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On $\mathbb{Z}$-flow-continuous maps and oriented colorings of cubic graphs

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Abstract

A $\mathbb{Z}$-flow in a graph G is an orientation $\vec{G}$ together with an assignment $\phi\colon E(\vec{G}) \to \mathbb{Z}$ such that at each vertex the sum of all incoming flow values equals the sum of all outgoing ones. A map $f\colon E(\vec{G}) \to E(\vec{H})$ between the edge sets of two oriented graphs is called $\mathbb{Z}$-flow-continuous if $\phi \circ f$ is a $\mathbb{Z}$-flow in $\vec{G}$ for every $\mathbb{Z}$-flow ϕ in $\vec{H}$. The existence of $\mathbb{Z}$-flow-continuous maps naturally defines a quasi-order $\succ_{\mathbb{Z}}$ on the class of finite graphs. The purpose of this note is to study the quasi-order $\succ_{\mathbb{Z}}$, give an operative description of such maps when restricted to cyclically 4-edge-connected cubic graphs and show that this quasi-order contains an infinite antichain of snarks with circular flow number 5 containing the Petersen graph $\mathcal{P}_{10}$.

1 Introduction

The graphs we consider are finite. They may contain multiple edges but no loops. Let M be an abelian group. An *M-flow* in a graph G is an orientation $\vec{G}$ of G together with an assignment $\phi\colon E(\vec{G}) \to M$ such that, at each vertex, the sum of all incoming flow values equals the sum of all outgoing ones. The existence of an M-flow in a graph does not depend on the fixed orientation $\vec{G}$, indeed if we reverse $e \in E(\vec{G})$ then ϕ is still an M-flow by changing $\phi(e)$ with its inverse. If k, d are two integers such that $k \geq 2d > 0$, a *circular nowhere-zero $\frac{k}{d}$-flow* (introduced in [4]) is a $\mathbb{Z}$-flow ψ such that $|\psi(e)| \in \{d, d+1, \ldots, k-d\}$, for every edge e. The *circular flow number* of a bridgeless graph G is the least number $\Phi_c(G)$ such that G has a nowhere-zero circular $\Phi_c(G)$-flow. It is well-known that graphs with a bridge do not

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admit any nowhere-zero flow and, on the other hand, it was proved in [4] that, if G is bridgeless, then $\Phi_c(G)$ is a minimum. The well-known Tutte's 5-flow Conjecture [14] claims that every bridgeless graph admits a nowhere-zero 5-flow.

A map $f\colon E(\vec{G}) \to E(\vec{H})$ between the edge sets of two oriented graphs G and H is called *M-flow-continuous* if every M-flow of H can be lifted to an M-flow of G in the given orientations, i.e. for every M-flow $\psi\colon E(\vec{H}) \to M$ the composition $\psi \circ f$ is still an M-flow. It is known that $\mathbb{Z}_2$-flow-continuous maps are exactly cycle-continuous maps, i.e. maps having the property that the pre-image of every cycle is a cycle, where by *cycle* we mean a graph having vertices of even degree. The interest for these maps comes from an outstanding conjecture by Jaeger claiming that every bridgeless graph has a cycle-continuous map to the Petersen graph $\mathcal{P}_{10}$ [8]. Indeed a positive answer to this conjecture would imply many other very important ones like the 5-cycle-double-cover, see [15], and Berge-Fulkerson conjectures [3].

As shown and studied in [1], M-flow-continuous maps naturally define quasi-orders on the class of finite graphs. We say that $G \succ_M H$ if there is an M-flow-continuous map between an orientation of G and an orientation of H (we remark that our notation is slightly different from [1] since we only need to specify the group on which the flow function takes values). Using this notation Jaeger's Conjecture can be stated as follows.

Conjecture 1.1 (Jaeger [8]). *Every bridgeless graph G satisfies $G \succ_{\mathbb{Z}_2} \mathcal{P}_{10}$.*

Jaeger's Conjecture can be reduced to cubic graphs. In this context it is also known as the Petersen Coloring Conjecture since it can be naturally stated in terms of graph colorings. A map $f\colon E(G) \to E(H)$ between two cubic graphs G and H is called an *H-coloring* of G if for every $v \in V(G)$ there is $v_h \in V(H)$ such that $f(\partial(v)) = \partial(v_h)$, where $\partial(v)$ denotes the set of edges incident to v. If G is cubic then $G \succ_{\mathbb{Z}_2} \mathcal{P}_{10}$ if and only if G has a $\mathcal{P}_{10}$-coloring. Hence Jaeger's Conjecture can be stated equivalently as follows.

Conjecture 1.2 (Jaeger [8]). *Every bridgeless cubic graph has a $\mathcal{P}_{10}$-coloring.*

In [13] an infinite antichain of cubic graphs in the $\mathbb{Z}_2$-flow-continuous quasi-order was presented, and the problem of finding an infinite antichain (in the same quasi-order) of cyclically 4-edge-connected cubic graphs was left for further research. Since $\mathbb{Z}$-flow-continuous maps are also cycle-continuous [1], the problem of finding such an infinite antichain in the $\mathbb{Z}$-flow-continuous quasi-order would be a weaker version of the previous one. The purpose of this note is to study the quasi-order $\succ_{\mathbb{Z}}$. More precisely, we first give an operative description of $\mathbb{Z}$-flow-continuous maps $f\colon E(\vec{G}) \to E(\vec{H})$, when H is a cyclically 4-edge-connected cubic graph, see Proposition 2.2. Then we show that there is an infinite antichain of snarks containing $\mathcal{P}_{10}$, where we recall that a *snark* is a cyclically 4-edge-connected cubic graph with girth at least 5 and not admitting a 3-edge-coloring.

2 Oriented Colorings

Given a positive integer k, a **multipole** consists of a set of vertices V and a set of edges E, which may contain also dangling edges, i.e. edges adjacent just to one vertex and having a dangling side. We call k**-pole** a multipole containing k dangling edges. A graph is a multipole having no dangling edge.

Let C be a connected 2-regular graph and let $\vec{C}$ be an orientation of C. $E(\vec{C})$ can be partitioned into two disjoint subsets A and B of edges oriented respectively clockwise and counterclockwise. We say that two edges of A (or B) have the same direction, but an edge of A and an edge of B have opposite direction. If one between A and B is empty we say that $\vec{C}$ is a **directed** cycle.

Let G be a multipole and $\vec{G}$ an orientation of G. Moreover let $x \in V(G)$ be a vertex incident to the edges $e_1, e_2 \in \partial(x)$. We say that x **reverses** the orientation of the path e_1xe_2 in $\vec{G}$ if e_1 and e_2 are both incoming or outgoing at x in $\vec{G}$. Otherwise we say that x **preserves** the orientation of the path e_1xe_2 in $\vec{G}$.

Definition 2.1. Let G and H be two multipoles on which we have fixed the orientations $\vec{G}$ and $\vec{H}$ respectively. A map $f\colon E(\vec{G}) \to E(\vec{H})$ is an H**-oriented-coloring** of G if

- for every vertex $v \in V(G)$ there is a vertex $v_h \in V(H)$ such that $f(\partial(v)) = \partial(v_h)$;
- for every $v \in V(G)$ the mutual orientation of pairs of edges $e_1, e_2 \in \partial(v)$ is the same with respect to $f(e_1), f(e_2) \in \partial(v_h)$; in other words if v preserves (resp. reverses) the orientation of the path e_1ve_2 in $\vec{G}$ then v_h preserves (resp. reverses) the orientation of the path $f(e_1)v_hf(e_2)$ in $\vec{H}$.

An H-oriented-coloring is first of all an H-coloring. Furthermore if, for an orientation $\vec{H}$ of H, there is an orientation $\vec{G}$ of G and a map $f\colon E(\vec{G}) \to E(\vec{H})$ that is an H-oriented-coloring then, for every orientation of H, there is an orientation of G and a map that is an H-oriented-coloring of G. Indeed, given such a map f, just notice that if we reverse the orientation of $e \in E(\vec{H})$ then it suffices to reverse the orientation of the set of edges $f^{-1}(e)$ and f remains an H-oriented-coloring of G.

The previous property holds also for $\mathbb{Z}$-flow-continuous maps $f\colon E(\vec{G}) \to E(\vec{H})$ from an orientation of G and an orientation of H. Indeed, if we reverse the orientation of an edge $e \in E(\vec{H})$, then f is still a $\mathbb{Z}$-flow-continuous map provided that we reverse the orientation of every edge of $f^{-1}(e)$.

Note also that an oriented coloring is $\mathbb{Z}$-flow-continuous and therefore, if there exists such a map $f\colon E(\vec{G}) \to E(\vec{H})$ between two graphs G and H, the following necessary condition holds: $\Phi_c(G) \leq \Phi_c(H)$. Indeed, if ψ is a circular nowhere-zero $\Phi_c(H)$-flow in H, $\psi \circ f$ is a circular nowhere-zero $\Phi_c(H)$-flow in G.

2.1 Oriented colorings of cubic multipoles

From now on we will focus on the study of oriented colorings of cubic multipoles. In [1] the authors prove that a map $f\colon E(G) \to E(\mathcal{P}_{10})$, where G is a cubic graph, is a $\mathcal{P}_{10}$-coloring if and only if it is cycle-continuous. A central role is played by the fact that $\mathcal{P}_{10}$ has only trivial 3-edge-cuts. Indeed this property still holds for cycle-continuous maps $G \to H$ of cubic graphs, whenever H has only trivial 3-edge-cuts. Our interest to oriented colorings of cubic graphs is motivated by the following proposition. Recall that a graph is **cyclically k-edge-connected**, if it does not have an edge-cut of cardinality less than k that separates two circuits of the graph.

Proposition 2.2. *Let G and H be two bridgeless cubic graphs and let H be cyclically 4-edge-connected. Suppose that they are endowed with the orientations $\vec{G}$ and $\vec{H}$ respectively. Then $f\colon E(\vec{G}) \to E(\vec{H})$ is an H-oriented-coloring of G if and only if f is a $\mathbb{Z}$-flow-continuous map.*

Proof. An oriented coloring is $\mathbb{Z}$-flow-continuous by definition.

On the other hand, let $f\colon E(\vec{G}) \to E(\vec{H})$ be a $\mathbb{Z}$-flow-continuous map. Since f is cycle-continuous and H is cyclically 4-edge-connected we get that f is an H-coloring of G. For all $u \in V(G)$, let u_h be the vertex of H such that $\partial(u_h) = f(\partial(u))$. Since H is bridgeless, it admits a strongly connected orientation $\vec{H}'$ [12]. Reorient suitable edges of $\vec{H}$ in such a way that $\vec{H}'$ is fixed. By previous observations we can reorient the corresponding edges of $\vec{G}$ (those whose image under f has been reoriented when passing from $\vec{H}$ to $\vec{H}'$) and keep f $\mathbb{Z}$-flow-continuous; call $\vec{G}'$ this new orientation of G. If we prove that $f\colon E(\vec{G}') \to E(\vec{H}')$ is an H-oriented-coloring the thesis follows because, similarly to the case of $\mathbb{Z}$-flow-continuous maps, we can pass from orientations $\vec{H}'$ and $\vec{G}'$ to $\vec{H}$ and $\vec{G}$ by keeping f an H-oriented-coloring. We have to show that for every $u \in V(G)$, the mutual orientation of pairs of edges in $\partial(u)$ is equal to the mutual orientation of their images under f in $\partial(u_h)$. Suppose by contradiction that this is not the case, meaning that there is a vertex v of G that does not satisfy the required property. Since $\vec{H}'$ is strongly connected, edges in $\partial(v_h)$ are not all incoming or all outgoing at v_h. Without loss of generality we can suppose that v_h has one incoming edge a and two outgoing edges b, c, otherwise we can reverse the orientation of every edge of $\vec{H}'$ and $\vec{G}'$. The set $\partial(v)$ is mapped onto the set $\{a, b, c\}$ by f, let us call e_i the edge of $\partial(v)$ such that $f(e_i) = i \in \partial(v_h)$. By our contradictory hypothesis the orientation of at least one path between $e_a v e_b$ and $e_a v e_c$ is reversed by v in $\vec{G}'$, assume that this holds for $e_a v e_b$. Since $\vec{H}'$ is strongly connected, there is a directed cycle C in $\vec{H}'$ containing the edges a and b. Let $\psi\colon E(\vec{H}') \to \mathbb{Z}$ be the $\mathbb{Z}$-flow such that $\psi(e) = 1$ for every $e \in C$ and $\psi = 0$ everywhere else. Then $\psi \circ f$ is not a $\mathbb{Z}$-flow in G, a contradiction. □

In [1] the authors prove that a graph G has circular flow number at most 4 if and only if $G \succ_{\mathbb{Z}} K_4$. When considering cubic graphs, this fact can be also stated as follows using oriented colorings. We will show in the Appendix an alternative proof of this result.

Theorem 2.3 ([1])**.** *Let G be a bridgeless cubic graph. Then $\Phi_c(G) \leq 4$ if and only if there is a K_4-oriented-coloring of G.*

For the case of bipartite cubic graphs another characterization is proved in [1]: a cubic graph G is bipartite if and only if $G \succ_{\mathbb{Z}} K_2^3$, where K_2^3 is the cubic loopless multigraph on 2 vertices and 3 edges. The following generalization, stated using oriented colorings, holds.

Theorem 2.4 ([1])**.** *Let G be a bridgeless cubic multipole. Then G is bipartite if and only if there is a K_2^3-oriented-coloring of G.*

Let G be a multipole. The multipole **induced** by $X \subseteq V(G)$ in G is the multipole whose vertex set is X and edge set consists of all edges adjacent to at least one vertex of X. In the following part, if $f\colon E(\vec{G}) \to E(\vec{H})$ is an oriented coloring of a cubic multipole G, consider the subgraph K of $\vec{H}$ induced by $f(E(\vec{G}))$. With a slight abuse of terminology, we will denote by $f(G)$ the undirected multipole induced by the vertices of degree 3 of K.

Corollary 2.5. *Let G be a non-bipartite cubic multipole, H a cubic graph and $f\colon E(\vec{G}) \to E(\vec{H})$ an oriented coloring. Then $f(G)$ is a non-bipartite multipole.*

Proof. Suppose by contradiction that $F = f(G)$ is bipartite and let $\vec{F}$ be the orientation that F inherits from $\vec{H}$. Then, by Theorem 2.4, there is a K_2^3-oriented-coloring $g\colon E(\vec{F'}) \to E(\vec{K}_2^3)$. We can reorient suitable edges of $\vec{G}$ in such a way that $f\colon E(\vec{G}) \to E(\vec{F'})$ remains an oriented coloring. Then the composition $g \circ f$ is a K_2^3-oriented-coloring of G. This is impossible by Theorem 2.4. □

The following remark holds.

Remark 2.6. *In the hypothesis of previous corollary, if G is the cubic multipole consisting of a k-cycle and k dangling edges, for an odd number k, then the girth of H is at most k. In particular, if it is exactly k, then $f(G)$ is isomorphic to G and its dangling edges are mapped to dangling edges of its image.*

Proof of the Remark. By Corollary 2.5, $f(G)$ must be a cubic multipole containing a cycle on t vertices for a suitable odd number $3 \leq t \leq k$. Moreover, let v_1, v_2 be two adjacent vertices of G. Let $v_{1,h}$ and $v_{2,h}$ be the vertices of H such that, for $i \in \{1, 2\}$, $f(\partial(v_i)) = \partial(v_{i,h})$. We have that either $v_{1,h} = v_{2,h}$ or they are adjacent vertices in H. Thus, if the girth of H is k it follows that $f(G)$ consists of a k-cycle and k dangling edges that are image of dangling edges of G. □

2.2 An infinite antichain of snarks in the quasi-order $\succ_{\mathbb{Z}}$

In this final part of the section we construct an infinite family of snarks that are pairwise incomparable in the quasi-order $\succ_{\mathbb{Z}}$. We are interested in studying the cubic 4-pole P obtained from the Petersen graph $\mathcal{P}_{10}$ by removing two adjacent vertices.

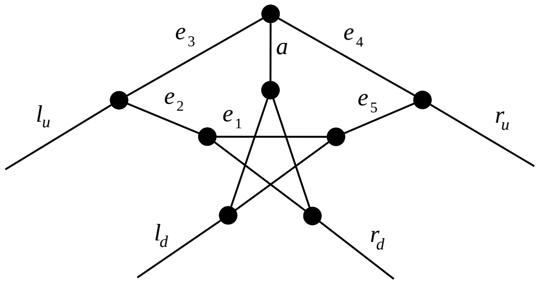

Figure 1: The multipole P.

The 4-pole P has 4 dangling edges and is depicted in Figure 1. In particular, we are interested in understanding what its image under an oriented coloring looks like.

We say that two (dangling) edges of a multipole are at **distance** k if the shortest path connecting two of their endvertices has length k, where the length of a path is the number of its edges.

Lemma 2.7. *Suppose that f is an H-oriented-coloring of P where H is a cyclically 4-edge-connected cubic graph of girth at least 5. Then $f(P)$ is isomorphic to a copy of P. Moreover dangling edges are mapped to dangling edges of the multipole $f(P)$ in such a way that both pairs $f(l_u), f(l_d)$ and $f(r_u), f(r_d)$ are at distance 3 in $f(P)$ (see Figure 1 as reference for the considered edges).*

Proof. Consider two 5-cycles C_1, C_2 in P such that $C_2 = e_1e_2e_3e_4e_5$, see Figure 1, and C_1 intersects C_2 just in e_1. Let M_i be the multipole induced by $V(C_i)$. By Corollary 2.5, $f(M_1)$ and $f(M_2)$ are both isomorphic to M_1 (and also to M_2), and dangling edges of M_1 are mapped to dangling edges of $f(M_1)$. Notice that $E(C_2) \cap E(M_1) = \{e_1, e_2, e_5\}$, and so $f(e_2)$ and $f(e_5)$ are dangling edges of $f(M_1)$. Therefore e_3 and e_4 are mapped to a couple of adjacent edges which are both adjacent to $f(a)$ and such that they are also adjacent to $f(e_2)$ and $f(e_5)$ respectively. Finally, the unique possibility is that the remaining two dangling edges l_u and r_u are mapped to dangling edges adjacent respectively to $f(e_2), f(e_3)$ and $f(e_4), f(e_5)$. □

The following corollary follows immediately from the main result of [10] by Mkrtchyan, claiming that if $\mathcal{P}_{10}$ has a G-coloring, for a connected bridgeless cubic graph G, then $\mathcal{P}_{10}$ is isomorphic to G.

Corollary 2.8. *Suppose that f is an H-oriented-coloring of $\mathcal{P}_{10}$, where H is a bridgeless cubic graph. Then $f(\mathcal{P}_{10})$ is isomorphic to $\mathcal{P}_{10}$.*

Proof. Follows from the fact that f is an H-coloring of $\mathcal{P}_{10}$. □

Other than P, we want to focus on the 5-pole P' shown in Figure 2.

Lemma 2.9. *There is no $\mathcal{P}_{10}$-oriented-coloring $f\colon E(\vec{P'}) \to E(\vec{\mathcal{P}}_{10})$ of P'.*

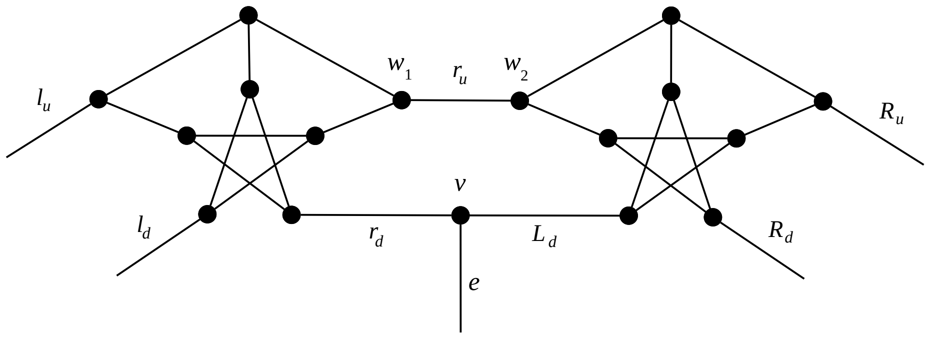

Figure 2: The multipole P'.

Proof. Suppose by contradiction that there is a $\mathcal{P}_{10}$-oriented-coloring f of P'. There are two distinct copies P_1 and P_2 of P inside P', which have a common dangling edge r_u and an other one adjacent to a new vertex v, see Figure 2 as reference for the considered edges. Without loss of generality we say that P_1 is the left copy of P and P_2 is the right one with respect to Figure 2. By previous lemma P_1 is sent to a copy isomorphic to P where l_u, l_d and r_u, r_d are mapped to pairwise adjacent edges. Let $z \in E(\mathcal{P}_{10}) \setminus f(P_1)$, i.e. z is adjacent to $f(l_u), f(l_d), f(r_u)$ and $f(r_d)$. In an analogous way P_2 is mapped to a copy isomorphic to P. Hence $f(L_d)$ must be adjacent to $f(r_u)$ and to $f(r_d)$ and therefore $f(L_d) = z$ and $f(e) = f(r_u)$. Thus P_2 is sent to $\mathcal{P}_{10} - f(r_d)$.

By definition of oriented coloring we have that the mutual orientation of every possible couple of edges of $\vec{C} = \{r_u, r_d, l_u, l_d\} \subseteq E(\vec{P'})$ must be the same with respect to its image in $f(\vec{C})$. The contradiction arises from the fact that, due to the presence of the vertex v, the mutual orientation of r_u and L_d is different to the mutual orientation of $f(r_u)$ and $f(L_d)$. □

All previous results lead us to the following

Theorem 2.10. *Let G be a bridgeless cubic graph obtained by joining dangling edges of a 5-pole C with dangling edges of P'. Then $G \not\prec_{\mathbb{Z}} \mathcal{P}_{10}$.*

Construction methods described in [2], [5] and [9] show that there are many snarks with circular flow number 5 that have the structure described by the previous theorem. Those snarks are examples of cubic graphs that are incomparable with $\mathcal{P}_{10}$ in the $\mathbb{Z}$-flow-continuous quasi-order as they contain the multipole P' and by Corollary 2.8. Every such snark S is also incomparable with K_4, indeed $S \not\prec_{\mathbb{Z}} K_4$ since $\Phi_c(S) > \Phi_c(K_4)$ and $K_4 \not\prec_{\mathbb{Z}} S$ since the girth of S is greater than the girth of K_4. In the following part we will show that some of these snarks with circular flow number 5 together with the Petersen graph form an infinite antichain in the $\mathbb{Z}$-flow-continuous quasi-order.

Definition 2.11. Consider $n \geq 3$ copies $P_1, P_2, \ldots, P_n$ of the multipole P. For $i = 1 \ldots, n$, let us denote by $l_{u,i}, l_{d,i}, r_{u,i}$ and $r_{d,i}$ the dangling edges of P_i with reference to Figure 1. Consider an n-cycle $c_1 c_2 \ldots c_n$. Call W_n the graph obtained

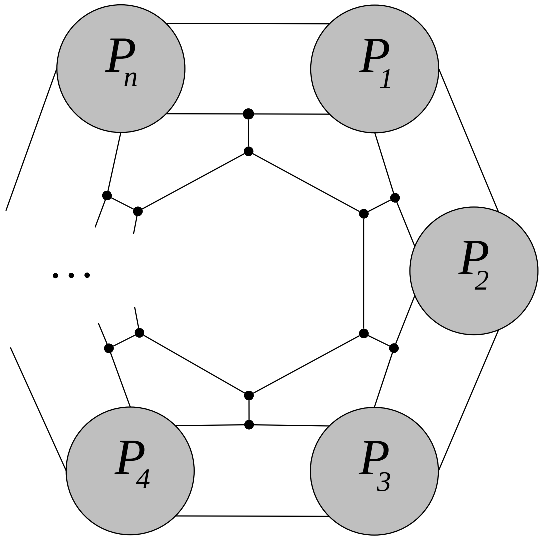

Figure 3: The graph W_n.

by identifying $r_{u,i}$ with $l_{u,i+1}$ and by making the vertex c_i and both dangling edges $r_{d,i}, l_{d,i+1}$ be adjacent to a new vertex v_i, where we compute the sum of indices modulo n. We will refer to the copy P_i inside W_n as P_i^n. In Figure 3 the graph W_n is depicted.

We remark that, when $n \geq 5$ is odd, W_n is a snark with circular flow number 5, see [2]. In order to make use of Lemma 2.7 and the equivalence given by Proposition 2.2, we will focus on graphs W_n with $n \geq 5$ and, in particular, we will be interested in graphs W_n, W_m such that n and m are coprime.

Proposition 2.12. *Consider two positive integers $n, m \geq 5$. There is a W_m-oriented-coloring of W_n if and only if m divides n.*

Proof. Let f be the W_m-oriented-coloring of W_n.

Claim 1. *Let R_i be the multipole isomorphic to P' induced by $V(P_i^n \cup P_{i+1}^n) \cup \{v_i\}$ in W_n. Then $f(R_i)$ is isomorphic to R_i.*

Proof of Claim 1. We take Figure 2 as a reference when considering edges and vertices, in particular we consider P_i^n to be the left copy of P and P_{i+1}^n the other one.

Since W_m is cyclically 4-edge-connected with girth at least 5, Lemma 2.7 implies that $f(P_i^n)$ is isomorphic to P. Suppose that P_{i+1}^n is sent to the same copy of P. Then $f(L_d) = f(r_d)$ and we get a contradiction because adjacent edges cannot have the same image. Notice that, if $f(\partial(w_2)) = f(\partial(w_1))$ then $f(P_i^n) = f(P_{i+1}^n)$, and we get the same contradiction. Hence we conclude that $f(\partial(w_2))$ is different from $f(\partial(w_1))$. Notice that, because of the structure of W_m, $f(\partial(w_2))$ does not contain $f(r_d)$. Therefore P_i^n and P_{i+1}^n are sent to different copies of P having just $f(r_u)$ in common, and the unique possibility for the oriented coloring to be defined is that $f(L_d)$ and $f(r_d)$ are adjacent to the unique vertex v_j in W_m which is adjacent to a dangling edge of both $f(P_i^n)$ and $f(P_{i+1}^n)$. □

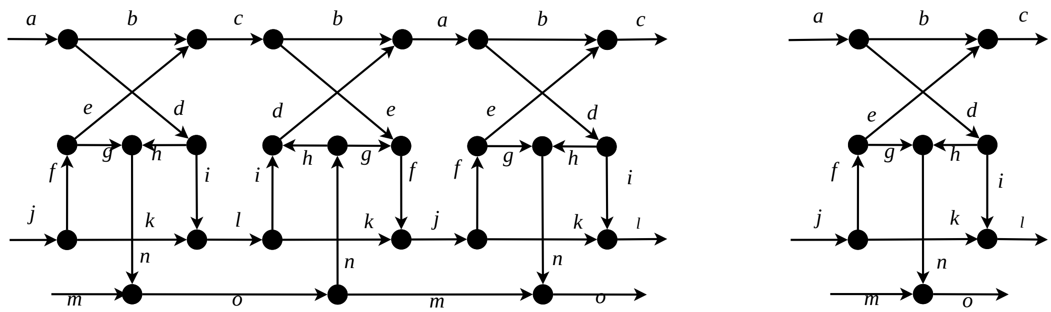

Figure 4: Oriented coloring of 6-poles of Goldberg snarks.

Claim 2. *Let Q_i be the multipole induced by $V(P_i^n \cup P_{i+1}^n \cup P_{i+2}^n) \cup \{v_i, v_{i+1}\}$ in W_n. Then $f(Q_i)$ is isomorphic to Q_i.*

Proof of Claim 2. Consider the multipole Q' isomorphic to P' induced by $V(P_i^n \cup P_{i+1}^n) \cup \{v_i\}$ inside W_n. Then, by Claim 1, $f(Q')$ is isomorphic to P' and so P_i^n and P_{i+1}^n must be sent to two adjacent copies P_j^m, P_{j+1}^m. Without loss of generality we can suppose that they are sent to P_i^m and P_{i+1}^m, and in particular that $f(P_k^n) = P_k^m$, for $k = i, i+1$. Using the same argument we notice that P_{i+1}^n and P_{i+2}^n must be sent to adjacent copies of P. In particular $f(P_{i+2}^n) = P_{i+2}^m$ for otherwise, if $f(P_{i+2}^n) = P_i^m$ we would get that dangling edges l_d and r_d (as well as l_u and r_u) of P_{i+1}^n would be mapped to the same edge a contradiction with Lemma 2.7. □

By previous claims we notice that $f(P_1^n), \dots, f(P_n^n)$ must be pairwise consecutive copies of P in W_m such that $f(P_i^n)$ is different from both $f(P_{i+1}^n)$ and $f(P_{i+2}^n)$, for every i. Therefore a necessary condition for f to be defined is that n is a multiple of m.

On the other hand, a W_m-oriented-coloring of W_{km} can be constructed in the natural way by identifying via identity map the multipoles M_{h+i}^{km} induced by $V(P_{h+i}^{km} \cup P_{h+i+1}^{km}) + c_{h+i} + c_{h+i+1} + v_{h+i} + v_{h+i+1}$ in W_{km} with the multipoles M_i^m induced by $V(P_i^m \cup P_{i+1}^m) + c_i + c_{i+1} + v_i + v_{i+1}$ in W_m, where $h \in \{0, m, 2m, \dots, (k-1)m\}$ and $i \in \{1, 2, \dots, m\}$. The orientation is naturally defined on every M_{h+i}^{km} (just set the very same orientation of M_i^m) by the chosen orientation on W_m. □

Theorem 2.13. *Let $\{p_j\}_{j\in\mathbb{N}}$ be the sequence of prime numbers greater than 3. The family $\mathcal{F} = \{\mathcal{P}_{10}, W_{p_1}, W_{p_2}, \dots\}$ is an antichain in the $\mathbb{Z}$-flow-continuous quasi-order $\succ_{\mathbb{Z}}$.*

Proof. By Proposition 2.12 for every couple of different prime numbers p_s and p_t we have that W_{p_s} and W_{p_t} are incomparable. Moreover $\mathcal{P}_{10}$ is incomparable with every other graph of $\mathcal{F}$ by Theorem 2.10. □

3 Further examples on oriented colorings

In this last section we show examples of construction of oriented colorings on classes of snarks. Consider the family of Goldberg snarks, introduced in [6]; they form an increasing chain in the $\mathbb{Z}$-flow-continuous quasi-order. The Goldberg snark G_{2k+1} can be constructed the following way. Consider a cycle $v_1v_2\dots v_{2k+1}$ of length $2k+1$. Remove each vertex v_i and substitute it with a copy P_i of the 6-pole obtained from the Petersen graph after the removal of two vertices at distance 2 (see Figure 4, right hand side). Finally for each couple of adjacent vertices v_iv_j of the initial cycle glue together P_i and P_j as shown in Figure 4, left hand side.

We show that $G_{2k+3} \succ_{\mathbb{Z}} G_{2k+1}$, for every positive integer k. Call $P_1, \dots, P_{2k+3}$ and $Q_1, \dots, Q_{2k+1}$ the consecutive 6-poles of G_{2k+3} and G_{2k+1} respectively. First we map the subgraph induced by P_1, P_2 and P_3 to Q_1 as shown in Figure 4, as well as fix on them the shown orientation. Then fix on the isomorphic subgraphs induced respectively by $P_4, \dots, P_{2k+3}$ and $Q_2, \dots, Q_{2k+1}$ the same orientation and map edges of the multipole P_{i+2} identically on the edges of Q_i, in the natural way. The defined map is a G_{2k+1}-oriented-coloring of G_{2k+3} and so a $\mathbb{Z}$-flow-continuous map as well. Hence we conclude that the family of Goldberg snarks $\{G_{2k+1}\}_{k\in\mathbb{N}}$ forms an increasing chain in $\succ_{\mathbb{Z}}$.

By following the very same method one can show that the family of Flower snarks, introduced in [7], also forms an increasing chain $J_3 \prec_{\mathbb{Z}} J_5 \prec_{\mathbb{Z}} J_7 \prec_{\mathbb{Z}} \dots$.

Appendix

In this appendix we show an alternative proof of Theorem 2.3 that makes use of oriented colorings.

Suppose that $\vec{C} = (A, B)$ is an oriented 2-regular graph and let $x, y, z \in V(C)$ be different vertices. Similarly to previous definitions, if $xy, yz \in A$ (or $xy, yz \in B$), we say that y **preserves** the orientation, vice versa if $xy \in A$ and $yz \in B$ (or $xy \in B$ and $yz \in A$) then we say that y **reverses** the orientation.

Proof of Theorem 2.3. If there is such an oriented coloring we obtain

$$\Phi_c(G) \le \Phi_c(K_4) = 4.$$

On the other hand let $\Phi_c(G) \le 4$. It is a well-known fact that, for cubic graphs, this is equivalent to having a 3-edge-coloring; see for example [11] and [14]. So G is 3-edge-colorable and has three pairwise disjoint perfect matchings M_1, M_2 and M_3 that partition its edge set. We are going to define an orientation $\vec{G}$ on G and a map $\phi\colon E(\vec{G}) \to E(\vec{K_4})$ with the required properties. Fix on K_4 the orientation shown in Figure 5.

Let $M_{ij} = M_i \cup M_j$ for every $i, j \in \{1, 2, 3\}$. Consider the two 2-factors M_{12} and M_{13}. Orient the edges of M_{12} in such a way that it becomes union of disjoint

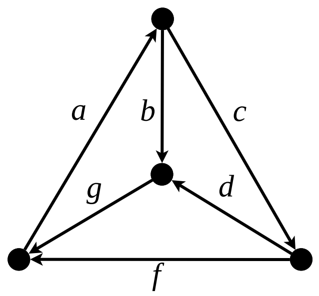

Figure 5: Orientation of K_4.

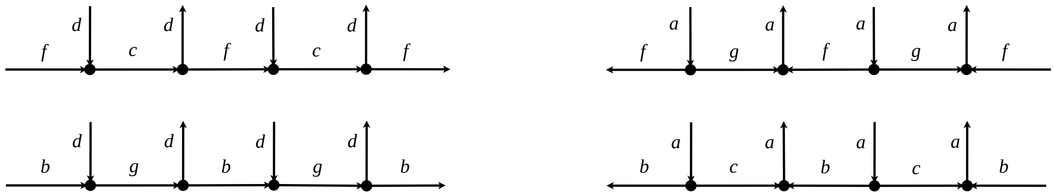

Figure 6: Possible assignments for sequences of d-edges (left hand side) and a-edges (right hand side).

directed cycles. Then orient the remaining edges in such a way that the edges with color 3 of each component of M_{13} have the same direction. Thus, for every directed component $\vec{C}$ of M_{13}, its edge set $E(\vec{C}) = A \cup B$, with $A \cap B = \emptyset$, and all edges of $\vec{C}$ colored by 3 are contained in A. For every connected component of M_{13} set

$$\begin{cases} \phi(e) = a, & \text{for every } e \in A \cap M_1, \\ \phi(e) = d, & \text{for every } e \in B \cap M_1. \end{cases}$$

Every connected component C of the 2-factor M_{23} is an oriented even cycle, hence C must contain an even number of vertices that reverse the orientation. Therefore it contains also an even number of vertices that preserve the orientation. Furthermore notice that vertices that reverse the orientation are incident to edges that are assigned a, call them a-edges. So the number of a-edges pointing towards C equals the number of a-edges pointing outwards C. The same property holds for d-edges incident to C since they are incident to vertices that preserve the orientation and since d-edges have the same orientation of 2-colored edges in M_{12}.

Now we prescribe an assignment also for edges of color 2 and 3. Notice that we can suppose without loss of generality that there are no even sequences of a-edges or d-edges since they can be labeled as in Figures 6. Hence the problem translates to the task of finding a proper assignment for the edges of an even cycle C' where there are no adjacent a-edges nor adjacent d-edges. By construction these edges have pairwise the same orientation, just notice that this holds for every edge e of color 3 of C', since they are adjacent to an a-edge (oriented coherently with respect to e) and a d-edge (having reversed orientation with respect to e). Then define the assignment as in Figure 7. □

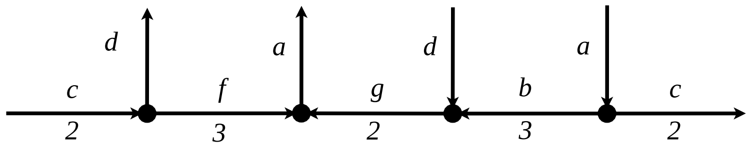

Figure 7: Extention of the map ϕ to the 2-factor M_{23}.

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