(v_{m(i)+1}) = 4$.

- $f_i(T) = \min\{i - n(i) + 2, m(i) - i + 1\}$, whenever $\tilde{d}(v_i) = 3$, $\tilde{d}(v_{n(i)-1}) = 4$ and $\tilde{d}(v_{m(i)+1}) = 1$.
- $f_i(T) = i - n(i) + 2$, whenever $\tilde{d}(v_i) = 3$, $\tilde{d}(v_{n(i)-1}) = 4$ and $\tilde{d}(v_{m(i)+1}) = 2$.
- $f_i(T) = m(i) - i + 2$, whenever $\tilde{d}(v_i) = 3$, $\tilde{d}(v_{n(i)-1}) = 2$ and $\tilde{d}(v_{m(i)+1}) = 4$.
- $f_i(T) = \min\{i - n(i), m(i) - i\} + 2$, whenever $\tilde{d}(v_i) = 3$, $\tilde{d}(v_{n(i)-1}) = 4$ and $\tilde{d}(v_{m(i)+1}) = 4$.
- $f_i(T) = 1$, whenever $\tilde{d}(v_i) = 4$.

We define $F(T) = \max_{1 \leq i \leq r} f_i(T)$. In the example of Fig. 1, $f = 01213210$, and thus $F = 3$, and in the example of Fig. 2, $f = 01212212112120$ and thus $F = 2$. Observe that in both examples the maximum is reached at some of the terms, meaning, according to the theorem below, that the maximum percolation time is reached only in the vertices corresponding to those terms.

Theorem 3.1. *If G is a caterpillar, then $\tau(T) = F(T)$.*

Proof. Let $P = v_1, v_2, \dots, v_k$ be a dominating path of T where v_1 and v_k are leaves of T . Let $s(T) = \tilde{d}_T(v_1)\tilde{d}_T(v_2) \cdots \tilde{d}_T(v_p)$. If $p \in \{1, 2\}$, the result holds.

For every $i \in \{1, \dots, p\}$, if $\tilde{d}(v_i) = 1$ or $\tilde{d}(v_i) = 4$ because the leaf set is included in any hull set, then $\tau_T(v_i) = 0$ or $\tau_T(v_i) = 1$, respectively. Thus $\tau(v_i) = f_i(T)$.

Let i be an integer such that $1 < i < k$. Assume that $\tilde{d}(v_i) = 3$. Notice that every hull set S of T verifies $\mathcal{L}(T) \subseteq S$. From now on, we are considering percolating set of a given vertex containing $\mathcal{L}(T)$. If $\tilde{d}(v_{n(i)-1}) = \tilde{d}(v_{m(i)+1}) = 1$, then $\mathcal{L}(T)$ is a hull set of T and thus $\tau_S(v_i) \leq \tau_{\mathcal{L}(T)}(v_i)$ for every hull set S of T . It is easy to prove that $\tau_{\mathcal{L}(T)}(v_i) = \min\{i - n(i), m(i) - i\} + 1$ and thus $\tau_T(v_i) = f_i(T)$. Suppose that $\tilde{d}(v_{n(i)-1}) = 1$ and $\tilde{d}(v_{m(i)+1}) = 2$. Notice that if S percolates v_i in T , since $\mathcal{L}(T) \subseteq S$, then $R(S) = S - \{v_{n(i)}, \dots, v_{m(i)+1}\}$ also percolates v_i . Hence $\tau_S(v_i) \leq \tau_{R(S)}(v_i) = i - n(i) + 1$ for every hull set S of T . Notice that $S = V(T) - \{v_{n(i)}, \dots, v_{m(i)+1}\}$ is a hull set such that $\tau_S(v_i) = i - n(i) + 1$. Hence $\tau_T(v_i) = f_i(T)$. Symmetrically, if $\tilde{d}(v_{n(i)-1}) = 2$ and $\tilde{d}(v_{m(i)+1}) = 1$, then $\tau_T(v_i) = m(i) - i + 1 = f_i(T)$. Analogously, if $\tilde{d}(v_{n(i)-1}) = 1$ and $\tilde{d}(v_{m(i)+1}) = 4$, S percolates v_i , then $R(S) = S - \{v_{n(i)}, \dots, v_{m(i)+1}\}$ percolates v_i , because $\mathcal{L}(T) \subseteq S$. Furthermore, $\tau_{R(S)}(v_i) = f_i(T) = \min\{i - n(i) + 1, m(i) - i + 2\}$. Hence, using a similar argument to the last case, we conclude that $\tau_T(v_i) = f_i(T)$. Symmetrically, $\tau_T(v_i) = f_i(T) = \min\{i - n(i) + 2, m(i) - i + 1\}$, if $\tilde{d}(v_{n(i)-1}) = 4$ and $\tilde{d}(v_{m(i)+1}) = 1$. The following cases, enumerated below, can be proved following the same line of argumentation and thus their proofs are omitted.

- If $\tilde{d}(v_{n(i)-1}) = 2$ and $\tilde{d}(v_{m(i)+1}) = 4$, then $\tau_T(v_i) = i - n(i) + 2$.
- If $\tilde{d}(v_{n(i)-1}) = 4$ and $\tilde{d}(v_{m(i)+1}) = 2$, then $\tau_T(v_i) = m(i) - i + 2$.
- If $\tilde{d}(v_{n(i)-1}) = 4$ and $\tilde{d}(v_{m(i)+1}) = 4$, then $\tau_T(v_i) = \min\{i - n(i) + 2, m(i) - i\} + 2$.

Suppose now that $\tilde{d}(v_{n(i)-1}) = \tilde{d}(v_{m(i)+1}) = 2$. Let S be a set such that percolates v_i . On the one hand, if $v_j \in S$ for some integer j such that $n(i) \leq j \leq m(i)$, then $\tau_S(v_i) \leq |i - j| \leq \max\{i - n(i) + 1, m(i) - i + 1\}$. On the other hand, if S percolates v_i and does not contain any vertex in $\{v_{n(i)}, \dots, v_{m(i)}\}$, then, by Remark 2.2, some of $v_{n(i)-1}$ or $v_{m(i)+1}$ belongs to S . In addition, if $v_{n(i)-1} \in S$ (respectively $v_{m(i)+1} \in S$), then $R(S) = S - \{v_{n(i)}, \dots, v_{m(i)+1}\}$ (respectively $R(S) = S - \{v_{n(i)-1}, \dots, v_{m(i)}\}$) also percolates v_i . Hence $\tau_S(v_i) \leq \tau_{R(S)} = i - n(i) + 1$ (respectively $\tau_S(v_i) \leq \tau_{R(S)} = m(i) - i + 1$). Consequently, $\tau_S(v_i) \geq \max\{i - n(i) + 1, m(i) - i + 1\}$ for every set S that percolates v_i . Since $V(T) - \{v_{n(i)}, \dots, v_{m(i)+1}\}$ and $V(T) - \{v_{n(i)-1}, \dots, v_{m(i)}\}$ are hull sets of T , $\tau_T(v_i) = \max\{i - n(i) + 1, m(i) - i + 1\} = f_i(T)$.

It remains to consider the case $\tilde{d}(v_i) = 2$. If $\tilde{d}(v_{i-1}) \in \{1, 2\}$ and $\tilde{d}(v_{i+1}) \in \{1, 2\}$, by Remark 2.2 and since $\mathcal{L}(T) \subseteq S$, $\tau_S(v_i) = 1$ for every set S that percolates v_i such that $v_i \notin S$.

From now on, we are considering a set S percolating v_i such that $\mathcal{L}(T) \subseteq S$. Suppose now that $\tilde{d}(v_{i-1}) \notin \{1, 2\}$ and $\tilde{d}(v_{i+1}) \in \{1, 2\}$. Notice that $v_{i+1} \in S$ for every set S percolating v_i such that $v_i \notin S$ because of Remark 2.2. If $\tilde{d}(v_{i-1}) = 4$, then $\tau_S(v_i) \leq 2$ for every set S percolating v_i such that $v_i \notin S$. Since $S' = V(T) - \{v_{i+1}, v_i\}$ percolates v_i and $\tau_{S'}(v_i) = 2$, $\tau_T(v_i) = 2$. Symmetrically, if $\tilde{d}(v_{i+1}) = 4$ and $\tilde{d}(v_{i-1}) \in \{1, 2\}$, then $\tau_T(v_i) = f_i(T)$. Assume now that $\tilde{d}(v_{i-1}) = 3$ and $\tilde{d}(v_{i+1}) \in \{1, 2\}$. Hence, if S is a hull set and thus percolates v_i and $v_i \notin S$, then $v_{i+1} \in S$ (see Remark 2.2). Besides, if $\mathcal{L}(T) \subseteq S$ then S percolates $v_{n(i)-1} \in S$ whenever $\tilde{d}(v_{n(i)-1}) \notin \{1, 2\}$. Hence $R(S) = S - \{v_{n(i)-1}, \dots, v_{i+1}\}$ also percolates v_i when $\tilde{d}(v_{n(i)-1}) = 4$, and $R(S) = S - \{v_{n(i)}, \dots, v_{i+1}\}$ also percolates v_i otherwise. Thus $\tau_S(v_i) \leq \tau_{R(S)} = f_i(T) + 1$ for every S percolating v_i . In addition, since $S' = V(T) - \{v_{n(i)-1}, \dots, v_i\}$ is a hull set of T if $\tilde{d}(v_{n(i)-1}) = 4$ and $S' = V(T) - \{v_{n(i)}, \dots, v_i\}$ is a hull set of T otherwise, and $\tau_{S'}(v_i) = f_i(T) + 1$, it follows that $\tau_T(v_i) = f_i(T) + 1$. By symmetry, if $\tilde{d}(v_{i-1}) \in \{1, 2\}$ and $\tilde{d}(v_{i+1}) \notin \{1, 2\}$, then $\tau_T(v_i) = f_{i+1}(T) + 1$. Following a similar line of argumentation it can be proved that if $\tilde{d}(v_{i-1}), \tilde{d}(v_{i+1}) \notin \{1, 2\}$, then $\tau_T(v_i) = \max\{f_i(T), f_{i+1}(T)\} + 1$.

Since $\tau(T)$ is the maximum $\tau_S(v)$ among all vertices in T and hull sets S of T , the result follows from the above discussions. □

Let us finish the section by presenting one extra example to illustrate the preceding theorem. Consider the caterpillar T of Fig. 3. Notice that $s(T) = 143333221$, $\ell(T) = 123333456$, $n(T) = 123333789$, $m(T) = 126666789$ and thus $f(T) = 012345610$. We have only two possible minimal, under inclusion, percolation sets R and R' , namely, the one including all leaves plus the first vertex v of degree two from left to right, and the one including all leaves plus the last vertex w of degree two from left to right. Therefore, $\tau_R(T) = 3$ and $\tau_{R'}(T) = 6 = F(T)$, (see Fig. 3).

3.2 Unit interval graphs

In [3] it was proved that the percolation time can be computed in polynomial time for chordal graphs. Although unit interval graphs are chordal, we present a quite simple

formula that profits from the existence of a unit interval order for the vertex set of a unit interval graph. In addition, we show a connection between this parameter and the diameter of a unit interval graph related to the considered unit interval graph.

Let $(G, <)$ be linear order of the vertices of a graph G such that $N[v]$ is an interval in that order, for every $v \in V(G)$. Denote by v_L and v_R the minimum and maximum vertex under that order, respectively.

Proposition 3.1. *Let G be a unit interval graph with a unit interval order $(G, <)$. If $u < v$, then every shortest path $u = y_1, \dots, y_r = v$ verifies that $y_i < y_{i+1}$ for each integer i such that $1 \leq i \leq r - 1$.*

Proof. Consider a shortest path $u = y_1, \dots, y_r = v$ in G . Since P is a shortest path, P turns out to be an induced path of G . Suppose, towards a contradiction, that there exists an integer j such that $y_j < u$ and let i be the biggest integer i such that $y_i < u$. Since $y_i < u < y_{i+1}$, u is adjacent to y_{i+1} , contradicting that P is an induced path. Hence $u < y_i$ for every $1 < i \leq r$. Analogously, it can be proved that $y_i < v$ for every $1 \leq i < r$. Now, suppose towards a contradiction that $y_{j+1} < y_j$ for some integer j such that $1 < j < r$. Since $y_j < y_r = v$, there exists an integer ℓ such that $j < j + 1 \leq \ell < \ell + 1 \leq r$ such that $y_\ell < y_j < y_{\ell+1}$. Hence y_j is adjacent to $y_{\ell+1}$, contradicting that P is an induced path of G . \square

Proposition 3.2. *Let G be a connected unit interval graph with a unit interval order $(G, <)$. If $u < v < w$, then $d(u, v) \leq d(u, w)$.*

Proof. Let $P = u = y_1, \dots, y_r = w$ be a shortest path. By Proposition 3.1, there exist an integer j such that $1 < j < r$ and $y_j \leq v < y_{j+1}$. If $y_j = v$, then $d(u, v) = j - 1 < r - 1 = d(u, w)$. Assume now that $y_j < v < y_{j+1}$ and thus v is adjacent to y_j . Since $u = y_1, \dots, y_j, v$ is a path, $d(u, v) \leq j \leq r - 1 = d(u, w)$. \square

Denote by $a_R(v)$ (respectively $a_L(v)$) to the rightmost (respectively leftmost) adjacent vertex of v .

Corollary 3.1. *Let G be a unit interval graph.*

1. *If $P: u = v_1, \dots, v_k = w$ is a path such that $v_{i+1} = a_R(v_i)$ for every $1 \leq i \leq k - 1$, then $d(u, w) = k - 1$; i.e., P is a shortest path between u and w .*
2. *If $P: u = v_1, \dots, v_k = w$ is a path such that $v_{i+1} = a_L(v_i)$ for every $1 \leq i \leq k - 1$, then $d(u, w) = k - 1$; i.e., P is a shortest path between u and w .*

Proof. Let $P': u = v'_1, \dots, v'_s = w$ be a shortest path; i.e., $s - 1 = d(u, w)$. Clearly, $v'_2 \leq a_R(v_1) = v_2$. Suppose, by the induction hypothesis, that $v'_h \leq v_h$ for every integer h such that $1 < h < k \leq s$. We will prove that $v'_k \leq v_k$. Suppose, towards a contradiction, that $v'_k > v_k$. By inductive hypothesis, $v'_{k-1} \leq v_{k-1} < v_k < v'_k$. Since v'_{k-1} is adjacent to v'_k , v_{k-1} is adjacent to v'_k . Hence $a_R(v_{k-1}) = v_k < v'_k \leq a_R(v_{k-1})$, a contradiction. This contradiction arose from supposing that $v'_k > v_k$. Consequently $v'_k \leq v_k$. We have already proved by induction that $v'_i \leq v_i$ for every $1 \leq i \leq s$

notice that, since H is a hull set, $A \neq \emptyset$ whenever $I^k[H] \neq V(G)$. Since A is the union of intervals having nonempty intersection with $I^k[H]$ which is an interval, it follows that $I^{k+1}[H] = A \cup I^k[H]$ is also an interval.

Claim 2: $I^2[H]$ is an interval.

First, suppose that $uv \notin E(G)$. Notice that if $x \in I^1[H] \setminus H$, then $u < x < v$, because otherwise u would be adjacent to v . In addition, $H_1 = I^1[H] \setminus H$ is a clique. Suppose, towards a contradiction, that there exists two nonadjacent vertices $x, y \in H_1$. Assume, without loss of generality, that $x < y$. Hence, since $x < y < v$ and $xv \in E(G)$, x is adjacent to y , reaching a contradiction. Since H_1 is a clique, it turns out to be an interval $[u_L, u_R]$ under the considered unit interval order. If there is no vertex x such that either $u < x < u_L$ or $u_R < x < v$, then $I^1[H]$ is an interval and the assertion follows by Claim 1. Suppose now that it is not the case, and there exists a vertex $x \notin I^1[H]$ such that $u < x < u_L$. Since u is adjacent to u_L , x is adjacent to u and u_L . Analogously, if there exists a vertex x such that $u_R < x < v$, then x is adjacent to u_R and v . Thus $[u, v] \subseteq I^2[H]$. In addition, if $z < y < u$ and $z \in I^2[H]$, then $y \in I^2[H]$. Because z is adjacent to some vertex $x \in H_1$ and thus z is also adjacent to u , which implies that y is adjacent to u and x . Symmetrically, if $v < y < z$ and $z \in I^2[H]$, then $y \in I^2[H]$. Therefore, $I^2[H]$ is an interval.

Finally, suppose that u is adjacent to v . Hence $x \in H_1$ if and only if $R = \{x, u, v\}$ is contained in a clique C in G . Therefore $I^1[H] = \bigcup C$ where the union is taking over all the cliques containing H . Since all such cliques are intervals, $I^1[H]$ turns out to be an interval which implies, by Claim 2, that $I^2[H]$ is also an interval.

On the one hand, since $H \neq \{v_L, v_R\}$, if $\tau_H(G) = 1$, then the result holds. On the other hand, if $k \geq 2$, since $I^k[H]$ is an interval for every $k \geq 2$ because of Claims 1 and 2, either $v_L \notin I^k[H]$ or $v_R \notin I^k[H]$ for every $1 \leq k < \tau_H(G)$. The result follows from these observations. □

Let “ $<$ ” be a unit interval order of a unit interval graph G . We use $v \downarrow$ and $v \uparrow$ to denote the vertex immediately before v and the vertex immediately after v , respectively. We denote by $V_k(H)$ the set of vertices with percolation time k under the hull set H . We denote $L(I)$ and $R(I)$ to the leftmost vertex and the rightmost vertex of the set $I \subseteq V(G)$ under the order “ $<$ ”.

Corollary 3.3. *If G is a unit interval graph such that $|V(G)| \geq 3$ every two maximal cliques with nonempty intersection have at least two common vertices, then $\tau(\{v_L, v_L \uparrow\}) = \tau(\{v_R \downarrow, v_R\}) = \text{diam}(G) = \tau(G)$.*

Proof. Notice that G is 2-connected. Otherwise, G would have a cut vertex v and thus, since G is chordal, $\{v\} = C \cap C'$ where C and C' are maximal cliques of G (see [17]). In addition, two maximal cliques with nonempty intersection have at least two common vertices and $|V(G)| \geq 3$, the only maximal clique containing v_L (respectively v_R) contains at least three vertices. Hence, by Theorem 3.2, $\{v_L, \uparrow v_L\}$ (respectively $\{\downarrow v_R, v_R\}$) is a hull set of G .

Since G is a 2-connected graph, it is easy to see, by applying Theorem 3.2, that $\tau(G) = 1$ if and only if G is a complete graph. From now on, we will assume that $\tau(G) \geq 2$. Let H be a hull set of G . We can assume by Theorem 3.2 and Lemma 3.1, without loss of generality, that $H = \{u, v\}$ for two vertices u and v having a common neighbor, and $\tau(G) = \tau_H(G) = \tau_H(v_R)$. Assume, without loosing generality, that $u < v$, and thus $v_L \leq u$ and $v_L \uparrow \leq v$. Hence $R(I^1[\{v_L, v_L \uparrow\}]) \leq R(I^1[H])$. Consequently, it can be proved by induction that $R(I^k[\{v_L, v_L \uparrow\}]) \leq R(I^k[H])$ for every $1 \leq k \leq \tau(G)$. Thus $\tau(G) = \tau_H(v_R) \leq \tau_{\{v_L, v_L \uparrow\}}(v_R)$. Therefore $\tau(G) = \tau(\{v_L, v_L \uparrow\})$. By symmetry, if $\tau(G) = \tau_H(G) = \tau_H(v_L)$, then $\tau(G) = \tau(\{v_R \downarrow, v_R\})$.

Notice that $Q_k = [a_L(R(I^k[H])), R(I^k[H])]$ is a maximal clique. Otherwise, there would exist a maximal clique $Q = [x, y]$ containing Q_k with $R(I^k[H]) < y$, since Q_k contains at least two vertices in $I^{k-1}[H]$, $y \in I^k[H]$. Since there are no two maximal cliques with nonempty intersection having at most one common vertex, $a_R(R(I^k[H])) = R(I^{k+1}[H])$. Otherwise, there would exist a vertex x adjacent to $R(I^k[H])$ and nonadjacent to $R(I^k[H]) \downarrow$ and thus the maximal clique $[R(I^k[H]), a_R(R(I^k[H]))]$ would be a maximal clique having exactly one vertex in common with the maximal clique $[a_L(R(I^k[H])), R(I^k[H])]$, which is precisely the vertex $R(I^k[H])$. By Corollary 3.2, the path $v_L, R(I^1[H]), \dots, R(I^{\tau(G)}), v_R$ is a diameter path. By symmetry, if $H' = \{v_R \downarrow, v_R\}$, then the path $v_R, L(I^1[H']), \dots, L(I^{\tau(G)}), v_L$ is also a diameter path. Finally, by Lemma 3.1 and the previous discussion, we conclude that $\tau(G) = \text{diam}(G) = \tau(\{v_L, v_L \uparrow\}) = \tau(\{v_R \downarrow, v_R\})$. \square

Reading carefully the first part of the proof of Corollary 3.3 we can derive the following remark, where the hypothesis of each pair of intersecting maximal cliques sharing at least two vertices is dropped.

Remark 3.1. If G is a 2-connected unit interval graph such that $|V(G)| \geq 3$, then either $\tau(G) = \tau(\{v_L, v_L \uparrow\})$ or $\tau(G) = \tau_G(\{v_R \downarrow, v_R\})$.

Theorem 3.3. If G is a 2-connected unit interval graph such that $|V(G)| \geq 3$, then there exists a 2-connected unit interval graph G^* such that every two nonempty-intersection maximum clique has at least two common vertices and $\tau(G) = \tau(G^*) = \text{diam}(G^*)$. Besides, $\tau(G) = \tau(\{v_L, v_L \uparrow\}) = \tau_G(\{v_R \downarrow, v_R\})$.

Proof. Since G is 2-connected and does not contain any induced cycle with at least four vertices, every maximal clique of G has at least three vertices. Indeed, if $C = [a, b]$ is clique with two vertices, then ab is a bridge of G , an edge whose removal disconnects the graph, and thus, since $|V(G)| \geq 3$, a or b is a cut vertex, contradicting that G is 2-connected. Hence, ab is an edge of a complete graph on three vertices. Let v_1, v_2, \dots, v_n be a unit interval order for $V(G)$. Recall that a maximal clique of G is a maximal interval of pairwise adjacent vertices. Besides, two vertices are adjacent if and only if they belong to at least one of these maximal intervals of pairwise adjacent vertices. In other words, these intervals define the adjacencies of G , once the unit interval order of $|V(G)|$ was established. We are going to define the maximal intervals of a graph G^* obtained from G by properly adding some vertices without modifying the relative order of those vertices in $V(G)$. See Fig. 4



Figure 4: We only draw the vertices of these two unit interval graphs, being two vertices adjacent if and only if they belong to the same ellipse. Ellipses stand for the maximal cliques of the graph G on the left and its corresponding graph G^* on the right. Numbers indicate the percolation time of each vertex given the hull set of those vertices labeled with 0.

First, set $w_i = v_i$ for each $1 \leq i \leq n$. We will traverse the vertices of G from w_1 to w_n . Set $G = G_1$; and any time we find a vertex w_h such that $\{w_h\} = C \cap C'$, where C and C' are maximal cliques of G_i , apply the following transformation; we will call such vertices singular vertices. Add a vertex w_{n+k} at the end, where $k - 1$ stands for the number of singular vertices previously processed, one for each graph G_j for every $1 \leq j \leq i - 1$, and replace each clique $C = [a, b]$ by $C' = [a, b]$ if $b \leq w_h$, by $C' = [a, b\uparrow]$ if $a < w_h < b$, by $C' = [a\uparrow, b\uparrow]$ if $a \geq w_h$. Notice that if $b = w_{n+k-1}$, then $b\uparrow = w_{n+k}$. The new cliques define a unit interval graph G^* with vertex set $\{v_1 \dots, v_{n+m}\}$, where m is the number singular vertices of G . Notice that, the number of maximal cliques of G and G^* agree. If C_1, \dots, C_r denote the set of maximal cliques of G , we denote by C_1^*, \dots, C_r^* the maximal cliques obtained by the transformation of G into G^* , where C_i^* is the clique corresponding to C_i for every $1 \leq i \leq k$, under the previously described transformation. Besides, the relative order of the left endpoints (respectively right endpoints) of the maximal cliques of G is not modified when transforming G into G^* , and $C_i^* \cap C_j^* \neq \emptyset$ if and only if $|C_i \cap C_j| \geq 2$.

It is easy to prove that two nonempty intersecting cliques in G^* have at least two vertices in common. In addition, G^* is 2-connected. Since each pair of intersecting cliques has at least two vertices in common, it suffices to prove that G^* is connected. We are going to proceed to prove it by induction. Recall that, if G^* has a cut vertex v , since G^* is chordal [17], then $\{v\} = C \cap C'$ where C and C' are maximal cliques of G^* . Suppose, towards a contradiction, that G^* is disconnected. Hence there exists a positive integer i such that $1 < i < n + m$ in G^* , such that every vertex v of G^* such that $v \leq w_i$ is nonadjacent to every vertex w of G^* such that $w_{i+1} \leq w$. Thus there exists an index j such that the maximal clique $[w_{i+1}, a_R(w_{i+1})]$ of G_{j+1} comes from the maximal clique $[w_i, a_R(w_i)]$ and thus $w_{i+1} = w_i\uparrow$, $a_R(w_{i+1}) = a_R(w_i)\uparrow$ and w_i is a singular vertex of G_j . Since G_j is 2-connected, by inductive hypothesis, and thus w_i is not a cut vertex of G_j , there exists a maximal clique $[x, y]$ in G_j such that $a_L(w_j) < x < w_j$ and $w_j < y < a_R(w_j)$. By construction, $x < w_j$ in G_{j+1} , $w_{j+1} < y\uparrow$ in G_{j+1} , and x is adjacent to $y\uparrow$ in G_{j+1} , a contradiction. Therefore, G_{j+1} is connected.

It remains to prove that $\tau(G^*) = \tau(G)$. We consider C_1, \dots, C_s and C_1^*, \dots, C_s^* the maximal cliques of G and G^* respectively ordered by their left endpoints. Since G and G^* are 2-connected, it suffices to prove that $\tau_{G^*}(\{w_1, w_2\}) = \tau_G(\{v_1, v_2\}) = \tau_G(\{v_{n-1}, v_n\}) = \tau_{G^*}(w_{n+m-1}, w_m)$ (see Corollary 3.3 and Remark 3.1). In order to

prove it, we have to note that $I^k[\{v_1, v_2\}] = I[[v_1, v_{k-1}]] = [v_1, v_k]$ and $I^k[\{w_1, w_2\}] = I[[w, w_{k-1}]] = [w_1, w_k]$, where $v_k = a_R(v_{k-1}\downarrow)$ and $w_k = a_R(w_k) = a_R(w_{k-1})$, and in addition, there exists an integer i such that $1 < i \leq s$ and if C and C^* are the maximal cliques of G and G^* respectively, containing $a_R(v_{k-1}\downarrow)$ and $a_R(w_k)$ as their leftmost endpoint, then $C = C_i$ and $C^* = C_i^*$. Therefore, $\tau_{G^*}(\{w_1, w_2\}) = \tau_G(\{v_1, v_2\})$. Recall that the relative order of the right endpoints and the left endpoints of the corresponding maximal cliques of G^* did not suffer any modifications respect to the maximal cliques of G . Therefore, by symmetry, $\tau_G(\{v_{n-1}, v_n\}) = \tau_{G^*}(\{w_{n+m-1}, w_{n+m}\})$. \square

Now we have to consider the case in which G has at least one cut vertex.

Proposition 3.3. *Let G be a connected graph. If C_1, \dots, C_r are the 2-connected components of G , then $\tau(G) \leq \sum_{i=1}^r \tau(C_i)$.*

Proof. Let S be a hull set of G . Denote by s_i the minimum $h > 0$ such that $V(C_i) \cap V_h[S]$ is a hull set of C_i ; and denote by t_i the minimum h such that $V(C_i) \subseteq I^h[S]$.

$$\tau(G) \leq \sum_{i=1}^r (t_i - s_i + 1) \leq \sum_{i=1}^r \tau(C_i). \quad \square$$

Theorem 3.4. *If G is a connected unit interval graph with no cut vertex of degree 2 and no vertex of degree 1, then $\tau(G) = \sum_{i=1}^r \text{diam}(C_i^*)$. In addition, $\tau(G) = \tau(v_L) = \tau(v_R)$.*

Proof. Since G has no vertices of degree 1, $d(v_1) \geq 2$ and $d(v_n) \geq 2$. Each 2-connected component is an interval of consecutive vertices in the unit interval order v_1, \dots, v_n of G , where both endpoints are cut vertices except for v_1 and v_n ; i.e., $V(C_i) = [v_{m_{i-1}}, v_{m_i}]$ for every $1 \leq i \leq r$, where $m_i = \sum_{j=1}^i |V(C_j)| - i + 1$ ($m_0 = 1$). Since G has no cut vertex of degree 2, it is easy to see, using Theorem 3.3, that $S = \{v_1, v_2, v_{m_1+1}, \dots, v_{m_{r-1}+1}\}$ (respectively $\{v_n, v_{n-1}, v_{m_{r-1}-1}, \dots, v_{m_1-1}\}$) is a hull set, and $\tau_G(S) = \sum_{i=1}^r \tau(C_i) = \tau(v_n)$ (respectively $\tau_G(S) = \sum_{i=1}^r \tau(C_i) = \tau(v_1)$). Therefore, by Proposition 3.3, $\tau(G) = \sum_{i=1}^r \text{diam}(C_i^*) = \tau(v_L) = \tau(v_R)$. \square

Let G be a connected unit interval graph whose 2-connected components are G_1, \dots, G_r and $x \leq y$ for every $x \in V(G_i)$ and $y \in V(G_j)$ such that $1 \leq i < j \leq r$. We define G^* as the unit interval graphs whose 2-connected components are G_1^*, \dots, G_r^* and $x \leq y$ for every $x \in V(G_i^*)$ and $y \in V(G_j^*)$ such that $1 \leq i < j \leq r$. Notice that in both cases the equality holds when $x = y$ is a cut vertex. In addition, if G has no vertex of degree 1 and it does not have cut vertices of degree 2, by Theorem 3.4, $\tau(G) = \tau(G^*) = \sum_{i=1}^r \text{diam}(G_i^*)$.

Let G be a connected unit interval graph with at least three vertices having a unit interval order of its vertices v_1, \dots, v_n . If u is a vertex of degree 1, since G is connected, $u = v_1$ or $u = v_n$. Besides, if u is a cut vertex of degree 2 whose only neighbors are v and w , then v is nonadjacent to w and either $v < u < w$ or $w < u < v$. We will call an *special subgraph* to a subgraph induced by those vertices in a maximal interval $[a, b]$ such that $2 < d(v)$ for every cut vertex v of G such that

$a < v < b$ and also one of the following assertions holds: (i) a (respectively b) belongs to $\{v_1, v_n\}$ or (ii) a (respectively b) is a cut vertex of degree 2 of G . Such an interval $[a, b]$ will be called *special interval*.

If H is a special subgraph of G induced by $[a, b]$ we define $t(H)$ as follows:

- $t(H) = 1$, if $|V(H)| = 2$.
- $t(H) = \tau(H^*)$, if $v_1 = a, v_n = b, 2 \leq d(v_1)$ and $2 \leq d(v_n)$.
- $t(H) = \tau((H - a)^*) + 1$, if $[(v_1 = a \text{ and } d(v_1) = 1) \text{ or } v_1 < a], v_n = b$, and $2 \leq d(v_n)$.
- $t(H) = \tau((H - b)^*) + 1$, if $v_1 = a, 2 \leq d(v_1)$ and $[(d(v_n) = 1 \text{ and } v_n = b) \text{ or } b < v_n]$.
- $t(H) = \tau((H - \{a, b\})^*) + 1$, if $v_1 = a, v_n = b, d(v_1) = d(v_n) = 1$.
- $t(H) = \tau((H - \{a, b\})^*) + 2$ if $(v_1 < a, v_n = b, \text{ and } d(v_n) = 1)$, or $(v_1 = a, b < v_n, \text{ and } d(v_1) = 1)$, or $(v_1 < a \text{ and } b < v_n)$.

We define $\epsilon(G)$ as the maximum $t(H)$ among all the special subgraphs H of G . Notice that if G is a path on n vertices with $n \geq 3$ then $\epsilon(G) = 1$. In Fig. 5 can be seen an example of a unit interval graph G with two special subgraphs and $\epsilon(G) = 6$.

Theorem 3.5. *If G is a connected unit interval graph such that $|V(G)| \geq 3$, then $\tau(G) = \epsilon(G)$.*

Proof. Suppose first that G has no cut vertex of degree 2. If G has no vertex of degree 1, the result follows by Theorem 3.4. Suppose now that G has at least one vertex of degree 1. Such a vertex could be either v_1 or v_n . Assume that $d(v_1) = 1$ and $d(v_n) \geq 2$. Hence $v_1 \in S$ for every hull set S of G . Since $G - v_1$ is a 2 connected unit interval graph and it has neither a cut vertex of degree 2 nor a vertex of degree 1, if $S - v_1$ is a hull set of $G - v_1$, then $\tau_S(G) = \tau_{S-v_1}(G - v_1) \leq \tau((G - v_1)^*) < t(G)$. Assume that $S - v_1$ is not a hull set of $G - v_1$. Thus there exist a vertex $u \in V(G - v_1)$ such that v_1 and u has a common neighbor in G , this vertex only can be v_2 ; i.e, $u = v_2$, which implies that $\{v_2, v_3\} \subseteq I_{P_3}^1[S]$. Consequently, by Theorem 3.4, $\tau_S(G) \leq \tau((G - v_1)^*) + 1 = t(G)$. Since $S = \{v_1, v_3\}$ is a hull set and $I_{P_3}^1[S] - \{v_1\} = \{v_2, v_3\}$, it follows that $\tau_S(G) = \tau((G - v_1)^*) + 1$. Therefore, $\tau(G) = t(G)$. Symmetrically, if $d(v_1) \geq 2$ and $d(v_n) = 1$, then $\tau(G) = t(G)$. Following the same line of argument, it can be proved that the result also holds if $d(v_1) = d(v_n) = 1$. That is why the details are omitted.

Suppose now that G has at least one cut vertex v of degree 2 and let $[a, b]$ be a special interval such that $|[a, b]| \geq 3$ and $H = G[[a, b]]$. Assume that one of a and b is a cut vertex of degree 2 in G , say a . Using the similar techniques to those of the above paragraph, it can be proved that:

- $\tau(H - a) = \max\{\tau_G(a\uparrow), \tau_G(b)\} = \max\{\tau((H - a)^*), \tau((H - a)^*) + 1\} = \tau((H - a)^*) + 1$, whenever $d(b) \geq 2$,

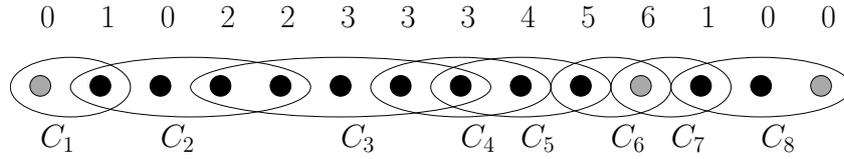


Figure 5: In this example the unit interval graph G contains two special intervals whose endpoints are grey vertices, considering the vertices ordered from left to right. For the edges of G we follow the convention of Fig. 4. Besides, $\tau(G) = 6$.

- $\tau(H - \{a, b\}) = \max\{\tau_G(a\uparrow), \tau_G(b\downarrow)\} = \max\{\tau((H - \{a, b\})^*) + 1, \tau((H - \{a, b\})^*) + 1\} = \tau((H - \{a, b\})^*) + 1$, whenever $d(b) = 1$ or b is cut vertex of degree 2.

Analogously, if b is a cut vertex of degree 2, then the following two conditions hold.

- $\tau(H - b) = \max\{\tau_G(a), \tau_G(b\downarrow)\} = \max\{\tau((H - b)^*) + 1, \tau((H - b)^*)\} = \tau((H - b)^*) + 1$, whenever $d(a) \geq 2$,
- $\tau(H - \{a, b\}) = \max\{\tau_G(a\uparrow), \tau_G(b\downarrow)\} = \max\{\tau((H - \{a, b\})^*) + 1, \tau((H - \{a, b\})^*) + 1\} = \tau((H - \{a, b\})^*) + 1$, whenever $d(a) = 1$ or a is a cut vertex of degree 2.

The result follows by combining these facts with the following: if v is a cut vertex of G such that $d(v) = 2$ then $\tau(v) = \max\{\tau(v\downarrow), \tau(v\uparrow)\}$. \square

4 A special graph class

A graph G satisfies the property \mathcal{P} if $I_{P_3}^2[S] = I_{P_3}[S]$ for every set $S \subset V(G)$. It is easy to see that this property is hereditary and every graph G belonging to this class satisfies $\tau(G) \leq 1$. The next result characterizes, by minimal forbidden induced subgraphs, those graphs satisfying property \mathcal{P} .

Theorem 4.1. *Let G be a graph. Then, G satisfies the property \mathcal{P} if and only if it does not contain as induced subgraph any graph depicted in Figure 6.*

Proof. It is easy to check that G_i does not satisfy the property \mathcal{P} for each $1 \leq i \leq 5$.

Conversely, if G does not satisfy the property \mathcal{P} , then G contains, as an induced subgraph, one of the graphs depicted in Figure 6. Let S be a subset of vertices of G such that $I_{P_3}[S]$ is properly contained in $I_{P_3}^2[S]$. Hence there exist two vertices u and v such that $u \in I_{P_3}[S] - S$ and $v \in I_{P_3}^2[S] - I_{P_3}[S]$. We will split the proof into two cases.

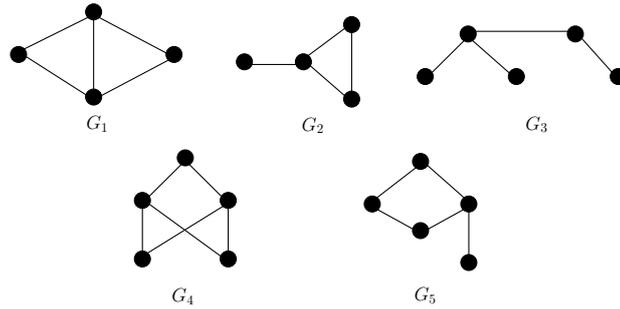


Figure 6: G_1 is the diamond, G_2 is the paw, G_3 is the chair, and G_4 is $K_{2,3}$.

Case 1: $N(v) \cap S \neq \emptyset$.

Assume first that there exists $z \in N(v) \cap N(u) \cap S$. Since $u \in I_{P_3}[S]$, there exists a vertex w distinct of z which is adjacent to u and nonadjacent to v , because otherwise $v \in I_{P_3}[S]$. If w is adjacent to z , then $\{z, u, v, w\}$ either induces G_1 (if v is adjacent to u) or $\{z, u, v, w\}$ induces G_2 (if v is nonadjacent to u). Hence w is nonadjacent to z . Besides, u is nonadjacent to v , because otherwise $\{u, v, w, z\}$ induces G_2 . Since $v \in I_{P_3}^2[S]$, there exists a vertex $x \in I_{P_3}^1[S] \setminus \{u\}$ which is adjacent to v . If x is adjacent to z , then $\{u, v, x, z\}$ either induces G_1 (if x is adjacent to u) or induces G_2 (if x is nonadjacent to u). Hence x is nonadjacent to z . On the one hand, if x is adjacent to u and w , then $\{u, v, w, x\}$ induces G_1 . On the other hand, if x is adjacent to u and nonadjacent to w , then $\{u, v, w, x, z\}$ induces G_5 . Thus x is nonadjacent to u . Since $x \in I_{P_3}[S] - S$, there exists a vertex $s_1 \in S \setminus \{w, z\}$ which is adjacent to x . If x is adjacent to w and w is nonadjacent to s_1 , then either $\{v, w, x, s_1, z\}$ induces G_3 . In addition, if x is adjacent to w and w is adjacent to s_1 , then $\{v, s_1, w, x\}$ induces G_2 . Hence x is nonadjacent to w . Notice also that s_1 is nonadjacent to z . If s_1 is adjacent to z and u , then $\{s_1, u, x, z\}$ induces G_2 . If s_1 is adjacent to z and nonadjacent to u , then $\{s_1, u, v, x, z\}$ induces G_5 . Hence s_1 is nonadjacent to z . Since x is nonadjacent to z and w , there exists a vertex $s \in S - \{s_1, w, z\}$ which is adjacent to x . By symmetry, s_2 is nonadjacent to v and z . If s_1 is adjacent to s_2 , then $\{s_1, s_2, v, x\}$ induces G_2 . Therefore, if s_1 is nonadjacent to s_2 , then $\{s_1, s_2, x, v, z\}$ induces G_3 , a contradiction.

Assume now that $N(v) \cap N(u) \cap S = \emptyset$ and thus there exists a vertex $z \in S$ adjacent to v but nonadjacent to u , and there exist two vertices $w_1, w_2 \in S$ adjacent to u and nonadjacent to v . Suppose, towards a contradiction, that u is adjacent to v . Hence w_1 is nonadjacent to w_2 , because otherwise $\{u, v, w_1, w_2\}$ induces G_2 . On the one hand, if z is adjacent to w_1 and w_2 , then $\{u, v, w_1, w_2, z\}$ induces G_4 . On the other hand, if z is adjacent to exactly one of w_1 and w_2 , then $\{u, v, w_1, w_2, z\}$ induces G_5 . Hence z is nonadjacent to w_1 and w_2 . Thus $\{u, v, w_1, w_2, z\}$ induces G_3 , a contradiction. Therefore, u is nonadjacent to v which implies that there exists a vertex $x \in I_{P_3}^1[S] - S$ adjacent to v . We may assume, by the discussion of the above paragraph, that x is nonadjacent to z , otherwise $N(v) \cap N(x) \cap S \neq \emptyset$. Consequently, there exist two vertices a and b in $S - \{z\}$ adjacent to x . Since G has no G_2 as an

induced subgraph, a is nonadjacent to b . Besides, since G has no G_4 neither G_5 as an induced subgraph, a and b are nonadjacent to z . Therefore, $\{a, b, x, v, z\}$ induces G_3 .

Case 2: $N(v) \cap S = \emptyset$.

There exist two vertices $u_1, u_2 \in I_{P_3}[S] \setminus S$ which are adjacent to v . Besides, there exist two vertices $x_1, x_2 \in S$ which are adjacent to u_1 . If x_1 is adjacent to x_2 , then $\{u_1, v, x_1, x_2\}$ induces G_2 . Hence x_1 is nonadjacent to x_2 . On the one hand, if x_1 is adjacent to u_2 and u_1 is adjacent to u_2 then $\{u_1, v, u_1, u_2\}$ induces G_1 . On the other hand, if x_1 is adjacent to u_2 and u_1 is nonadjacent to u_2 then $\{u_1, v, u_1, u_2\}$ induces G_2 . Thus x_1 is nonadjacent to u_2 . By symmetry, x_2 is nonadjacent to u_2 . Consequently, u_1 is nonadjacent to u_2 , because otherwise $\{u_1, u_2, v, x\}$ induces G_2 and thus u_1 is nonadjacent to u_2 . Therefore, $\{u_1, u_2, v, x_1, x_2\}$ induces G_3 .

We have already proved that in all possible cases the graph G which does not satisfy the property \mathcal{P} contains one of the graphs depicted in Figure 6 as an induced subgraph. \square

Corollary 4.1. *Let G be a graph. If G satisfies the property \mathcal{P} , then $g_{P_3}(G) = h_{P_3}(G)$.*

Corollary 4.1 shows that Theorem 4.1 is a characterization of a subclass of those graphs G such that $h(H) = g(H)$ for every induced subgraph of G , characterized in [9].

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