Gallai-Ramsey numbers of $C_{10}$ and $C_{12}$

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Abstract

A Gallai coloring is a coloring of the edges of a complete graph without rainbow triangles, and a Gallai $k$-coloring is a Gallai coloring that uses at most $k$ colors. Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the Gallai-Ramsey number $GR(H_1, \ldots, H_k)$ is the least integer $n$ such that every Gallai $k$-coloring of the complete graph $K_n$ contains a monochromatic copy of $H_i$ in color $i$ for some $i \in \{1, \ldots, k\}$. When $H = H_1 = \cdots = H_k$, we simply write $GR_k(H)$. We continue to study Gallai-Ramsey numbers of even cycles and paths. For all $n \geq 3$ and $k \geq 1$, let $G_i = P_{2i+3}$ be a path on $2i + 3$ vertices for all $i \in \{0, 1, \ldots, n-2\}$ and $G_{n-1} \in \{C_{2n}, P_{2n+1}\}$. Let $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in \{1, \ldots, k\}$ with $i_1 \geq i_2 \geq \cdots \geq i_k$. Song recently conjectured that $GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j$. This conjecture has been verified to be true for $n \in \{3, 4\}$ and all $k \geq 1$. In this paper, we prove that the aforementioned conjecture holds for $n \in \{5, 6\}$ and all $k \geq 1$. Our result implies that for all $k \geq 1$, $GR_k(C_{2n}) = GR_k(P_{2n}) = (n-1)k + n + 1$ for $n \in \{5, 6\}$ and $GR_k(P_{2n+1}) = (n-1)k + n + 2$ for $1 \leq n \leq 6$.

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1 Introduction

In this paper we consider graphs that are finite, simple and undirected. Given a graph $G$ and a set $A \subseteq V(G)$, we use $|G|$ to denote the number of vertices of $G$, and $G[A]$ to denote the subgraph of $G$ obtained from $G$ by deleting all vertices in $V(G) \setminus A$. A graph $H$ is an induced subgraph of $G$ if $H = G[A]$ for some $A \subseteq V(G)$. We use $P_n$, $C_n$ and $K_n$ to denote the path, cycle and complete graph on $n$ vertices, respectively. For any positive integer $k$, we write $[k]$ for the set $\{1, \ldots, k\}$.

Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the classical Ramsey number $R(H_1, \ldots, H_k)$ is the least integer $n$ such that every $k$-coloring of the edges of $K_n$ contains a monochromatic copy of $H_i$ in color $i$ for some $i \in [k]$. Ramsey numbers are notoriously difficult to compute in general. In this paper, we study Ramsey numbers of graphs in Gallai colorings, where a Gallai coloring is a coloring of the edges of a complete graph without rainbow triangles (that is, a triangle with all its edges colored differently). Gallai colorings naturally arise in several areas including: information theory [17]; the study of partially ordered sets, as in Gallai’s original paper [12] (his result was restated in [15] in the terminology of graphs); and the study of perfect graphs [5]. There are now a variety of papers which consider Ramsey-type problems in Gallai colorings (see, e.g., [2, 3, 4, 6, 10, 13, 14, 16, 21, 24]). These works mainly focus on finding various monochromatic subgraphs in such colorings. More information on this topic can be found in [9, 11].

A Gallai $k$-coloring is a Gallai coloring that uses at most $k$ colors. Given an integer $k \geq 1$ and graphs $H_1, \ldots, H_k$, the Gallai-Ramsey number $GR(H_1, \ldots, H_k)$ is the least integer $n$ such that every Gallai $k$-coloring of $K_n$ contains a monochromatic copy of $H_i$ in color $i$ for some $i \in [k]$. When $H = H_1 = \cdots = H_k$, we simply write $GR_k(H)$ and $R_k(H)$. Clearly, $GR_k(H) \leq R_k(H)$ for all $k \geq 1$ and $GR(H_1, H_2) = R(H_1, H_2)$. In 2010, Gyárfás, Sárközy, Sebő and Selkow [14] proved the general behavior of $GR_k(H)$.

**Theorem 1.1** ([14]) Let $H$ be a fixed graph with no isolated vertices and let $k \geq 1$ be an integer. Then $GR_k(H)$ is exponential in $k$ if $H$ is not bipartite, linear in $k$ if $H$ is bipartite but not a star, and constant (does not depend on $k$) when $H$ is a star.

It turns out that for some graphs $H$ (e.g., when $H = C_3$), $GR_k(H)$ behaves nicely, while the order of magnitude of $R_k(H)$ seems hopelessly difficult to determine. It is worth noting that finding exact values of $GR_k(H)$ is far from trivial, even when $|H|$ is small. We will utilize the following important structural result of Gallai [12] on Gallai colorings of complete graphs.

**Theorem 1.2** ([12]) For any Gallai coloring $c$ of a complete graph $G$ with $|G| \geq 2$, $V(G)$ can be partitioned into nonempty sets $V_1, \ldots, V_p$ with $p \geq 2$ so that at most two colors are used on the edges in $E(G) \setminus (E(G[V_1]) \cup \cdots \cup E(G[V_p]))$ and only one color is used on the edges between any fixed pair $(V_i, V_j)$ under $c$.

The partition given in Theorem 1.2 is a Gallai-partition of the complete graph.
Conjecture 1.3 ([9]) For all integers \(k \geq 1\) and \(t \geq 3\),
\[
\text{GR}_k(K_t) = \begin{cases}
(R_2(K_t) - 1)^{k/2} + 1 & \text{if } k \text{ is even} \\
(t - 1)(R_2(K_t) - 1)^{(k-1)/2} + 1 & \text{if } k \text{ is odd}.
\end{cases}
\]

The first case of Conjecture 1.3 follows from a result of Chung and Graham [6] from 1983. A simpler proof of this case can be found in [14]. The case when \(t = 4\) was recently settled in [18]. Conjecture 1.3 remains open for all \(t \geq 5\). The next open case, when \(t = 5\), involves \(R_2(K_5)\). Angeltveit and McKay [1] recently proved that \(R_2(K_5) \leq 48\). It is widely believed that \(R_2(K_5) = 43\) (see [1]). It is worth noting that Schiermeyer [20] recently observed that if \(R_2(K_5) = 43\), then Conjecture 1.3 fails for \(K_5\) when \(k = 3\). More recently, Gallai-Ramsey numbers of odd cycles on at most 15 vertices have been completely settled by Fujita and Magnant [10] for \(C_5\), Bruce and Song [4] for \(C_7\), Bosse and Song [2] for \(C_9\) and \(C_{11}\), and Bosse, Song and Zhang [3] for \(C_{13}\) and \(C_{15}\). Very recently, the exact values of \(\text{GR}_k(C_{2n+1})\) for \(n \geq 8\) has been solved by Zhang, Song and Chen [23]. We summarize these results below.

Theorem 1.4 ([2, 3, 4, 23]) For all \(n \geq 3\) and \(k \geq 1\), \(\text{GR}_k(C_{2n+1}) = n \cdot 2^k + 1\).

In this paper, we continue to study Gallai-Ramsey numbers of even cycles and paths. For all \(n \geq 3\) and \(k \geq 1\), let \(G_{n-1} \in \{C_{2n}, P_{2n+1}\}\), \(G_i := P_{2i+3}\) for all \(i \in \{0, 1, \ldots, n-2\}\), and \(i_j \in \{0, 1, \ldots, n-1\}\) for all \(j \in [k]\). We want to determine the exact values of \(\text{GR}(G_{i_1}, \ldots, G_{i_k})\). By reordering colors if necessary, we assume that \(i_1 \geq \cdots \geq i_k\). Song and Zhang [22] recently proved that

Proposition 1.5 ([22]) For all \(n \geq 3\) and \(k \geq 1\),
\[
\text{GR}(G_{i_1}, \ldots, G_{i_k}) \geq |G_{i_1}| + \sum_{j=2}^{k} i_j.
\]

In the same paper, Song [22] further made the following conjecture.

Conjecture 1.6 ([22]) For all \(n \geq 3\) and \(k \geq 1\),
\[
\text{GR}(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.
\]
To completely solve Conjecture 1.6, one only needs to consider the case $G_{n-1} = C_{2n}$.

**Proposition 1.7 ([22])** For all $n \geq 3$ and $k \geq 1$, if Conjecture 1.6 holds for $G_{n-1} = C_{2n}$, then it also holds for $G_{n-1} = P_{2n+1}$.

Let $M_n$ denote a matching of size $n$ on $2n$ vertices. As observed in [22], the truth of Conjecture 1.6 implies that $GR_k(C_{2n}) = GR_k(P_{2n}) = GR_k(M_n) = (n-1)k + n + 1$ for all $n \geq 3$ and $k \geq 1$, and $GR_k(P_{2n+1}) = (n-1)k + n + 2$ for all $n \geq 1$ and $k \geq 1$. It is worth noting that Dzido, Nowik and Szuca [7] proved that $R_k(C_{2n}) \geq 4n$ for all $n \geq 3$. The truth of Conjecture 1.6 implies that $GR_3(C_{2n}) = 4n - 2 < R_3(C_{2n})$ for all $n \geq 3$. Conjecture 1.6 has recently been verified to be true for $n \in \{3, 4\}$ and all $k \geq 1$.

**Theorem 1.8 ([22])** For $n \in \{3, 4\}$ and all $k \geq 1$, let $G_i = P_{2i+3}$ for all $i \in \{0, 1, \ldots, n-2\}$, $G_{n-1} = C_{2n}$, and $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in [k]$ with $i_1 \geq \cdots \geq i_k$. Then
\[
GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.
\]

In this paper, we continue to establish more evidence for Conjecture 1.6. We prove that Conjecture 1.6 holds for $n \in \{5, 6\}$ and all $k \geq 1$.

**Theorem 1.9** For $n \in \{5, 6\}$ and all $k \geq 1$, let $G_i = P_{2i+3}$ for all $i \in \{0, 1, \ldots, n-2\}$, $G_{n-1} = C_{2n}$, and $i_j \in \{0, 1, \ldots, n-1\}$ for all $j \in [k]$ with $i_1 \geq \cdots \geq i_k$. Then
\[
GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.
\]

We prove Theorem 1.9 in Section 2. Applying Theorem 1.9 and Proposition 1.7, we obtain the following.

**Corollary 1.10** Let $G_i = P_{2i+3}$ for all $i \in \{0, 1, 2, 3, 4, 5\}$. For every integer $k \geq 1$, let $i_j \in \{0, 1, 2, 3, 4, 5\}$ for all $j \in [k]$ with $i_1 \geq \cdots \geq i_k$. Then
\[
GR(G_{i_1}, \ldots, G_{i_k}) = |G_{i_1}| + \sum_{j=2}^{k} i_j.
\]

**Corollary 1.11** For all $k \geq 1$,

(a) $GR_k(P_{2n+1}) = (n-1)k + n + 2$ for all $n \in [6]$.

(b) $GR_k(C_{2n}) = GR_k(P_{2n}) = (n-1)k + n + 1$ for $n \in \{5, 6\}$.

Finally, we shall make use of the following results on 2-colored Ramsey numbers of cycles and paths in the proof of Theorem 1.9.
Theorem 1.12 ([19]) For all \( n \geq 3 \), \( R_2(C_{2n}) = 3n - 1 \).

Theorem 1.13 ([8]) For all integers \( n, m \) satisfying \( 2n \geq m \geq 3 \), \( R(P_m, C_{2n}) = 2n + \lfloor \frac{m}{2} \rfloor - 1 \).

2 Proof of Theorem 1.9

We are ready to prove Theorem 1.9. Let \( n \in \{5, 6\} \). By Proposition 1.5, it suffices to show that \( GR(G_{i_1}, \ldots, G_{i_k}) \leq |G_{i_1}| + \sum_{j=2}^{k} i_j \).

By Theorem 1.8 and Proposition 1.7, we may assume that \( i_1 = n - 1 \). Then \( |G_{i_1}| = 2n \). By Theorem 1.12 and Theorem 1.13, we have \( GR(G_{i_1}, G_{i_2}) = R(G_{i_1}, G_{i_2}) = 2n + i_2 \). So we may assume \( k \geq 3 \). Let \( N := |G_{i_1}| + \sum_{j=2}^{k} i_j \). Then \( N \geq 2n \). Let \( G \) be a complete graph on \( N \) vertices and let \( c : E(G) \to [k] \) be any Gallai coloring of \( G \) using at least three colors. We next show that \( G \) contains a monochromatic copy of \( G_{i_j} \) in color \( j \) for some \( j \in [k] \). Suppose \( G \) contains no monochromatic copy of \( G_{i_j} \) in color \( j \) for any \( j \in [k] \) under \( c \). Such a Gallai \( k \)-coloring \( c \) is called a bad coloring. Among all complete graphs on \( N \) vertices with a bad coloring, we choose \( G \) with \( N \) minimum, taken over all \( n - 1 \geq i_1 \geq \cdots \geq i_k \geq 0 \).

By Theorem 1.2, we may consider a Gallai-partition of \( G \) with parts \( A_1, \ldots, A_p \), where \( p \geq 2 \). We may assume that \( |A_1| \geq \cdots \geq |A_p| \geq 1 \). Let \( \mathcal{R} \) be the reduced graph of \( G \) with vertices \( a_1, \ldots, a_p \), where \( a_i \in A_i \) for all \( i \in [p] \). By Theorem 1.2, assume that the edges of \( \mathcal{R} \) are colored either red or blue. Since \( c \) uses at least three colors, we see that \( \mathcal{R} \neq G \) and so \( |A_1| \geq 2 \). By abusing the notation, we use \( i_b \) to denote \( i_j \) when the color \( j \) is blue. Similarly, we use \( i_r \) (respectively, \( i_g \)) to denote \( i_j \) when the color \( j \) is red (respectively, green). Let

\[
A_k := \{a_i \in \{a_2, \ldots, a_p\} \mid a_i a_1 \text{ is colored blue in } \mathcal{R}\},
A_r := \{a_j \in \{a_2, \ldots, a_p\} \mid a_j a_1 \text{ is colored red in } \mathcal{R}\}.
\]

Then \( |A_b| + |A_r| = p - 1 \). Let \( B := \bigcup_{a_i \in A_b} A_i \) and \( R := \bigcup_{a_j \in A_r} A_j \). Then \( |A_1| + |R| + |B| = N \) and \( \max\{|B|,|R|\} \neq 0 \) because \( p \geq 2 \). Thus \( G \) contains a blue \( P_3 \) between \( B \) and \( A_1 \), or a red \( P_3 \) between \( R \) and \( A_1 \), and so \( \max\{i_b, i_r\} \geq 1 \). We next prove several claims.

Claim 1. Let \( r \in [k] \) and let \( s_1, s_r \) be nonnegative integers with \( s_1 + \cdots + s_r \geq 1 \). If \( i_{r_1} \geq s_1, \ldots, i_{r_r} \geq s_r \) for colors \( j_1, \ldots, j_r \in [k] \), then for any \( S \subseteq V(G) \) with \( |S| \geq |G| - (s_1 + \cdots + s_r) \), \( G[S] \) must contain a monochromatic copy of \( G_{i_{q_j}} \) in color \( j_q \) for some \( j_q \in \{j_1, \ldots, j_r\} \), where \( i_{j_q} = i_{j_q} - s_q \).

Proof. Let \( i_{j_1}^* := i_{j_1} - s_1, \ldots, i_{j_r}^* := i_{j_r} - s_r \) and \( i_j^* := i_j \) for all \( j \in [k] \setminus \{j_1, \ldots, j_r\} \). Let \( i_j^* := \max\{i_j^* \mid j \in [k]\} \). Then \( i_j^* \leq i_j \). Let \( N^* := |G_{i_j^*}| + |\sum_{j=1}^{k} i_j^*| - i_j^* \). Then \( N^* \geq 3 \) and \( N^* \leq N - (s_1 + \cdots + s_r) \) because \( s_1 + \cdots + s_r \geq 1 \). Since \( |S| \geq N - (s_1 + \cdots + s_r) \geq N^* \) and \( G[S] \) does not have a monochromatic copy of \( G_{i_j} \)
in color \( j \) for all \( j \in [k] \setminus \{j_1, \ldots, j_r\} \) under \( c \), by minimality of \( N \), \( G[S] \) must contain a monochromatic copy of \( G_{j_q}^* \) in color \( j_q \) for some \( j_q \in \{j_1, \ldots, j_r\} \).

**Claim 2.** \( |A_1| \leq n - 1 \), and so \( G \) does not contain a monochromatic copy of a graph on \( |A_1| + 1 \leq n \) vertices in color \( m \), where \( m \in [k] \) is a color that is neither red nor blue.

**Proof.** Suppose \( |A_1| \geq n \). We first claim that \( i_b \geq |B| \) and \( i_r \geq |R| \). Suppose \( i_b \leq |B| - 1 \) or \( i_r \leq |R| - 1 \). Then we obtain a blue \( G_{i_b} \) using the edges between \( B \) and \( A_1 \), or a red \( G_{i_r} \) using the edges between \( R \) and \( A_1 \), a contradiction. Thus \( i_b \geq |B| \) and \( i_r \geq |R| \), as claimed. Let \( i^*_b := i_b - |B| \) and \( i^*_r := i_r - |R| \). Since \( |A_1| = N - |B| - |R| \), by Claim 1 applied to \( i_b \geq |B| \), \( i_r \geq |R| \), and \( A_1 \), \( G[A_1] \) must have a blue \( G_{i^*_b} \) or a red \( G_{i^*_r} \), say the latter. Then \( i_r > i^*_r \). Thus \( |R| > 0 \) and \( G_{i^*_r} \) is a red path on \( 2i^*_r + 3 \) vertices. Note that

\[
|A_1| = |G_{i_1}| + \sum_{j=2}^{k} i_j - |B| - |R|
\]

\[
\geq \begin{cases} 
|G_{i_r}| + i_b - |B| - |R| & \text{if } i_r \geq i_b \\
|G_{i_b}| + i_r - |B| - |R| & \text{if } i_r < i_b,
\end{cases}
\]

\[
\geq \begin{cases} 
|G_{i_r}| + i^*_b - |R| & \text{if } i_r \geq i_b \\
2i^*_b + 2 + i_r - |B| - |R| \geq i^*_b + (2i^*_r + 3) - |R| & \text{if } i_r < i_b,
\end{cases}
\]

\[
\geq |G_{i_r}| - |R|.
\]

Then

\[
|A_1| - |G_{i^*_r}| \geq |G_{i_r}| - |G_{i^*_r}| - |R|
\]

\[
= \begin{cases} 
(3 + 2i_r) - (3 + 2i^*_r) - |R| = |R| & \text{if } i_r \leq n - 2 \\
(2 + 2i_r) - (3 + 2i^*_r) - |R| = |R| - 1 & \text{if } i_r = n - 1.
\end{cases}
\]

But then \( G[A_1 \cup R] \) contains a red \( G_{i_r} \) using the edges of the \( G_{i^*_r} \) and the edges between \( A_1 \setminus V(G_{i^*_r}) \) and \( R \), a contradiction. This proves that \( |A_1| \leq n - 1 \). Next, let \( m \in [k] \) be any color that is neither red nor blue. Suppose \( G \) contains a monochromatic copy of a graph, say \( J \), on \( |A_1| + 1 \) vertices in color \( m \). Then \( V(J) \subseteq A_{\ell} \) for some \( \ell \in [p] \). But then \( |A_{\ell}| \geq |A_1| + 1 \), contrary to \( |A_{\ell}| \geq |A_{n-1}| \).

For two disjoint sets \( U, W \subseteq V(G) \), we say \( U \) is blue-complete (respectively, red-complete) to \( W \) if all the edges between \( U \) and \( W \) are colored blue (respectively, red) under \( c \). For convenience, we say \( u \) is blue-complete (respectively, red-complete) to \( W \) when \( U = \{u\} \).

**Claim 3.** \( \min\{|B|, |R|\} \geq 1 \), \( p \geq 3 \), and \( B \) is neither red- nor blue-complete to \( R \) under \( c \).

**Proof.** Suppose \( B = \emptyset \) or \( R = \emptyset \). By symmetry, we may assume that \( R = \emptyset \). Then \( B \neq \emptyset \) and so \( i_b \geq 1 \). By Claim 2, \( |A_1| \leq n - 1 \leq 5 \) because \( n \in \{5,6\} \). Then
$|A_1| \leq i_b + 4$. If $i_b \leq |A_1| - 1$, then $i_b \leq n - 2$ by Claim 2. But then we obtain a blue $G_{i_b}$ using the edges between $B$ and $A_1$. Thus $i_b \geq |A_1|$. Let $i_b^* = i_b - |A_1|$. By Claim 1 applied to $i_b \geq |A_1|$ and $B$, $G[B]$ must have a blue $G_{i_b^*}$. Since $|B| \geq n + 1 + i_b^*$, we see that $G$ contains a blue $G_{i_b}$, a contradiction. Hence $R \neq \emptyset$, and similarly $B \neq \emptyset$, and so $p \geq 3$ for any Gallai-partition of $G$. It follows that $B$ is neither red- nor blue-complete to $R$, otherwise $\{B \cup A_1, R\}$ or $\{B, R \cup A_1\}$ yields a Gallai-partition of $G$ with only two parts.

Claim 4. Let $m \in [k]$ be a color that is neither red nor blue. Then $i_m \leq n - 4$. In particular, if $i_m \geq 1$, then $G$ contains a monochromatic copy of $P_{2i_m + 1}$ in color $m$ under $c$.

Proof. Note that $i_m \leq n - 4$ is trivially true when $i_m = 0$ because $n \in \{5, 6\}$ and $n - 4 \geq 1$. Suppose $i_m \geq 1$. By Claim 2, $|A_1| \leq n - 1$ and $G$ contains no monochromatic copy of $P_{|A_1| + 1}$ in color $m$ under $c$. Let $i_m^* := i_m - 1$. By Claim 1 applied to $i_m^* \geq 1$ and $V(G)$, $G$ must have a monochromatic copy of $G_{i_m^*}$ in color $m$ under $c$. Since $n \in \{5, 6\}$, $|A_1| \leq n - 1$ and $G$ contains no monochromatic copy of $P_{|A_1| + 1}$ in color $m$, we see that $i_m^* \leq n - 5$. Thus $i_m \leq n - 4$ and $G$ contains a monochromatic copy of $P_{2i_m + 1}$ in color $m$ under $c$ if $i_m \geq 1$.

By Claim 3 and the fact that $|A_1| \geq 2$, $G$ has a red $P_3$ and a blue $P_3$. Thus $\min\{i_b, i_r\} \geq 1$. By Claim 4, $\max\{i_b, i_r\} = i_1 = n - 1$. Then $|G| = |G_{i_1}| + \sum_{j=2}^{k} i_j \geq 2n + 1$. For the remainder of the proof of Theorem 1.9, we choose $p \geq 3$ to be as large as possible.

Claim 5. $\min\{|B|, |R|\} \leq n - 1$ if $|A_1| \geq n - 3$.

Proof. Suppose $|A_1| \geq n - 3$ but $\min\{|B|, |R|\} \geq n$. By symmetry, we may assume that $|B| \geq |R| \geq n$. Let $B := \{x_1, x_2, \ldots, x_{|B|}\}$ and $R := \{y_1, y_2, \ldots, y_{|R|}\}$. Let $H := (B, R)$ be the complete bipartite graph obtained from $G[B \cup R]$ by deleting all the edges with both ends in $B$ or in $R$. Then $H$ has no blue $P_5$ with both ends in $B$ and no red $P_5$ with both ends in $R$, else we obtain a blue $C_{2n}$ or a red $C_{2n}$ because $|A_1| \geq n - 3$. We next show that $H$ has no red $K_{3,3}$.

Suppose $H$ has a red $K_{3,3}$. We may assume that $H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ is a red $K_{3,3}$ under $c$. Since $H$ has no red $P_7$ with both ends in $R$, $\{y_4, \ldots, y_{|R|}\}$ must be blue-complete to $\{x_1, x_2, x_3\}$. Thus $H[\{x_1, x_2, x_3, y_4, y_5\}]$ has a blue $P_5$ with both ends in $\{x_1, x_2, x_3\}$ and $H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]$ has a red $P_5$ with both ends in $\{y_1, y_2, y_3\}$. If $|A_1| \geq n - 2$ or $\min\{i_b, i_r\} \leq n - 2$, then we obtain a blue $G_{i_b}$ or a red $G_{i_r}$, a contradiction. It follows that $|A_1| = n - 3$ and $i_b = i_r = n - 1$. Then $|G| = |G_{i_1}| + \sum_{j=2}^{k} i_j \geq 2n + (n - 1) = 3n - 1$. Thus $|B \cup R| = |G| - |A_1| \geq 2n + 2$. If $|R| \geq 6$, then $\{y_4, y_5, y_6\}$ must be red-complete to $\{x_4, x_5, x_6\}$, else $H$ has a blue $P_7$ with both ends in $B$. But then we obtain a red $C_{2n}$ in $G$. Thus $|R| = 5$, $n = 5$, and so $|B| \geq 7$. Let $A_1 = \{a_1, a_1^*\}$. For each $j \in \{4, 5, 6, 7\}$ and every $W \subseteq \{x_1, x_2, x_3\}$ with $|W| = 2$, no $x_j$ is red-complete to $W$ under $c$, else, say, $x_4$ is red-complete to $\{x_1, x_2\}$, then we obtain a red $C_{10}$ with vertices $a_1, y_1, x_1, x_4, x_2, y_2, x_3, y_3, a_1^*, y_4$.
in order, a contradiction. We may assume that \(x_4x_1, x_5x_2\) are colored blue. But then we obtain a blue \(C_{10}\) with vertices \(a_1, x_4, x_1, y_4, x_3, y_5, x_2, x_5, a_1', x_6\) in order, a contradiction. This proves that \(H\) has no red \(K_{3,3}\).

Let \(X := \{x_1, x_2, \ldots, x_5\}\) and \(Y := \{y_1, y_2, \ldots, y_5\}\). Let \(H_b\) and \(H_r\) be the spanning subgraphs of \(H[X \cup Y]\) induced by all the blue edges and red edges of \(H[X \cup Y]\) under \(c\), respectively. By the Pigeonhole Principle, there exist at least three vertices, say \(x_1, x_2, x_3\), in \(X\) such that either \(d_{H_b}(x_i) \geq 3\) for all \(i \in [3]\) or \(d_{H_r}(x_i) \geq 3\) for all \(i \in [3]\). Suppose \(d_{H_r}(x_i) \geq 3\) for all \(i \in [3]\). We may assume that \(x_1\) is red-complete to \(\{y_1, y_2, y_3\}\). Since \(|Y| = 5\) and \(H\) has no red \(P_7\) with both ends in \(R\), we see that \(N_{H_r}(x_1) = N_{H_r}(x_2) = N_{H_r}(x_3) = \{y_1, y_2, y_3\}\). But then \(H[\{x_1, x_2, x_3, y_1, y_2, y_3\}]\) is a red \(K_{3,3}\), contrary to \(H\) has no red \(K_{3,3}\). Thus \(d_{H_b}(x_i) \geq 3\) for all \(i \in [3]\). Since \(|Y| = 5\), we see that any two of \(x_1, x_2, x_3\) have a common neighbor in \(H_b\). Furthermore, two of \(x_1, x_2, x_3\), say \(x_1, x_2\), have at least two common neighbors in \(H_b\). It can be easily checked that \(H\) has a blue \(P_5\) with ends in \(\{x_1, x_2, x_3\}\), and there exist three vertices, say \(y_1, y_2, y_3\), in \(Y\) such that \(y_ix_i\) is blue for all \(i \in [3]\) and \(\{x_1, x_2, x_3, y_1, y_2, y_3\}\) is red-complete to \(\{y_1, y_2, y_3\}\). Then \(H\) has a blue \(P_5\) with both ends in \(\{x_1, x_2, x_3\}\) and a red \(P_5\) with both ends in \(\{y_1, y_2, y_3\}\). If \(|A_1| \geq n - 2\) or \(\min\{i_b, i_r\} \leq n - 2\), then we obtain a blue \(G_{i_b}\) or a red \(G_{i_r}\), a contradiction. It follows that \(|A_1| = n - 3\) and \(i_b = i_r = n - 1\). Thus \(|B \cup R| \geq 1 + n + i_b + i_r - |A_1| = 2n + 2\). Then \(|B| \geq n + 1\) and so \(H[\{x_4, x_5, x_6, y_1, y_2, y_3\}]\) is a red \(K_{3,3}\), contrary to the fact that \(H\) has no red \(K_{3,3}\). 

Claim 6. \(|A_1| \geq 3\).

Proof. Suppose \(|A_1| = 2\). Then \(G\) has no monochromatic copy of \(P_5\) in color \(j\) for any \(j \in \{3, \ldots, k\}\) under \(c\). By Claim 4, \(i_3 = \cdots = i_k = 0\) and so \(N = 1 + n + i_b + i_r\).

We may assume that \(|A_1| = \cdots = |A_i| = 2\) and \(|A_{i+1}| = \cdots = |A_p| = 1\) for some integer \(t\) satisfying \(p \geq t \geq 1\). Let \(A_i = \{a_i, b_i\}\) for all \(i \in [t]\). By reordering if necessary, each of \(A_1, \ldots, A_t\) can be chosen as the largest part in the Gallai-partition \(A_1, A_2, \ldots, A_p\) of \(G\). For all \(i \in [t]\), let

\[
A_i^b := \{a_j \in V(R) \mid a_ja_i\text{ is colored blue in } R\},
\]

\[
A_i^r := \{a_j \in V(R) \mid a_ja_i\text{ is colored red in } R\}.
\]

Let \(B^i := \bigcup_{a_j \in A_i^b} A_j\) and \(R^i := \bigcup_{a_j \in A_i^r} A_j\). Then \(|B^i| + |R^i| = 2n - 2 + \min\{i_b, i_r\} = n - 1 + i_b + i_r\). Let

\[
E_B := \{a_ib_i \mid i \in [t]\text{ and } |R^i| < |B^i|\},
\]

\[
E_R := \{a_ib_i \mid i \in [t]\text{ and } |B^i| < |R^i|\},
\]

\[
E_Q := \{a_ib_i \mid i \in [t]\text{ and } |B^i| = |R^i|\}.
\]

Let \(c^*\) be obtained from \(c\) by recoloring all the edges in \(E_B\) blue, all the edges in \(E_R\) red, and all the edges in \(E_Q\) either red or blue. Then all the edges of \(G\) are colored red or blue under \(c^*\). Note that \(|G| = n + 1 + i_b + i_r = R(G_{i_b}, G_{i_r})\). By
Theorem 1.12 and Theorem 1.13, we see that $G$ must contain a blue $G_{i_b}$ or a red $G_{i_r}$ under $c^*$. By symmetry, we may assume that $G$ has a blue $H := G_{i_b}$ under $c^*$. Then $H$ contains no edges of $E_R$ but must contain at least one edge of $E_B \cup E_Q$, else we obtain a blue $H$ in $G$ under $c$. We choose $H$ so that $|E(H) \cap (E_B \cup E_Q)|$ is minimal. We may further assume that $a_1b_1 \in E(H) \cap (E_B \cup E_Q)$, so that $|B^1| \geq |R^1|$. Since $|B^1| + |R^1| = 2n - 2 + \min \{i_b, i_r\} \geq 2n-2+1$, we see that $|B^1| \geq n \geq 5$ and $|R^1| \leq n-1 + \left\lfloor \frac{\min \{i_b, i_r\}}{2} \right\rfloor \leq 7$. So $i_b \geq 2$. By Claim 5, $|R^1| \leq 4$ when $n = 5$. Let $W := V(G) \setminus V(H)$.

We next claim that $i_b = n - 1$. Suppose $i_b \leq n - 2$. Then $H = P_{2i_b + 3}$, $i_r = n - 1$, $|G| = 2n + i_b$ and $|W| = 2n - 3 - i_b \geq n - 1$. Let $x_1, x_2, \ldots, x_{2i_b + 3}$ be the vertices of $H$ in order. We may assume that $x_1x_{i+1} = a_1b_1$ for some $i \in [2i_b + 2]$. If a vertex $w \in W$ is red-complete to $\{a_1, b_1\}$, then we obtain a blue $H' := G_{i_b}$ under $c^*$ with vertices $x_1, x_{i+1}, x_{2i_b+2}$ and so $W \cup \{x_1, x_{2i_b+3}\}$ is red-complete to $\{a_1, b_1\}$. If $n = 5$, then $4 \geq |R^1| \geq |W \cup \{x_1, x_{2i_b+3}\}| \geq 6$, a contradiction. Thus $n = 6$ and $7 \geq |R^1| \geq |W \cup \{x_1, x_{2i_b+3}\}| \geq 7$. It follows that $R^1 \cap V(H) = \{x_1, x_{2i_b+3}\}$ and thus either $x_{i+2}, x_{i-1}$ or $x_{i+3}$ is blue-complete to $\{a_1, b_1\}$. In either case, we obtain a blue $H' := G_{i_b}$ under $c^*$ such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, a contradiction. This proves that $i_b = n - 1$. Then $x_1x_3$ is colored blue under $c$ because $A_1 = \{a_1, b_1\}$. Similarly, for all $j \in \{3, \ldots, 2i_b + 2\}$, $\{x_j, x_{j+1}\}$ is not blue-complete to $\{a_1, b_1\}$, else we obtain a blue $H' := G_{i_b}$ with vertices $x_1, x_j, x_{j+1}, x_{2i_b+3}$ in order under $c^*$ such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$. It follows that $x_4 \in R^1$ and so $|R^1| \geq 2n-3$, so $4 \geq |R^1| \geq 7$ (when $n = 5$) or $7 \geq |R^1| \geq 9$ (when $n = 6$), a contradiction. This proves that $i_b = n - 1$.

Since $i_b = n - 1$, we see that $H = C_{2n}$. Then $|G| = 2n + i_r$ and so $|W| = i_r$. Let $a_1, x_1, \ldots, x_{2n-2}, b_1$ be the vertices of $H$ in order and let $W := \{w_1, \ldots, w_{i_r}\}$. Then $x_1b_1$ and $a_1x_{2n-2}$ are colored blue under $c$ because $A_1 = \{a_1, b_1\}$. Suppose $\{x_j, x_{j+1}\}$ is blue-complete to $\{a_1, b_1\}$ for some $j \in [2n-3]$. We then obtain a blue $H' := C_{2n}$ with vertices $a_1, x_1, \ldots, x_{2n-3}, x_{2n-2}, x_{j+1}$ in order under $c^*$ such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, contrary to the choice of $H$. Thus, for all $j \in [2n-3]$, $\{x_j, x_{j+1}\}$ is not blue-complete to $\{a_1, b_1\}$. Since $\{x_1, x_{2n-2}\}$ is blue-complete to $\{a_1, b_1\}$ under $c$, we see that $x_2, x_{2n-3}$ is blue-complete to $\{a_1, b_1\}$ under $c$, and so $4 \geq |R^1| \geq 4$ (when $n = 5$) and $7 \geq 5 + \left\lfloor \frac{\min \{i_b, i_r\}}{2} \right\rfloor \geq |R^1| \geq |R^1 \cap V(H)| \geq 5$ (when $n = 6$). Thus, when $n = 5$, the distinct cases are $R^1 = \{x_2, x_4, x_5, x_7\}$ or $R^1 = \{x_2, x_4, x_6, x_7\}$, as depicted in Figure 1(a) and Figure 1(b); when $n = 6$, we
have $R^1 \cap V(H) = \{x_2, x_3\} \cup \{x_j \mid j \in J\}$, where $J = \{4, 6, 8\}, \{4, 6, 7\}, \{3, 4, 6, 7\}, \{3, 5, 6, 7\}, \{4, 5, 6, 7\}, \{4, 6, 7, 8\}, \{3, 3, 5, 7, 8\}, \{3, 4, 5, 6, 7\}, \{3, 4, 5, 6, 8\}, \{3, 4, 5, 7, 8\}$.

Figure 1: Two cases of $R^1$ when $i_b = 4$ and $n = 5$.

Since $|R^1| \geq n - 1$ and $R^1$ is red-complete to $\{a_1, b_1\}$ under $c$, we see that $i_r \geq 2$. Let $W' := W \setminus R^1$. Then $W' \subseteq B^1$. Since $|B^1| \geq |R^1|$, it follows that $|W'| \geq \left\lceil \frac{n}{2} \right\rceil \geq 1$. We may assume $W' = \{w_1, \ldots, w_{|W'|}\}$. We claim that $E(H) \cap (E_B \cup E_Q) = \{a_1b_1\}$. Suppose, say $a_2b_2 \in E(H) \cap (E_B \cup E_Q)$. Since $\{x_1, x_2\} \neq A_i$ and $\{x_{2n-3}, x_{2n-2}\} \neq A_i$ for all $i \in \{t\}$, we may assume that $a_2 = x_j$ and $b_2 = x_{j+1}$ for some $j \in \{2, \ldots, 2n-4\}$. Then $x_{j-1}x_{j+1}$ and $x_jx_{j+2}$ are colored blue under $c$. But then we obtain a blue $H' := C_{2n}$ under $c'$ with vertices $a_1, x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{2n-2}, b_1, w_1$ in order such that $|E(H') \cap (E_B \cup E_Q)| < |E(H) \cap (E_B \cup E_Q)|$, contrary to the choice of $H$. Thus $E(H) \cap (E_B \cup E_Q) = \{a_1b_1\}$, as claimed.

(*) Let $w \in W'$. For $j \in \{1, 2n-2\}$, if $\{x_j, w\} \neq A_i$ for all $i \in \{t\}$, then $x_jw$ is colored red. For $j \in \{2, \ldots, 2n-3\}$, if $\{x_j, w\} \neq A_i$ for all $i \in \{t\}$ and $x_{j-2}$ or $x_{j+2} \in B^1$, then $x_jw$ is colored red.

**Proof.** Suppose there is some $j \in [2n-2]$ such that $\{x_j, w\} \neq A_i$ for all $i \in \{t\}$, and $x_{j-2}$ or $x_{j+2} \in B^1$ if $j \in \{2, \ldots, 2n-3\}$, but $x_jw$ is colored blue. Then we obtain a blue $C_{2n}$ under $c$ with vertices $a_1, x_1, \ldots, x_{2n-2}$ (when $j = 1$) or $a_1, x_1, \ldots, x_{2n-2}, w$ (when $j = 2n-2$) in order if $j \in \{1, 2n-2\}$, and with vertices $b_1, x_{2n-2}, x_{2n-3}, \ldots, x_{j+2}, a_1, w, x_j, \ldots, x_1$ in order (when $x_{j+2} \in B^1$) or $a_1, x_1, \ldots, x_{j-2}, b_1, w, x_j, \ldots, x_{2n-2}$ in order (when $x_{j-2} \in B^1$) if $j \in \{2, \ldots, 2n-3\}$, a contradiction.

(**) For $j \in [2n-4]$, $x_jx_{j+2}$ is colored red if $\{x_j, x_{j+2}\} \neq A_i$ for all $i \in \{t\}$.

**Proof.** Suppose $x_jx_{j+2}$ is colored blue for some $j \in [2n-4]$. Then we obtain a blue $C_{2n}$ under $c$ with vertices $a_1, x_1, \ldots, x_j, x_{j+2}, \ldots, x_{2n-2}, b_1, w_1$ in order, a contradiction.

We claim that $n = 6$. Suppose $n = 5$. Then $R^1 = \{x_2, x_4, x_\alpha, x_\beta\}$, where $(\alpha, \beta) \in \{(5, 7), (7, 6)\}$. Thus $W' = W$ and $x_{\alpha+1}, x_{\alpha-2} \in B^1$. Since $\{x_{\alpha-1}, w_j\} \neq A_i$
and \( \{x_o, w_j\} \neq A_i \) for all \( w_j \in W \) and \( i \in [t] \), it follows from (*) that \( \{x_{a-1}, x_a\} \) must be red-complete to \( W \) under \( c \). Then for any \( w_j \in W \), \( \{x_{a-2}, w_j\} \neq A_i \) and \( \{x_{a+1}, w_j\} \neq A_i \) for all \( i \in [t] \) since \( x_{a-1}x_{a-2} \) and \( x_ax_{a+1} \) are colored blue under \( c \). Thus \( \{x_{a-2}, x_{a+1}\} \) is red-complete to \( W \) by (*). So \( \{x_{a-2}, x_{a-1}, x_a, x_{a+1}\} \) is red-complete to \( W \) under \( c \). But then we obtain a red \( P_9 \) under \( c \) (when \( i_r \leq 3 \)) with vertices \( x_2, a_1, x_{a-1}, b_1, x_a, w_1, x_{a-2}, w_2, x_{a+1}, w_3, x_a \) in order, or a red \( C_{10} \) under \( c \) (when \( i_r = 4 \)) with vertices \( a_1, x_2, b_1, x_{a-1}, w_1, x_{a-2}, w_2, x_{a+1}, w_3, x_a \) in order, a contradiction. This proves that \( n = 6 \), as claimed. By (*), we may assume \( x_1 \) is red-complete to \( W \setminus w_1 \) and \( x_{10} \) is red-complete to \( W \setminus w_{10} \) because \( |A_1| = 2 \). Recall that \( 5 \leq |R^i \cap V(H)| \leq 7 \) when \( n = 6 \). We next consider three cases based on the value of \( |R^i \cap V(H)| \).

**Case 1.** \( |R^i \cap V(H)| = 5 \). Then \( R^i \cap V(H) = \{x_2, x_4, x_6, x_a, x_b\} \), where \( (\alpha, \beta) \in \{(9,8), (7,9)\} \). Then \( x_{a+1}, x_{a-2} \in B^i \). Since \( \{x_{a-1}, w_j\} \neq A_i \) and \( \{x_a, w_j\} \neq A_i \) for all \( w_j \in W \) and \( i \in [t] \), \( \{x_{a-1}, x_{a-2}\} \) must be red-complete to \( W \) under \( c \) by (*). Then for any \( w_j \in W \), \( \{x_{a-2}, w_j\} \neq A_i \) and \( \{x_{a-1}, w_j\} \neq A_i \) for all \( i \in [t] \) since \( x_{a-1}x_{a-2} \) are colored blue under \( c \). Thus \( \{x_{a-2}, x_{a+1}\} \) is red-complete to \( W \) by (*). So \( \{x_{a-2}, x_{a-1}, x_a, x_{a+1}\} \) is red-complete to \( W \) under \( c \). We see that \( G \) has a red \( P_7 \) with vertices \( x_{a-1}, w_1, x_a, a_1, b_1, x_4 \) in order, and so \( i_r \geq 3 \) and \( |W'| \geq \left\lceil \frac{7}{2} \right\rceil \geq 2 \). Moreover, \( x_{a-1}x_{a+1} \) and \( x_{a-2}x_a \) are colored red by (**). Then \( G \) has a red \( P_{11} \) with vertices \( x_1, w_2, x_{a-1}, w_{a-1}, w_{a-2}, x_2, a_1, b_1, x_4 \) in order under \( c \). Thus \( i_r = 5 \) and so \( |W'| \geq \left\lceil \frac{7}{2} \right\rceil \geq 3 \). Since \( |A_1| = 2 \) and \( x_{a-6} \in B^1 \), by (*), we may assume \( x_{a-4} \) is red-complete to \( W \setminus w_2 \). But then we obtain a red \( C_{12} \) with vertices \( a_1, x_2, x_{a-2}, w_1, x_{a-4}, w_3, x_1, w_2, x_{a+1}, x_a, x_{a+1}, b_1, x_2 \) in order under \( c \), a contradiction.

**Case 2.** \( |R^i \cap V(H)| = 6 \). We claim that \( i_r \geq 3 \). Suppose \( i_r = 2 \). Then \( |B^1| = |R^i| = 6 \) and \( G[B^1 \cup R^i] \) contains no red \( P_9 \) with at least one end in \( R^i \), else we obtain a red \( P_7 \). By Claim 3, \( B^1 \) is not blue-complete to \( R^i \). Let \( x \in B^1 \) and \( y \in R^i \) such that \( xy \) is colored red. Then \( x \) is blue-complete to \( R^i \setminus y \) and there exists at most one vertex \( w \in B^1 \) such that \( x \) is blue-complete to \( B^1 \setminus \{x, w\} \) because \( G[B^1 \cup R^1] \) contains no red \( P_3 \) with at least one end in \( R^i \). Let \( i^*_r := 1, i^*_r := 0, i^*_j := 0 \) for all colors \( j \) other than red and blue. Let \( N^* := |G_{v_r}^*| + |(\sum_{j=1}^k i^*_j) - i^*_r| = 5 \). Observe that \( |R^i \setminus y| = 5 = N^* \), by minimality of \( N \), \( G[R^i \setminus y] \) contains a blue \( P_5 \). Let \( y_1, y_2, \ldots, y_5 \) be the vertices of the \( P_5 \) in order. Then \( y \) is blue-complete to \( \{y_j, y_{j+1}\} \) for some \( j \in [4] \) and \( x_1 \in B^1 \setminus x \) is not red-complete to \( \{y_1, y_5\} \) because \( G[B^1 \cup R^1] \) contains no red \( P_3 \) with at least one end in \( R^i \) and \( |A_1| = 2 \). So we may assume \( x_1y_1 \) is colored blue. But then we obtain a blue \( C_{12} \) under \( c \) with vertices \( a_1, x_1, y_1, \ldots, y_j, y, y_{j+1}, \ldots, y_5, x, x_2, b_1, x_3 \) in order, where \( x_2, x_3 \in B^1 \setminus \{x, x_1, w\} \), a contradiction. Thus \( i_r \geq 3 \), as claimed. Note that \( |B^1 \cap V(H)| = 4 \), so \( |W'| \geq 3 \). We may further assume that \( \{x_1, w_2\} \neq A_i \) and \( \{x_1, w_3\} \neq A_i \) for all \( i \in [t] \); and \( \{x_{10}, w_1\} \neq A_i \) and \( \{x_{10}, w_2\} \neq A_i \) for all \( i \in [t] \). By (*), \( x_1 \) is red-complete to \( \{w_2, w_3\} \) under \( c \); and \( x_{10} \) is red-complete to \( \{w_1, w_2\} \) under \( c \). Let \( (\alpha, \beta, \gamma) \in \{(5,2,4), (4,7,5)\} \). Suppose \( R^i \cap V(H) = \{x_2, x_3, x_5, x_6, x_7, x_9\} \). Since \( \{x_{\beta}, w_{\gamma}\} \neq A_i \), \( \{x_3, w_{\gamma}\} \neq A_i \) and \( \{x_6, w_{\gamma}\} \neq A_i \) for all \( w_j \in W \) and \( i \in [t] \), by (*), \( \{x_{\beta}, x_3, x_6\} \) must be red-complete to
$W'$ under $c$. By (**), $x_1$ is red-complete to $\{x_{r-2}, x_{r+2}\}$. But then we obtain a red $C_{12}$ under $c$ with vertices $a_1, x_2, x_4, x_6, w_1, x_{10}, w_2, x_1, x_3, x_5, b_1, x_5$ (when $\alpha = 5$) or $a_1, x_3, x_5, x_7, w_1, x_{10}, w_2, x_1, x_3, x_6, b_1, x_4$ (when $\alpha = 4$) in order, a contradiction. Let $(\alpha, \beta, \gamma, \delta) \in \{(3, 5, 6), (3, 5, 7, 8), (4, 6, 8, 2)\}$. Suppose $R^1 \cap V(H) = V(H) \setminus \{a_1, b_1, x_1, x_{10}, x_\alpha, x_\beta, x_\delta, x_\gamma, x_\delta\}$. Since $\{x_1, w_2\} \neq A_i$ and $\{x_3, w_2\} \neq A_i$ for all $w_j \in W'$ and $i \in \{t\}$, $\{x_1, x_2\}, \{x_\gamma, x_\delta\}$ must be red-complete to $W'$ under $c$ by (**). Moreover, $x_\gamma x_\delta$ and $x_3 x_4$ are colored red by (**). Since $|A_1|=2$, at least one of $x_{10}, x_\alpha, x_\beta$ is red-complete to $\{w_1, w_2, w_3\}$ by (**). So we may assume $x_\alpha$ is red-complete to $W' \setminus w_2$ and $x_\beta$ is red-complete to $\{w_1, w_2, w_3\}$. But then we obtain a red $C_{12}$ with vertices $a_1, x_2, x_4, x_6, w_1, x_{10}, w_2, x_1, x_3, x_5, b_1, x_5$ in order if $(\alpha, \beta, \gamma, \delta) \in \{(3, 5, 6), (4, 6, 8, 2)\}$ and $a_1, x_7, x_5, w_1, x_3, x_1, w_2, x_{10}, x_8, b_1, x_6$ in order if $(\alpha, \beta, \gamma, \delta) = (3, 5, 7, 8)$, a contradiction. Finally if $R^1 \cap V(H) = \{x_2, x_3, x_5, x_6, x_8, x_9\}$. By (**), $R^1 \cap V(H)$ is red-complete to $W'$. Then $G$ has a red $P_{11}$ with vertices $x_2, a_1, x_3, b_1, x_5, w_1, x_6, w_2, x_8, w_3, x_9$ in order. Thus $i_r = 5$ and so $|W'| \geq 4$. But then we obtain a red $C_{12}$ with vertices $a_1, x_2, w_1, x_3, x_5, w_2, x_6, w_1, x_8, b_1, x_9$ in order, a contradiction.

**Case 3.** $R^1 = |R^1 \cap V(H)| = 7$, then $i_r \geq 4$ and $|W'| = |W| = i_r$. Let $(\alpha, \beta, \gamma, \delta) \in \{(6, 5), (7, 4)\}$. Suppose $R^1 = \{x_2, x_3, x_4, x_5, w_2, x_8, x_9\}$. Since $\{x_3, w_2\} \neq A_i$, $\{x_3, w_2\} \neq A_i$ and $\{x_8, w_2\} \neq A_i$ for all $i \in \{t\}$ and any $w_j \in W'$, $\{x_3, x_5, x_8\}$ must be red-complete to $W'$ under $c$ by (**). But then we obtain a red $C_{12}$ with vertices $a_1, x_2, w_1, x_{10}, w_2, x_1, x_3, x_5, w_2, x_8, x_1, x_2$ in order, a contradiction. Finally if $R^1 = \{x_2, x_3, x_4, x_5, x_6, x_7, x_9\}$. Since $\{x_3, w_3\} \neq A_i$ and $\{x_6, w_3\} \neq A_i$ for all $i \in \{t\}$ and any $w_j \in W'$, $\{x_3, x_6\}$ must be red-complete to $W'$ under $c$ by (**). We may assume $x_8$ is red-complete to $W' \setminus w_2$ by (**). But then we obtain a red $C_{12}$ with vertices $a_1, x_3, w_1, x_{10}, w_2, x_1, x_3, x_5, w_4, x_6, b_1, x_2$ in order, a contradiction. This proves that $|A_1| \geq 3$.

**Claim 7.** For any $A_i$ with $3 \leq |A_i| \leq 4$, $G[A_i]$ has a monochromatic copy of $P_3$ in some color $m \in \{k\}$ other than red and blue.

**Proof.** Suppose there exists a part $A_i$ with $3 \leq |A_i| \leq 4$ but $G[A_i]$ has no monochromatic copy of $P_3$ in any color $m \in \{k\}$ other than red and blue. We may assume $i = 1$. Since $GR_k(P_3) = 3$, we see that $G[A_1]$ must contain a red or blue $P_3$, say blue. We may assume $a_1, b_1, c_1$ are the vertices of the blue $P_3$ in order. Then $|A_1| = 4$, else $\{b_1, \{a_1, c_1\}, A_2, \ldots, A_p\}$ is a Gallai partition of $G$ with $p + 1$ parts. Let $z_1 \in A_1 \setminus \{a_1, b_1, c_1\}$. Then $z_1$ is not blue-complete to $\{a_1, c_1\}$, else $\{a_1, c_1\}, \{a_1, z_1\}, A_2, \ldots, A_p$ is a Gallai partition of $G$ with $p + 1$ parts. Moreover, $b_1 z_1$ is not colored blue, else $\{b_1, \{a_1, c_1, z_1\}, A_2, \ldots, A_p\}$ is a Gallai partition of $G$ with $p + 1$ parts. If $b_1 z_1$ is colored red, then $a_1 z_1$ and $c_1 z_1$ are colored either red or blue because $G$ has no rainbow triangle. Similarly, $z_1$ is not red-complete to $\{a_1, c_1\}$, else $\{z_1, \{a_1, b_1, c_1\}, A_2, \ldots, A_p\}$ is a Gallai partition of $G$ with $p + 1$ parts. Thus, by symmetry, we may assume $a_1 z_1$ is colored blue and $c_1 z_1$ is colored red, and so $a_1 c_1$ is colored blue or red because $G$ has no rainbow triangle. But then $\{a_1, \{b_1, \{c_1, z_1\}, A_2, \ldots, A_p\}$ is a Gallai partition of $G$ with $p + 3$ parts, a contradiction. Thus $b_1 z_1$ is colored neither red nor blue. But then $a_1 z_1$ and $c_1 z_1$ must be
colored blue because $G[A_1]$ has neither rainbow triangle nor monochromatic $P_3$ in any color $m \in [k]$ other than red and blue, a contradiction.

For the remainder of the proof of Theorem 1.9, we assume that $|B| \geq |R|$.

By Claim 5, $|R| \leq n - 1$. Let $\{a_i, b_i, c_i\} \subseteq A_i$ if $|A_i| \geq 3$ for any $i \in [p]$. Let $B := \{x_1, \ldots, x_{|B|}\}$ and $R := \{y_1, \ldots, y_{|R|}\}$. We next show that

**Claim 8.** $i_r \geq |R|$.

**Proof.** Suppose $i_r \leq |R| - 1 \leq n - 2$. Then $i_b = n - 1$, $i_r \geq 3$, $|A_1| \leq 4$, else we obtain a red $G_i$, because $R$ is not blue-complete to $B$ and $|A_1| \geq 3$. By Claim 7, $G[A_1]$ has a monochromatic, say green, copy of $P_3$. By Claim 4, $i_g = 1$. We have $|G| \geq n + 1 + i_b + i_r + i_g \geq 2n + 4$. This implies that there exist two independent edges between $B$ and $R$, say $x_1y_1, x_2y_2$, that are colored red, else we obtain a blue $C_2$. Then $G[A_1 \cup R \cup \{x_1, x_2\}]$ has a red $P_3$, it follows that $n = 6$, $i_r = 4$ and $|R| = 5$. Then $|A_1 \cup B| = |G| - |R| \geq 7 + i_b + i_r + i_g - |R| = 12$, and so $G[B]$ has no blue $G_{i_b - |A_1|}$, else we obtain a blue $C_{12}$. Let $i_r^* := i_b - |A_1| \leq 2$, $i_r^* := i_r - |R| + 2 = 1$, $i_g^* := i_g \leq 2$ for all color $j \in [k]$ other than red and blue. Let $i_g^* := max \{i_r^* \mid j \in [k]\}$. Then $i_r^* \leq i_1$. Let $N^* := |G_{i_r^*}| + (\sum_{j=1}^{k} i_j^*) - i_g^*$. Observe that $|B| \geq N^*$. By minimality of $N$, $G[B]$ has a red $G_{i_r^*} = P_5$ with vertices, say $x_1, \ldots, x_5$, in order. Because there is a red $P_5$ with both ends in $R$ by using edges between $A_1$ and $R$, we see that $R$ is blue-complete to $\{x_1, x_2, x_3, x_4\}$, else $G[A_1 \cup R \cup \{x_1, \ldots, x_5\}]$ has a red $P_{11}$. But then we obtain a blue $C_{12}$ under condition with vertices $a_1, x_1, x_2, y_2, y_3, x_4, y_5, b_1, x_3, c_1, x_6$ in order, a contradiction.

**Claim 9.** $i_b > |A_1|$ and so $|A_1| \leq n - 2$.

**Proof.** Suppose $i_b \leq |A_1| - 1$, then $i_b \leq n - 2$ by Claim 2 and so $i_r = n - 1$. Thus $|B| \geq 2 + i_b$ because $|B| + |R| = |G| - |A_1| \geq n + 1 + i_b + (i_r - |A_1|) \geq 3 + 2i_b$. But then $G$ has a blue $G_{i_b}$ using edges between $A_1$ and $B$, a contradiction. Thus $i_b = |A_1|$. By Claims 5 and 8, $|R| \leq n - 1$ and $i_r \geq |R|$. Observe that $|B| \geq 1 + n + i_r - |R| \geq 1 + n$. Then $G[B \cup R]$ has no blue $P_5$ with both ends in $B$, else we obtain a blue $G_{i_b}$ in $G$. Let $i_r^* := i_b - |A_1| = 0$, $i_r^* := i_r - |R|$, and $i_g^* := i_g \leq n - 4$ for all colors $j \in [k]$ other than blue and red. Let $i_r^* := max \{i_r^* \mid j \in [k]\}$. Then $i_r^* \leq i_1$. Let $N^* := |G_{i_r^*}| + (\sum_{j=1}^{k} i_j^*) - i_g^*$. Then $3 < N^* < N$. Suppose first that $|R| \geq 2$. Since $B$ is not red-complete to $R$, we may assume that $y_1x$ is colored blue for some $x \in B$. Note that $i_r^* \leq n - 3$ and $|B \setminus x| = N - |A_1| - |R| - 1 \geq N^*$. By minimality of $N$, $G[B \setminus x]$ must have a red $G_{i_r^*} = P_{2i_r^* + 3}$ with vertices, say $x_1, \ldots, x_q$, in order, where $q = 2i_r^* + 3$. Since $G[B \cup R]$ contains no blue $P_3$ with both ends in $B$ and $xy_1$ is colored blue, we see that $y_1$ must be red-complete to $B \setminus x$ and $y_2$ is not blue-complete to $\{x_1, x_2\}$. We may assume that $x_qy_2$ is colored red in $G$. Then $n = 6$, $i_r = |R| = 5$ and $i_b = |A_1| = 3$, else we obtain a red $G_{i_r}$ using vertices in $V(P_{2i_r^* + 3}) \cup R \cup A_1$. Let $x' \in B \setminus \{x, x_1, x_2, x_3\}$. Then $\{x, x'\} \not\subseteq A_i$ and $\{x, x_1\} \not\subseteq A_i$ for all $i \in [p]$ because $y_1x$ is colored blue and $y_1x_1$ are colored red, and so $xx'$ and $xx_1$ are colored red, else $G[A_1 \cup B \cup \{y_1\}]$ has a blue $P_9$. But then we obtain a red
Claim 10. If $|A_1| = 3$, then $|A_2| = 3$, $|A_3| \leq 2$, and $i_j = 0$ for all colors $j \in [k]$ other than red, blue and green.

Proof. We may assume that the first three colors in $[k]$ are red, blue, and green. Assume $|A_1| = 3$. To prove $|A_2| = 3$, we show that $G[B \cup R]$ has a green $P_3$. Suppose $G[B \cup R]$ has no green $P_3$. By Claim 9, $i_b \geq |A_1| + 1 = 4$. Let $i^*_b := 0$ and $i^*_j := i_j$ for all $j \in [k]$ other than green. Let $N^* := (\sum_{j=1}^{k} i^*_j) - i^*_b$. Then $N^* = N - 1$ and $|G \setminus a_1| = N - 1 = N^*$. But then $G \setminus a_1$ has no monochromatic copy of $G_{i^*_j}$ in color $j$ for all $j \in [k]$, contrary to the minimality of $N$. Thus $G[B \cup R]$ has a green $P_3$ and so $|A_2| = 3$. For the rest of the proof of Claim 10, we do not use the condition $|B| \geq |R|$ because we make no use of Claim 8 and Claim 9.

Suppose $|A_3| = 3$. For all $i \in [3]$, let 
\[A_i := \{a_j \in V(R) \mid a_j a_i \text{ is colored blue in } R\},\]
\[A'_i := \{a_j \in V(R) \mid a_j a_i \text{ is colored red in } R\}.\]

Let $B^i := \bigcup_{a_j \in A_i} A_j$ and $R^i := \bigcup_{a_j \in A'_i} A_j$. Since each of $A_1, A_2, A_3$ can be chosen as the largest part in the Gallai-partition $A_1, A_2, \ldots, A_p$ of $G$, by Claim 5, either $|B^i| \leq 5$ or $|R^i| \leq 5$ for all $i \in [3]$. Without loss of generality, we may assume that $A_2$ is blue-complete to $A_1 \cup A_3$. Let $X := V(G) \setminus (A_1 \cup A_2 \cup A_3) = \{v_1, \ldots, v_\ell\}$. Then $|X| \geq 1 + n + i_b + i_r + i_g - 9 = 2n - 8 + \min\{i_b, i_r, i_g\}$. Suppose $|X \cap B^1| \geq 2$. We may assume $v_1, v_2 \in X \cap B^1$. Then $G$ has a blue $C_{10}$ with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, b_2, b_3, c_2$ in order and a blue $P_{11}$ with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, b_3, b_3, c_2, c_3$ in order, and so $n = 6$ and $i_g = 5$. Moreover, $X \setminus \{v_1, v_2\} \subseteq R^3$, else, say $v_3$ is blue-complete to $A_3$, then we obtain a blue $C_{12}$ under $c$ with vertices $a_1, v_1, b_1, v_2, c_1, a_2, a_3, b_3, b_3, c_3, c_2$ in order. Thus $|R^3| \geq |X \setminus \{v_1, v_2\}| \geq 2 + i_r$, and so $i_r \geq 3$, else $G$ has a red $G_{i_r}$ using the edges between $A_3$ and $R^3$. Then there exist at least two vertices in $X \setminus \{v_1, v_2\}$, say $v_3, v_4$, such that $\{v_3, v_4\}$ is blue-complete to $A_1$, else $G[A_1 \cup A_3 \cup (X \setminus \{v_1, v_2\})]$ contains a red $G_{i_r}$. Thus $|B^1| \geq |A_2 \cup \{v_1, \ldots, v_4\}| = 7$ and so
$|R^1| \leq 5$. Moreover, $\{v_1, v_2\} \subset R^3$, else, say $v_1$ is blue-complete to $A_3$, we then obtain a blue $C_{12}$ under $c$ with vertices $a_1, v_3, b_1, v_4, c_1, a_2, a_3, v_1, b_3, b_2, c_3, c_2$ in order. Then $X \subseteq R^3$ and $|R^3| \geq |X| \geq 4 + i_r \geq 7$, and so $|B^3| \leq 5$ and $A_4$ is red-complete to $A_3$. Furthermore, $G[B^1 \setminus A_2]$ has no blue $P_3$, else, say $v_1, v_2, v_3$ is such a blue $P_3$ in order, we obtain a blue $C_{12}$ with vertices $a_1, v_1, v_2, v_3, b_1, v_4, c_1, a_2, a_3, b_2, b_3, c_2$ in order. Therefore for any $U \subseteq B^1 \setminus A_2$ with $|U| \geq 4$, $G[U]$ contains a red $P_3$ because $|A_1| = 3$ and $GR_k(P_3) = 3$. Since $|R^1| \leq 5$ and $A_3 \subseteq R^1$, we may assume $v_1, \ldots, v_{|X|-2} \in B^1 \setminus A_2$. Then $G[\{v_1, \ldots, v_4\}]$ must contain a red $P_3$ with vertices, say $v_1, v_2, v_3$, in order. We claim that $X \subseteq B^1$. Suppose $v_{|X|} \in R^1$. Then $v_{|X|}$ is red-complete to $A_1$ and so $G$ has a red $P_{11}$ with vertices $c_1, v_{|X|}, a_1, a_3, b_1, b_3, v_1, v_2, c_3, v_4$ in order, it follows that $i_r = 5$. Thus $|X| \geq 9$, and $G[\{v_4, \ldots, v_7\}]$ has a red $P_3$ with vertices, say $v_4, v_5, v_6$, in order. But then we obtain a red $C_{12}$ with vertices $a_1, v_1, v_2, v_3, b_3, v_4, v_5, v_6, c_3$ in order, a contradiction. Thus $X \subseteq B^1$ as claimed. Since $|X| \geq 7, G[\{v_4, \ldots, v_7\}]$ contains a red $P_3$ with vertices, say $v_4, v_5, v_6$, in order. Then $G$ has a red $P_{11}$ with vertices $a_1, a_3, b_1, b_3, v_1, v_2, v_3, c_3, v_4, v_6, v_5, v_5, v_6$ in order, and so $i_r = 5$, $|X| \geq 9$. Suppose $G[\{v_4, \ldots, v_9\}]$ has no red $P_3$. Then $G[\{v_4, \ldots, v_9\}]$ contains at most one part of the Gallai-partition with order three, say $A_4$, and we may assume $G[A_4]$ has a monochromatic $P_3$ in some color $m$ other than red and blue if $|A_4| = 3$ by Claim 7. Let $i_r^* := 1, i_m^* := 1, i_j^* := 0$ for all color $j \in [k]\{m\}$ other than red. Let $N^* := |G[i_r^*]| + \left| \left( \sum_{j=1}^k i_j^* \right) - i_r^* \right| = 6 < N$. Then $G[\{v_1, \ldots, v_9\}]$ has no monochromatic copy of $G[i_j^*]$ in any color $j \in [k]$, which contradicts the minimality of $N$. Thus $G[\{v_4, \ldots, v_9\}]$ has a red $P_3$ with vertices, say $v_4, \ldots, v_8$, in order. But then we obtain a red $C_{12}$ with vertices $a_3, v_1, v_2, v_3, b_3, v_4, v_5, v_6, c_3, v_9$ in order, a contradiction. Therefore, $|X \cap B^1| \leq 1$. By symmetry, $|X \cap B^3| \leq 1$. Let $w \in X \cap B^1$ when $X \cap B^1 \neq \emptyset$ and $w' \in X \cap B^3$ when $X \cap B^3 \neq \emptyset$. Then $A_1 \cup A_3$ is red-complete to $X \setminus \{w, w'\}$. It follows that $n = 5$ and $|X \cap B^1| = |X \cap B^3| = 1$, else $G[A_1 \cup A_3 \cup (X \setminus \{w, w'\})]$ has a red $G_4$, because $|X| \geq 2n - 8 + \min\{i_r, i_m\}$, a contradiction. But then we obtain a blue $C_{10}$ with vertices $a_2, a_1, w, b_1, b_2, a_3, w', b_3, c_2, c_3$ in order, a contradiction. This proves that $|A_3| \leq 2$ and so $G[A_4]$ has no monochromatic copy of $P_3$ for all $i \in [p]$ with $i \geq 3$. Since $G[R \cup B]$ has a green $P_3$, it follows that $G[A_2]$ has a green $P_3$, so $i_j = 0$ for all color $j \in [k]$ other than red, blue and green by Claim 4.

Claim 11. If $i_b = |A_1| + 1$, then $|R| \leq 2$.

Proof. Suppose $i_b = |A_1| + 1$ but $|R| \geq 3$. By Claim 8, $i_r \geq |R|$, it follows that $|B| \geq 1 + n + i_b + i_r + i_g - |A_1| - |R| \geq 3 + n$. Thus $G[B \cup R]$ has no blue $P_3$ with both ends in $B$, else we obtain a blue $G_{i_b}$. Let $i_b^* := i_b - |A_1| = 1, i_r^* := i_r - |R| + 1$ (when $n = 5$) or $i_r^* := \max\{i_r - |R| + 1, 2\}$ (when $n = 6$), $i_j^* := i_j$ for all $j \in [k]$ other than red and blue. Let $i_r^* := \max\{i_j^* \mid j \in [k]\}$ and $N^* := |G[i_r^*]| + \left[ \left( \sum_{j=1}^k i_j^* \right) - i_r^* \right]$. Then $3 < N^* < N$. Observe that $|B| \geq N^*$. By minimality of $N$, $G[B]$ has a red $G_4 = P_{2k+3}$ with vertices, say $x_1, \ldots, x_q$, in order, where $q = 2i_r^* + 3$. If $R$ is blue-complete to $\{x_1, x_q\}$, then $R$ is red-complete to $B \setminus \{x_1, x_q\}$ because $G[B \cup R]$ has no blue $P_3$ with both ends in $B$. But then $G[A_1 \cup R \cup \{x_2, \ldots, x_{q-1}\}]$ has a red $G_n$, a contradiction. Thus $R$ is not blue-complete to $\{x_1, x_q\}$, and so we
may assume $y_i x_1$ is colored red. Then $i_r = n - 1$ and $R \{ y_i \}$ is blue-complete to
\{ $x_{q-2}, x_q$ \}, else $G[A_1 \cup R \cup \{ x_1, \ldots, x_p \}]$ has a red $G_{i_r}$. So
$R \{ y_i \}$ is red-complete to $B \{ x_{q-2}, x_q \}$ because $G[B \cup R]$ has no blue $P_5$ with both ends in $B$. But then
$G[A_1 \cup R \cup \{ x_2, \ldots, x_{q-1} \}]$ has a red $G_{i_r} = C_{2n}$, a contradiction.

Claim 12. $i_b = n - 1$.

Proof. Suppose $i_b \leq n - 2$. By Claim 6 and Claim 9, $|A_1| \geq 3$ and $i_b \geq |A_1|$, it
follows that $n = 6$, $i_r = n - 1 = 5$, $i_b = 4$, and $|A_1| = 3$. By Claim 10, $|A_2| = 3$, $|A_3| \leq 2$, $i_j = 0$ for all colors $j \in [k]\{3\}$. By Claim 11, $|R| \leq 2$ and so $A_2 \subset B$. It
follows that $|B| = 7 + i_b + i_r + i_q - |A_1 \cup R| = 14 - |R| \geq 12$. Then $G[B \cup R]$ has no blue $P_5$
with both ends in $B$, else $G$ has a blue $P_{11}$ because $|A_1| = 3$. Thus there exists a
set $W$ such that $(B \cup R) \backslash (A_2 \cup W)$ is red-complete to $A_2$, where $W \subset (B \cup R) \backslash A_2$
with $|W| \leq 1$. Let $i_b' := i_b - |A_1| = 1$, $i_r' := 2$, $i_j' := 0$ for all $j \in [k]$ other than
red and blue. Let $N^* := |G| + \sum_{j=1}^{k} i_j' = 8$. Then $N^* < N$. Observe that $|W| \leq 3$ such that $W$ is blue-complete to $A_2$ and $(B \cup R) \backslash (A_2 \cup W)$ is red-complete to $A_2$. Since $|B\backslash (A_2 \cup W)| \geq 4$, we see that there is a red $P_7$ using edges between $A_2$
and $B\backslash (A_2 \cup W)$, so $i_r \geq 3$ and $i_r - |B| \geq 1$. Let $i_b' := 2$ (when $|B \cap W| \leq 1$) or
$i_b'' := 0$ (when $|B \cap W| \geq 2$), $i_r' := \min\{ i_r - |R| - 1, 2 \}$, $i_j' := 0$ for all colors $j \in [k]$ other
than red and blue. Let $i_r^* := \max\{ i_r' \mid j \in [k] \}$ and $N^* := |G| + \sum_{j=1}^{k} (i_r^* - i_j') = 3 + \max\{ i_b', i_r' \} + i_b + i_r$. Observe that $|B\backslash (A_2 \cup W)| = 7 + i_r - |R \cup W| \geq N^*$. By
minimality of $N$, $G[B\backslash (A_2 \cup W)]$ has a red $G_{i_r} = P_{2n+3}$ because $G[B]$
has neither blue $P_5$ nor blue $P_5$ and $|A_1| \leq 2$. But then $G[(B \cup R) \backslash W]$ has a red $G_{i_r}$
because $|(B \cup R) \backslash W| \geq 7 + i_r \geq |G|$, so $A_2$ is red-complete to $(B \cup R) \backslash (A_2 \cup W)$,
a contradiction. Therefore, $3 \leq |R| \leq 5$ and so $i_r \geq 3$.

We claim that $i_r = 5$. Suppose $3 \leq i_r \leq 4$. Let $i_b'' := 2$, $i_r' := 2$, $i_j' := i_j$ for
all colors $j \in [k]$ other than red and blue, and $N^* := |G| + \sum_{j=1}^{k} (i_r' - i_j') = 10$.
Observe that $|B| \geq 10 = N^*$. Since $G[B]$ has no blue $P_7$, by minimality of $N$, $G[B]$ has a red $P_7$ with vertices, say $x_1, \ldots, x_7$, in order. Then $R$ is blue-complete to
\{ $x_1, \ldots, x_7$ \}, else $G[A_1 \cup R \cup \{ x_1, \ldots, x_7 \}]$ has a red $G_{i_r} = P_{2n+3}$. But then
$G[B \cup R]$ has a blue $P_7$ with vertices $x_1, y_1, x_2, y_2, x_3, y_3, x_5$ in order, a contradiction.
Thus $i_r = 5$ and so $|G| = 18$, $|B| = 15 - |R|$.
We next consider the case $|R| = 3$. Suppose first $A_2 = R$. Since $R$ is not red-complete to $B$, we may assume that $A_2$ is blue-complete to $x_1$. Let $i^*_b := 2$, $i^*_r := 3$, $i^* := 0$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i^*}| + \left[ (\sum_{j=1}^k i^*_j) - i^*_r \right] = 11$. Observe that $|B \setminus x_1| = 11 = N^*$. By minimality of $N$, $G[B \setminus x_1]$ has a red $P_3$ with vertices, say $x_2, \ldots, x_{10}$, in order. We claim that $A_2$ is blue-complete to $\{x_2, x_{10}\}$, else, say $x_2$ is red-complete to $A_2$. Then $A_2$ is blue-complete to $\{x_8, x_{10}\}$, else $G[A_1 \cup A_2 \cup \{x_2, \ldots, x_{10}\}]$ has a red $C_{12}$. Thus $A_2$ is red-complete to $B \setminus \{x_1, x_8, x_{10}\}$ because $G[B \cup R]$ has no blue $P_7$ with both ends in $B$. But then we obtain a red $C_{12}$ with vertices $a_1, a_2, a_3, \ldots, a_9, b_2, b_1, c_2$ in order, a contradiction. Thus, $A_2$ is blue-complete to $\{x_1, x_2, x_8, x_{10}\}$, and so $A_2$ is red-complete to $B \setminus \{x_1, x_2, x_8, x_{10}\}$ because $G[B \cup R]$ has no blue $P_7$ with both ends in $B$. But then we obtain a red $C_{12}$ with vertices $a_1, a_2, a_3, \ldots, x_9, b_2, b_1, c_2$ in order, a contradiction. This proves that $A_2 \subset B$. Then there exists a set $W \subset (B \cup R) \setminus A_2$ with $|W \cap B| \leq 3$ such that $W$ is blue-complete to $A_2$ and $(B \cup R) \setminus (A_2 \cup W)$ is red-complete to $A_2$. Then $|W| \leq 3$ and $|W \cap B| \leq 3$ or $|W| = 4$ and $|W \cap B| = 1$ because $G[B \cup R]$ has no blue $P_7$ with both ends in $B$. Let $i^*_b := 2 - |W|$, $i^*_r := 2$ when $|W| \in \{0, 1\}$, $i^*_b := 0$, $i^*_r := 2$ when $|W| \geq 2$ and $|W \cap B| \leq 2$, $i^*_b := 0$, $i^*_r := 1$ when $|W| = |W \cap B| = 3$, $i^*_b := 0$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i^*}| + \left[ (\sum_{j=1}^k i^*_j) - i^*_r \right] = 3 + 2i^*_r + i^*_b$. Observe that $|B \setminus (A_2 \cup W)| \geq N^*$. By minimality of $N$, $G[B \setminus (A_2 \cup W)]$ has a red $G_{i^*} = P_{2i^*_r+3}$ because $G[B \cup R]$ has neither blue $P_2$ nor blue $P_3 \cup P_3$ with all ends in $B$ and $|A_3| \leq 2$. If $|W| \leq 3$ and $|W \cap B| \leq 2$, then $G[(B \cup R) \setminus W]$ has a red $C_{12}$ because $|(B \cup R) \setminus W| \geq 12$ and $A_2$ is red-complete to $(B \cup R) \setminus (A_2 \cup W)$. Thus $|W| = |W \cap B| = 3$ or $|W| = 4$ and $|W \cap B| = 1$. For the former case, $G[B \setminus (A_2 \cup W)]$ has a red $P_5$ with vertices, say $x_1, \ldots, x_5$, in order. Let $W := \{w_1, w_2, w_3\} \subset B$. Then $A_2$ is blue-complete to $W$ and red-complete to $\{x_1, \ldots, x_5\}$, and so $W$ is red-complete to $\{x_1, \ldots, x_5\}$ because $G[B]$ has no blue $P_7$. But then we obtain a red $C_{12}$ with vertices $a_2, x_1, w_1, x_2, w_2, x_3, w_3, x_4, b_2, x_5, c_2, x_6$ in order, where $x_6 \in B \setminus (A_2 \cup W \cup \{x_1, \ldots, x_5\})$, a contradiction. For the latter case, $G[B \setminus (A_2 \cup W)]$ has a red $P_7$ with vertices, say $x_1, \ldots, x_7$, in order. Let $W \cap B := \{w\}$. Then $w$ is red-complete to $\{x_1, \ldots, x_7\}$ because $G[B]$ has no blue $P_7$. But then we obtain a red $C_{12}$ with vertices $a_2, x_1, w, x_2, w_2, b_2, x_7, c_2, x_8$ in order, where $x_8 \in B \setminus (A_2 \cup W \cup \{x_1, \ldots, x_7\})$, a contradiction. This proves that $|R| \in \{4, 5\}$.

We claim that $G[E(B, R)]$ has no blue $P_5$ with both ends in $B$. Suppose there is a blue $H := P_3$ with vertices, say $x_1, y_1, x_2, y_2, x_3$, in order. Then $G[(B \cup R) \setminus V(H)]$ has no blue $P_3$ with both ends in $B$. Let $i^*_b := 0$, $i^*_r := r - |R| + 1 = 6 - |R|$, $i^*_r := i^*_r$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i^*}| + \left[ (\sum_{j=1}^k i^*_j) - i^*_r \right] = 3 + 2(6 - |R|) + 1 = 16 - 2|R|$. Observe that $|B \setminus \{x_1, x_2, x_3\}| = 12 - |R| \geq N^*$ since $|R| \in \{4, 5\}$. By minimality of $N$, $G[B \setminus \{x_1, x_2, x_3\}]$ has a red $G_{i^*}$ with vertices, say $x_4, \ldots, x_q$, in order, where $q = 2i^*_r + 6$. Then $y_3$ is not blue-complete to $\{x_4, x_q\}$ because $G[(B \cup R) \setminus V(H)]$ has no blue $P_3$ with both ends in $B$. We may assume $x_qy_3$ is colored red.
Then $R\setminus \{y_1, y_2, y_3\}$ is blue-complete to $x_8$, else say if $x_8 y_4$ is colored red, we obtain a red $C_{12}$ with vertices $a_1, y, x_4, \ldots, x_8, y_4, b_1, y_1, c_1, y_2$ in order, a contradiction. Since $G[(B \cup R) \setminus V(H)]$ has no blue $P_3$ with both ends in $B$, we see that $R\setminus \{y_1, y_2, y_3\}$ is red-complete to $\{x_4, \ldots, x_9\}$ in order, a contradiction. But then we obtain a red $C_{12}$ with vertices $a_1, y, x_4, \ldots, x_9, y_4, b_1, y_1, c_1, y_2$ (when $|B| = 4$), or $a_1, y_4, x_5, x_6, y_4, x_7, y_5, b_1, y_1, c_1, y_2$ (when $|B| = 5$) in order, a contradiction. Thus, $G[E(B, R)]$ has no blue $P_3$ with both ends in $B$. Let $i^*_i := 2, i^*_i := 2, i^*_j := i_j$ for all colors $j \in [k]$ other than red and blue, and $N^* := |G_{i^*_i}| + |(\sum_{j=1}^k i^*_j) - i^*_i| = 10$. Observe that $|B| \geq 10 = N^*$. By minimality of $N$, $G[B]$ has a red $P_7$ with vertices, say $x_1, \ldots, x_7$, in order. We claim that $x_1$ is blue-complete to $R$. Suppose $x_1 y_1$ is colored red. Then $R\setminus y_1$ is blue-complete to $\{x_5, x_7\}$, else $G[A_1 \cup R \setminus \{x_5, x_7\}]$ has a red $C_{12}$. Thus $R\setminus y_1$ is red-complete to $B \setminus \{x_5, x_7\}$ because $G[E(B, R)]$ has no blue $P_3$ with both ends in $B$. But then we obtain a red $C_{12}$ with vertices $a_1, y, x_2, \ldots, x_6, y_3, b_1, y_4, c_1, y_1$ in order, a contradiction. Therefore, $x_1$ is blue-complete to $R$. By symmetry, $x_7$ is blue-complete to $R$. Then $R$ is red-complete to $B \setminus \{x_1, x_7\}$ because $G[E(B, R)]$ has no blue $P_3$ with both ends in $B$. But then we obtain a red $C_{12}$ with vertices $a_1, y_2, x_2, \ldots, x_6, y_3, b_1, y_4, c_1, y_1$ in order, a contradiction. This proves that $|A_1| = n - 2$.  

By Claims 12, 13 and 8, $i_b = n - 1$, $|A_1| = n - 2$, $i_r \geq |R|$. By Claim 11, $|R| \leq 2$. Then $|B| \geq 3 + n + i_r - |R| \geq 3 + n$, and so $G[B \cup R]$ has no blue $P_3$ with both ends in $B$, else there is a blue $C_{2n}$.  

Claim 14. $i_r = n - 1$.  

Proof. Suppose $i_r \leq n - 2$. By Claim 3, $B$ is not blue-complete to $R$. Let $x \in B$ and $y \in R$ such that $xy$ is colored red. Let $i_0 := i_b - |A_1| = 1$ and $i_r := i_r - |R| \leq n - 3$, $i^*_j := i_j \leq n - 4$ for all colors $j \in [k]$ other than red and blue. Let $N^* := |G_{i^*_i}| + |(\sum_{j=1}^k i^*_j) - i^*_i|$. Then $3 < N^* < N$ and $|B\setminus x| = N - |A_1| - |R| - 1 \geq N^*$. By minimality of $N$, $G[B\setminus x]$ must have a red $P_{2i^*+3}$ with vertices, say $x_1, x_2, \ldots, x_{2i^*+3}$, in order. Then $\{x_1, x_{2i^*+3}\}$ must be blue-complete to $\{x, y\}$ and $x x_2$ must be colored blue under $c$, else we obtain a red $P_{2i^*+3}$ using vertices in $V(P_{2i^*+3}) \cup \{x, y\} \cup A_1$. But then $G[B \cup R]$ has a blue $P_3$ with vertices $x_1, x, y, x_{2i^*+3}$ in order, a contradiction.  

Recall that $|A_1| = n - 2$, $G[A_1]$ has a green $P_3$, and $i_g = 1$. We next show that $|A_0| \geq 3$. Suppose $|A_2| \leq 2$. Then by Claim 10, $|A_1| = 4$ and so $n = 6$. Let $A_1 := \{a_1, b_1, c_1, z_1\}$. Let $i^*_0 := i_b - |A_1| = 1$, $i^*_r := i_r - |R| + 1 = 6 - |R| \geq 4$, $i^*_g := i_g - 1 = 0$ and $i^*_0 := i_j$ for all $j \in [k]$ other than red, blue and green. Let $i^*_j := \max\{i^*_j \mid j \in [k]\}$ and $N^* := |G_{i^*_i}| + |(\sum_{j=1}^k i^*_j) - i^*_i|$. Then $3 < N^* < N$ and $|B| = |G| - |A_1| - |R| = N^*$. By minimality of $N$, $G[B]$ must contain a red $G_{i^*_i}$. It follows that $|R| = 2$ and $G_{i^*_i} = P_{11}$. Let $x_1, x_2, \ldots, x_{11}$ be the vertices of the red $P_{11}$ in order. If $R$ is blue-complete to $\{x_1, x_{11}\}$, then $R$ is red-complete to $B \setminus \{x_1, x_{11}\}$ because $G[B \cup R]$ has no blue $P_3$ with both ends in $B$. But then $G$ has a red $C_{12}$ with vertices $a_1, y_1, x_2, \ldots, x_{10}, y_2$ in order, a contradiction. Thus, $R$ is not blue-complete.
to \( \{x_1, x_{11}\} \) and we may assume \( x_1y_1 \) is colored red. Then \( x_{11}y_1 \) and \( x_9y_2 \) are colored blue, else \( G[\{x_1, \ldots, x_{11}\} \cup R \cup A_1] \) has a red \( C_{12} \). If \( x_{11}y_2 \) is colored red, then \( x_{11}y_2 \) and \( x_3y_1 \) are colored blue by the same reasoning. But then we obtain a blue \( C_{12} \) with vertices \( a_1, x_1, y_2, b_1, x_3, y_1, x_{11}, c_1, x_2, z_1, x_9 \) in order, a contradiction. Thus \( x_{11}y_2 \) is colored blue. Then \( y_1 \) is red-complete to \( B \backslash \{x_9, x_{11}\} \), else, say \( y_1w \) is colored blue with \( w \in B \backslash \{x_9, x_{11}\} \), then \( G[B \cup R] \) has a blue \( P_5 \) with vertices \( w, y_1, x_{11}, y_2, x_9 \) in order. It follows that \( \{x_{11}, w\} \not\subseteq A_j \) for all \( j \in [p] \), where \( w \in B \backslash \{x_9, x_{11}\} \). Moreover, \( x_2y_2 \) is colored blue, else \( G \) has a red \( C_{12} \) with vertices \( a_1, y_2, x_2, \ldots, x_{10}, y_1 \) in order, a contradiction. Thus \( G[B \backslash \{x_2, x_9\}] \) has no blue \( P_3 \), else \( G[A_1 \cup B \cup \{y_2\}] \) has a blue \( C_{12} \). Therefore, \( x_{11}x_1 \) is colored red for some \( i \in \{3, \ldots, 7\} \). But then we obtain a red \( C_{12} \) with vertices \( y_1, x_1, \ldots, x_i, x_{11}, x_{10}, \ldots, x_{i+1} \) in order, a contradiction. Thus \( 3 \leq |A_2| \leq n - 2 \) and \( A_2 \subset B \) because \( |R| \leq 2 \).

Since \( G[B \cup R] \) has no blue \( P_5 \) with both ends in \( B \), there exists at most one vertex, say \( w \in (B \cup R) \backslash A_2 \), such that \( (B \cup R) \backslash (A_2 \cup \{w\}) \) is red-complete to \( A_2 \), and \( w \) is blue-complete to \( A_2 \). Suppose \( 3 \leq |A_3| \leq n - 2 \). Then \( n = 6 \) and \( |A_1| = 4 \) by Claim 10. \( A_3 \subseteq B \) and \( A_3 \) must be red-complete to \( A_2 \), so \( w \not\subseteq A_3 \). Since \( G[B \cup R] \) has no blue \( P_5 \) with both ends in \( B \), there exists at most one vertex, say \( w' \in (B \cup R) \backslash (A_2 \cup A_3) \), such that \( (B \cup R) \backslash (A_2 \cup A_3 \cup \{w'\}) \) is red-complete to \( A_3 \). Note that we may have \( w' = w \). Since \( |(B \cup R) \backslash \{w, w'\}| \geq |G| - |A_1| - 2 = 18 - 4 - 2 = 12 \), we see that \( G[\{B \cup R\} \backslash \{w, w'\}] \) has a red \( C_{12} \), a contradiction. Thus \( |A_3| \leq 2 \) and so \( G[B \backslash A_2] \) has no monochromatic copy of \( P_3 \) in color \( j \) for all \( j \in [k] \) other than red and blue. Let \( i_j^* := 1, i_j^* := n - 1 - |A_2|, \) and \( i_j^* := 0 \) for all colors \( j \in [k] \) other than red and blue. Let \( N^* := |G_{i_j^*}| + \left( \sum_{j=1}^k i_j^* \right) - i_j^* \). Then \( 3 < N^* < N \) and \( |B \backslash (A_2 \cup \{w\})| \geq 2n + 1 - |R| - |A_2| \geq N^* \). By minimality of \( N \), \( G[B \backslash (A_2 \cup \{w\})] \) has a red \( G_{i_j^*} = P_{2i_j^*+3} \). But then \( G[(B \cup R) \{w\}] \) has a red \( C_{2n} \), a contradiction.

This completes the proof of Theorem 1.9.

Acknowledgements

The authors would like to thank Christian Bosse for many helpful comments and discussion.

Lei was partially supported by the National Natural Science Foundation of China (No. 12001296) and the Fundamental Research Funds for the Central Universities, Nankai University (No. 63201163). Shi was partially supported by the National Natural Science Foundation of China (Nos. 11771221, 11922112).
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(Received 23 Dec 2019; revised 3 Feb 2021)