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On increasing and invariant parking sequences

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Abstract

The notion of parking sequences is a new generalization of parking functions introduced by Ehrenborg and Happ. In the parking process defining the classical parking functions, instead of each car only taking one parking space, we allow the cars to have different sizes and each takes up a number of adjacent parking spaces after a trailer T parked on the first $z-1$ spots. A preference sequence in which all the cars are able to park is called a parking sequence. In this paper, we study increasing parking sequences and count them via bijections to lattice paths with right boundaries. Then we study two notions of invariance in parking sequences and present various characterizations and enumerative results.

1 Introduction

Classical parking functions were first introduced by Konheim and Weiss [4]. The original concept involves a linear parking lot with n available spaces and n labeled cars each with a pre-fixed parking preference. Cars enter one-by-one in order. Each car attempts to park in its preferred spot first. If a car finds its preferred spot occupied, it moves towards the exit and takes the next available spot. If there is no space available, the car exits without parking. A *parking function* of length n is a preference sequence for the cars in which all cars are able to park (not necessarily

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in their preferred spaces). A formal definition for parking functions can be stated as follows.

Definition 1.1. Let n be a positive integer and $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a sequence of positive integers. Assume $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ is the non-decreasing rearrangement of $\vec{a}$. Then the sequence $\vec{a}$ is a parking function of length n if and only if $a_{(i)} \leq i$ for all indices i. Equivalently, $\vec{a}$ is a parking function (of length n) if and only if for all $1 \leq i \leq n$,

$$\#\{j : a_j \leq i\} \geq i. \tag{1}$$

For example, the preference sequences (1, 2, 3, 4), (2, 1, 3, 4) or (1, 2, 4, 1) are all parking functions, while (2, 2, 4, 2) is not since it will have one car leave un-parked. It is well-known that the number of classical parking functions is $(n+1)^{n-1}$. An elegant proof by Pollak (see [7]) uses a circle with $(n+1)$ spots where the parking functions are the preference sequences that can park all n cars without using the $(n+1)$-th spot.

One well-known generalization of Definition 1.1 is the notion of vector parking functions, or $\vec{u}$-parking functions. Let $\vec{u}$ be a non-decreasing sequence $(u_1, u_2, u_3, \ldots)$ of positive integers. A $\vec{u}$-parking function of length n is a sequence $(x_1, x_2, \ldots, x_n)$ of positive integers whose non-decreasing rearrangement $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ satisfies $x_{(i)} \leq u_i$. Equivalently, $(x_1, \ldots, x_n)$ is a $\vec{u}$-parking function if and only if for all $1 \leq i \leq n$,

$$\#\{j : x_j \leq u_i\} \geq i. \tag{2}$$

Denote by $\mathsf{PF}_n(\vec{u})$ the set of all $\vec{u}$-parking functions of length n. When $u_i = i$ we obtain the classical parking functions. When $u_i = a + b(i-1)$ for some $a, b \in \mathbb{Z}_+$, it is known that the number of $\vec{u}$-parking functions is $a(a+bn)^{n-1}$; see e.g. [5].

The set of parking functions is a basic object lying in the center of combinatorics, with many connections and applications to other branches of mathematics and disciplines, such as storage problems in computer science, graph searching algorithms, interpolation theory, diagonal harmonics, and sandpile models. Because of their rich theories and applications, parking functions and their variations have been studied extensively in the literature. See [9] for a comprehensive survey on the combinatorial theory of parking functions.

There is a particular generalization of parking functions that was recently introduced by Ehrenborg and Happ [2, 3], called parking sequences. Again, there are n cars trying to park in a linear parking lot. In this new model the car C_i has length $y_i \in \mathbb{Z}_+$ for each $i = 1, 2, \ldots, n$. Call $\vec{y} = (y_1, y_2, \ldots, y_n)$ the *length vector*. There is a trailer T of length $z - 1$ parked at the beginning of the street after which the n cars park with car C_i taking up y_i adjacent parking spaces. Given a sequence $\mathbf{c} = (c_1, ..., c_n) \in \mathbb{Z}_+^n$, for $i = 1, 2, \ldots, n$ the cars enter the street in order, and car C_i looks for the first empty spot $j \geq c_i$. If the spaces j through $j + y_i - 1$ are all empty, then car C_i parks in these spots. If j does not exist or any of the spots $j + 1$ through $j + y_i - 1$ is already occupied, then there will be a collision and the car cannot park and has to leave the street. In this case, we say the parking fails.

Definition 1.2. Assume there are $z-1+\sum_{i=1}^{n} y_i$ parking spots along a street, with the first $z-1$ occupied by a trailer. The sequence $\mathbf{c} = (c_1, ..., c_n)$ is called a *parking sequence for* $(\vec{y}, z)$ where $\vec{y} = (y_1, ..., y_n)$ if all n cars can park without any collisions. We denote the set of all such parking sequences by $\mathsf{PS}(\vec{y}; z)$.

For example, $\mathbf{c} = (3, 7, 5, 3)$ is a parking sequence for $(\vec{y}; z)$ where $\vec{y} = (1, 2, 2, 3)$ and $z = 4$. Figure 1 shows how the cars $C_1, \ldots, C_4$ would park along the street with the preference sequence $\mathbf{c}$. As given in [3], the number of parking sequences in $\mathsf{PS}(\vec{y}; z)$ is

$$z \cdot (z + y_1 + n - 1) \cdot (z + y_1 + y_2 + n - 2) \cdots (z + y_1 + \cdots + y_{n-1} + 1). \qquad (3)$$

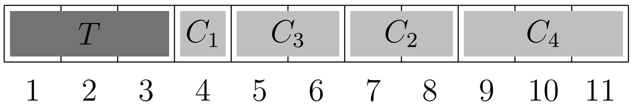

Figure 1: $\mathbf{c} = (3, 7, 5, 3)$ is a parking sequence for $\vec{y} = (1, 2, 2, 3)$ and $z = 4$.

From (1) and (2) it is easy to see that any permutation of a $\vec{u}$-parking function is also a $\vec{u}$-parking function. This is however not true for parking sequences. Consider as an example a one-way street with 4 spots and 2 cars with fixed length vector $\vec{y} = (2, 2)$ and $z = 1$, (no trailer). Then whereas $\mathbf{c} = (1, 2)$ is a parking sequence for $(\vec{y}; z)$, $\mathbf{c}' = (2, 1)$ is not. Thus, it is natural to ask which parking sequences $\mathbf{c}$ are invariant for $(\vec{y}; z)$, that is, $\mathbf{c}$ is still a parking sequence for $(\vec{y}; z)$ after the entries of $\mathbf{c}$ are permuted. Another question is which sequence remains a parking sequence when the cars enter the street in different orders. In other words, we want to know which preference sequence allows all the cars to park when the length vector $\vec{y}$ is permuted to $(y_{\sigma(1)}, y_{\sigma(2)}, \ldots, y_{\sigma(n)})$ for an arbitrary permutation σ of length n.

There are several basic notions and variations associated with parking functions and their generalizations. Usually, these notions lead to a study of special classes of parking functions that have some interesting property. One of such special classes is the set of *increasing parking functions*, which have non-decreasing entries and are counted by the ubiquitous Catalan numbers. It is only natural to ask for a generalization of this class in the set of parking sequences.

We study these questions in the present work. The rest of the paper is organized as follows. In Section 2, we discuss increasing parking sequences and their connection to lattice paths. In Section 3, we fix the length vector $\vec{y}$ and characterize all permutation-invariant parking sequences when $\vec{y}$ has some special characteristics. Then in Section 4, we characterize all parking sequences that remain valid for all permutations of $\vec{y}$. We finish the paper with some closing remarks in Section 5.

2 Increasing Parking Sequences

In this section, we consider all non-decreasing parking sequences for any given pair $(\vec{y}; z)$. By convention, we write $[x] = \{1, 2, \ldots, x\}$ and the interval $[x, y] = \{x, x +$

$1, \ldots, y\}$, where $x, y \in \mathbb{Z}_+$ and $x \leq y$. Given any sequence $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}_+^n$, let $\mathbf{b}_{inc} = (b_{(1)}, \ldots, b_{(n)})$ be the non-decreasing rearrangement of the entries of $\mathbf{b}$ and the i^{th} entry $b_{(i)}$ of $\mathbf{b}_{inc}$ is called the *i-th order statistic* of $\mathbf{b}$. Next, we define the final parking configuration for any given parking sequence.

Definition 2.1. Let $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$. The *final parking configuration* of $\mathbf{c}$ is the arrangement of cars $C_1, C_2, \ldots, C_n$ following the trailer T encoding their relative order on the street after they are done parking using the preference sequence $\mathbf{c}$.

For example, in Figure 1, the final parking configuration of $\mathbf{c} = (3, 7, 5, 3)$ is T, C_1, C_3, C_2, C_4.

The following inequalities analogous to (1) give a necessary condition for being a parking sequence.

Lemma 2.1. *Suppose* $\mathbf{c} = (c_1, ..., c_n) \in \mathsf{PS}(\vec{y}; z)$ *where* $\vec{y} = (y_1, ..., y_n)$. *Then* $\#\{j \in [n] : c_j \leq z\} \geq 1$ *and for each* $1 \leq t \leq n-1$,

$$\#\{j : c_j \leq z + \sum_{i=0}^{t-1} y_{(n-i)}\} \geq t + 1. \tag{4}$$

Proof. We have $\#\{j : c_j \leq z\} \geq 1$ because otherwise, there is no car whose preference is less than or equal to z, thus no car parks on spot z and we obtain a contradiction. Suppose for some $t \in [1, n-1]$, $\#\{j : c_j \leq z + \sum_{i=0}^{t-1} y_{(n-i)}\} \leq t$. Then in the final parking configuration on spots $[1, z + \sum_{i=0}^{t-1} y_{(n-i)}]$, there are at most 1 trailer and t cars occupying a total of at most $z - 1 + y_{(n)} + y_{(n-1)} + \cdots + y_{(n-t+1)}$ spots. Thus, not all spots are used in the final parking configuration and this contradicts the fact that $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$. □

Corollary 2.2. *Let* $\mathbf{c} = (c_1, ..., c_n) \in \mathsf{PS}(\vec{y}; z)$ *where* $\vec{y} = (y_1, ..., y_n)$. *Then* $c_{(1)} \leq z$ *and for* $j = 2, \ldots, n$,

$$c_{(j)} \leq z + \sum_{i=0}^{j-2} y_{(n-i)}. \tag{5}$$

We note that the conditions of Lemma 2.1 are not sufficient. Using the same example as before, even though $\mathbf{c} = (1, 2)$ and $\mathbf{c}' = (2, 1)$ both satisfy (4) for $\vec{y} = (2, 2)$, $\mathbf{c}' \notin \mathsf{PS}(\vec{y}; 1)$. In addition, for a parking sequence $\mathbf{c} \in PS(\vec{y}; z)$, its rearrangement $\mathbf{c}_{inc}$ is not necessarily a parking sequence. Consider the following example for $\vec{y} = (1, 1, 4)$ and $z = 1$. $\mathbf{c} = (5, 6, 1)$ is in $\mathsf{PS}(\vec{y}; z)$ but $\mathbf{c}_{inc} = (1, 5, 6)$ is not.

Definition 2.2. A sequence $\mathbf{c} = (c_1, ..., c_n) \in \mathsf{PS}(\vec{y}; z)$ is an *increasing parking sequence for* $(\vec{y}; z)$ if $c_1 \leq c_2 \leq \cdots \leq c_n$. We denote the set of all increasing parking sequences for $(\vec{y}; z)$ by $\mathsf{IPS}(\vec{y}; z)$.

When $\vec{y} = (1, 1, ..., 1)$ and $z = 1$ (i.e. the trailer of length 0), Definition 2.2 leads to the classical increasing parking functions, which are counted by the Catalan numbers. It is well-known that classical increasing parking functions of length n are in one-to-one correspondence with Dyck paths of semilength n, which are lattice paths from $(0,0)$ to (n,n) that lie strictly above and not touching the line $y = x - 1$. This result can be generalized to increasing parking sequences.

First we show that an analog of (1) is enough to characterize increasing parking sequences.

Proposition 2.3. *Let* $(\vec{y}; z) = (y_1, ..., y_n; z)$. *Then* $\mathbf{c} = (c_1, ..., c_n) \in \mathsf{IPS}(\vec{y}; z)$ *if and only if* $c_1 \le c_2 \le \cdots \le c_n$ *and for all* $i \in [n]$,

$$c_i \le z + \sum_{j=1}^{i-1} y_j. \tag{6}$$

Proof. Observe that if $\mathbf{c}$ is a non-decreasing preference sequence satisfying (6), then the cars will park in the final configuration $T, C_1, \ldots, C_n$. Hence $\mathbf{c}$ is in $\mathsf{IPS}(\vec{y}; z)$.

Conversely, for a non-decreasing sequence $\mathbf{c}$ that allows all the cars to park, we need to prove that it satisfies (6). First by Corollary 2.2, $c_1 \le z$. Thus, car C_1 parks right after the trailer leaving no gap. By the rules of the parking process, if $c_i \le c_{i+1}$ and both cars C_i and C_{i+1} are able to park, then C_{i+1} will park after C_i. Hence for a non-decreasing $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$, the final parking configuration must be $T, C_1, C_2, \ldots, C_n$. It follows that the first spot occupied by car C_i is $z + y_1 + \cdots + y_{i-1}$, which must be larger than or equal to c_i. □

Proposition 2.3 allows us to enumerate increasing parking sequences for any given length vector $\vec{y}$ and $z \in \mathbb{Z}_+$ using results in lattice path counting. Recall that a *lattice path* from $(0,0)$ to (p,q) is a sequence of p east steps and q north steps. It can be represented by a sequence of non-decreasing integers $(x_1, x_2, \ldots, x_q)$ such that the north steps are at $(x_i, i-1) \to (x_i, i)$, for $i = 1, \ldots, q$. The lattice path is said to have strict right boundary $(b_1, b_2, \ldots, b_q)$ if $0 \le x_i < b_i$ for all $1 \le i \le q$. Let $\mathsf{LP}_{p,q}(b_1, b_2, \ldots, b_q)$ denote the set of all lattice paths from $(0,0)$ to (p,q) with strict right boundary $(b_1, b_2, \ldots, b_q)$. Figure 2 shows an example of a lattice path (2,3,3,7) from (0,0) to (8,4) with strict right boundary $\vec{b} = (3, 4, 5, 8)$.

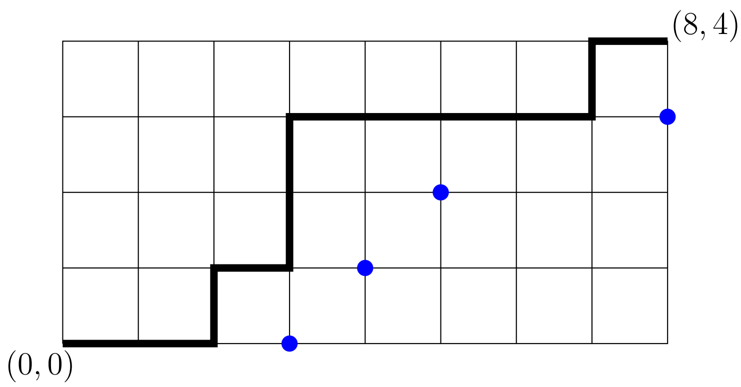

Figure 2: A lattice path (2,3,3,7) with strict right boundary at $(3, 4, 5, 8)$.

We can represent increasing parking sequences in terms of lattice paths with strict right boundary as follows: Let $(\vec{y}; z) = (y_1, ..., y_n; z)$ and $M = z - 1 + y_1 + y_2 + \cdots +$

$y_{n-1}+y_n$. By Proposition 2.3 there is a bijection from $\mathsf{IPS}(\vec{y};z)$ to the set of lattice paths from $(0,0)$ to (M,n) with strict right boundary $(z, z+y_1, z+y_1+y_2, \dots, z+y_1+y_2+\cdots+y_{n-1})$. (The boundary is strict because in the lattice path, x_i can be 0 while in $\mathbf{c}\in\mathsf{IPS}(\vec{y},z)$, $c_i\geq 1$.)

There are well-known determinant formulas to count the number of lattice paths with general boundaries, see, for example, Theorem 1 of [6, Chap.2], which leads to the following determinant formula.

Corollary 2.4. *Suppose $M = z-1+y_1+y_2+\cdots+y_{n-1}+y_n$. Then*

$$\begin{aligned}\#\mathsf{IPS}(\vec{y};z) &= \#\mathsf{LP}_{M,n}(z, z+y_1, z+y_1+y_2, \dots, z+y_1+y_2+\cdots+y_{n-1})\\ &= \det\left[\binom{b_i}{j-i+1}\right]_{1\leq i,j\leq n}\end{aligned}$$

where $b_1 = z$ and $b_i = z+y_1+y_2+\cdots+y_{i-1}$ for $i=2,\dots,n$.

For the special case that the length vector has constant entries, there are nicer closed formulae for the determinant. Specifically, when $\vec{y}=(k^n)=(k,k,\dots,k)$ and $M=z+kn-1$, $\mathsf{LP}_{M,n}(z, z+k, z+2k, \dots, z+(n-1)k)$ is the set of lattice paths from $(0,0)$ to $(z+kn-1,n)$ which never touch the line $x=z+ky$. Using the formula (1.11) of [6, Chap.1], we have

Corollary 2.5. *Suppose $(\vec{y},z)=((k^n);z)$ and $M=z+kn-1$. Then*

$$\begin{aligned}\#\mathsf{IPS}(\vec{y};z) &= \#\mathsf{LP}_{M,n}(z, z+k, z+2k, \dots, z+(n-1)k)\\ &= \frac{z}{z+n(k+1)}\binom{z+n(k+1)}{n}.\end{aligned}$$

This specializes to the Fuss-Catalan numbers when $z=1$.

Corollary 2.6. *Suppose $\vec{y}=(k,k,\dots,k)\in\mathbb{Z}_+^n$. Then*

$$\#\mathsf{IPS}(\vec{y};1) = \frac{1}{kn+1}\binom{(k+1)n}{n}.$$

3 Invariance for Fixed Length Vectors

In this section, we study the first of two types of invariance for parking sequences. Fixing the length vector $\vec{y}\in\mathbb{Z}_+^n$ and a positive integer z, we investigate which parking sequence remains in the set $\mathsf{PS}(\vec{y};z)$ after its entries are arbitrarily rearranged.

Definition 3.1. Fix $\vec{y}=(y_1,...,y_n)$ and $z\in\mathbb{Z}_+$. Let $\mathbf{c}\in\mathsf{PS}(\vec{y};z)$. We say that $\mathbf{c}$ is a *permutation-invariant parking sequence for* $(\vec{y};z)$ if for any rearrangement $\mathbf{c}'$ of $\mathbf{c}$, we have $\mathbf{c}'\in\mathsf{PS}(\vec{y};z)$. We denote the set of all permutation-invariant parking sequences for $(\vec{y};z)$ by $\mathsf{PS}_{inv}(\vec{y};z)$.

For example, for $\vec{y}=(1,2)$ and $z=1$, we have $\mathsf{PS}(\vec{y};1)=\{(1,1),(1,2),(3,1)\}$ and $\mathsf{PS}_{inv}(\vec{y};z)=\{(1,1)\}$. First, we describe a subset of the invariant parking sequences.

Proposition 3.1. *For any* $\mathbf{c} = (c_1, \ldots, c_n) \in [z]^n$*, we have* $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$*.*

Proof. For any preference sequence $\mathbf{c}$, if $c_i \leq z$ for all i, then we obtain the final parking configuration $T, C_1, C_2, \ldots, C_n$, which means $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$. Since the condition $c_i \in [z]$ for all i does not depend on the order of c_i, we have that $\mathbf{c}$ is permutation-invariant. □

In general, $\mathsf{PS}_{inv}(\vec{y}; z)$ is larger than the set $[z]^n$, and the situation can be more complicated. The following two examples show that $\mathsf{PS}_{inv}(\vec{y}; z)$ depends not only on the relative order of the y_i's, but also on the difference of y_i's.

Example 1. Let $\vec{y} = (y_1, y_2)$ and $z = 1$. If $y_1 < y_2$, then $\mathsf{PS}_{inv}(\vec{y}; 1) = \{(1, 1)\}$. On the other hand, if $y_1 \geq y_2$, we have $\mathsf{PS}_{inv}(\vec{y}; 1) = \{(1, 1), (1, y_2 + 1), (y_2 + 1, 1)\}$.

Example 2. Suppose $\vec{y} = (4, 3, 2)$ and $\vec{t} = (4, 3, 1)$. It is easy to check that

$$\mathsf{PS}_{inv}(\vec{y}; 1) = \{(1, 1, 1), (1, 1, 4), (1, 4, 1), (4, 1, 1)\}$$

and

$$\mathsf{PS}_{inv}(\vec{t}; 1) = \{(1, 1, 1), (1, 1, 4), (1, 4, 1), (4, 1, 1), (1, 1, 5), (1, 5, 1), (5, 1, 1)\}.$$

Note that the relative orders for the vectors $\vec{y}$ and $\vec{t}$ are the same, (both have the pattern 321), but the invariant sets are not similar.

In the following we characterize the invariant set for some families of $\vec{y}$. First, we consider the case where the length vector is strictly increasing. Next, we look at the case where $\vec{y}$ is a constant sequence. Lastly, given $a, b \in \mathbb{Z}_+$, we consider the case where the length vector is of the form $\vec{y} = (a, \ldots, a, b, \ldots, b)$, where $a < b$.

3.1 Strictly increasing length vector

When $\vec{y}$ is a strictly increasing sequence, we show that Proposition 3.1 gives all the permutation-invariant parking sequences.

Theorem 3.2. *Let* $(\vec{y}; z) = (y_1, y_2, ..., y_n; z)$ *where* $y_1 < y_2 < \cdots < y_n$*. Then*

$$\mathsf{PS}_{inv}(\vec{y}; z) = [z]^n.$$

Proof. By Proposition 3.1, $[z]^n \subseteq \mathsf{PS}_{inv}(\vec{y}; z)$. Conversely, suppose $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ is a parking sequence for $(\vec{y}; z)$ with some $c_i \notin [z]$. We claim that $\mathbf{c}$ is not permutation-invariant. To see this, let $x = \min\{c_i \in \mathbf{c} \mid c_i > z\}$. Then we can consider $\mathbf{c}_{inc} = (c_{(1)}, c_{(2)}, \ldots, c_{(r)}, x, c_{(r+2)}, \ldots, c_{(n)})$, where $c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(r)} \leq z < x \leq c_{(r+2)} \leq \cdots \leq c_{(n)}$ and $r \geq 1$ by Corollary 2.2. Then by Proposition 2.3, x satisfies the inequality: $z < x \leq z + \sum_{i=1}^{r} y_i$. Thus, we can choose the maximum s such that $x > z + \sum_{i=1}^{s} y_i$, where $0 \leq s < r$. Consider the preference

$$\mathbf{c}' = (c_{(1)}, c_{(2)}, \ldots, c_{(s)}, x, c_{(s+1)}, \ldots, c_{(r)}, c_{(r+2)}, \ldots, c_{(n)})$$

We try to park according to $\mathbf{c}'$. Clearly, the first s cars park in order after the trailer T without any gaps in between them. Then the car C_{s+1} has preference x and parks after car C_s with h unoccupied spots in between C_s and C_{s+1}, where $h = x - (z + \sum_{i=1}^{s} y_i) \geq 1$ and $h \leq y_{s+1}$ by the maximality of s. Among the unparked cars $C_{s+2}, \ldots, C_n$, the minimal length is y_{s+2}, where $y_{s+2} > y_{s+1} \geq h$. Thus no car can fill in these h unoccupied spots. It follows that $\mathbf{c}' \notin \mathsf{PS}(\vec{y}; z)$, and hence $\mathbf{c} \notin \mathsf{PS}_{inv}(\vec{y}; z)$. ☐

Corollary 3.3. *Let $(\vec{y}; z) = (y_1, y_2, ..., y_n; z)$ where $y_1 < y_2 < \cdots < y_n$. Then*

$$\#\mathsf{PS}_{inv}(\vec{y}; z) = z^n.$$

3.2 Constant length vector

In this subsection, we investigate the case where $\vec{y}$ is of the form $(k^n) = (k, k, ..., k)$.

Theorem 3.4. *Suppose $(\vec{y}; z) = ((k^n); z)$ where $k \in \mathbb{Z}_+$ and $k > 1$. Then $\mathsf{PS}_{inv}(\vec{y}; z)$ is the set of all sequences $(c_1, \ldots, c_n)$ such that for each $1 \leq i \leq n$,*

(i) $c_{(i)} \leq z + (i-1)k$, *and*

(ii) $c_i \in \{1, 2, \ldots, z, z+k, z+2k, \ldots, z+(n-1)k\}$.

Proof. Let $\mathbf{c} = (c_1, \ldots, c_n)$ be a sequence such that for each $1 \leq i \leq n$, $c_{(i)} \leq z + (i-1)k$, and $c_i \leq z$ or $c_i = z + sk$ for some $s = 0, 1, \ldots, n-1$. We claim that $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$. Since these conditions are independent of the arrangement of the terms c_i's, this implies $\mathbf{c}$ is permutation-invariant.

We attempt to park using $\mathbf{c}$. First, C_1 either parks right after the trailer if $c_1 \leq z$, or on spots $[c_1, c_1 + k - 1]$ if $c_1 = z + sk$ for some $s \geq 0$. We assume for our inductive hypothesis, that the first r cars are parked already, (where $1 \leq r \leq n-1$). At this stage in the parking process, any car already parked on the street occupies spots of the form $[z + ks, z + k(s+1) - 1]$ where $s \in \{0, 1, \ldots, n-1\}$. This means that for any maximal interval of unoccupied spots, the length is a multiple of k and the interval starts at $z + km$ for some $m \in \{0, 1, \ldots, n-1\}$.

Now car C_{r+1} comes in with preference $c_{r+1} \leq z$ or $c_{r+1} = z + kl$. If $c_{r+1} \leq z$, clearly C_{r+1} can park. If $c_{r+1} = z + kl$ for some $l \geq 1$, then there are two possibilities:

- if spot $(z + kl)$ is empty, then C_{r+1} parks on spots $[z + kl, z + k(l+1) - 1]$.
- if spot $(z + kl)$ is non-empty, then C_{r+1} drives forward to park in the first open interval ahead. Such an open interval must exist. Otherwise, assume that the last open spot after C_r parked is x. By the inductive hypothesis, $x = z + sk - 1$ for some $s < l$. So all the spots from $x + 1$ to the end of the street $(z + nk - 1)$ are occupied, by $n - s$ cars. These $n - s$ cars, as well as C_{r+1}, all have preference greater than x. In other words, there are at least $n - s + 1$ cars having preference $c_i \geq z + sk$, which implies $c_{(s)} \geq z + sk$, a contradiction.

This exhausts all possible cases for C_{r+1}. Thus, by induction, all cars can park and $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$.

Conversely, suppose for a contradiction that there is a parking sequence $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$ not satisfying Condition (ii). (By Corollary 2.2 Condition (i) holds for all $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$.) Then there is some $j \in [n]$ such that $c_j = z + sk + t$ for some $s \in \{0, 1, \ldots, n-1\}$ and $1 \le t < k$. Consider the following rearrangement of $\mathbf{c}$ given by $\mathbf{c}' = (c_j, c_1, c_2, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n)$. By our assumption, $\mathbf{c}' \in \mathsf{PS}(\vec{y}; z)$. We attempt to park using this preference. First, C_1 parks on $[z + sk + t, z + (s+1)k + t - 1]$. However, between the trailer and C_1, there is now an unoccupied interval of $(z + sk + t) - z = sk + t$ spots, which is clearly nonempty and not a multiple of k. Thus, no matter what preferences the remaining cars have, it is impossible to park all cars on this street. This yields a contradiction to our assumption. ☐

Recall that a $\vec{u}$-parking function of length n is an integer sequence $(x_1, x_2, \ldots, x_n)$ satisfying $1 \le x_{(i)} \le u_i$. We can use the results of vector parking functions to enumerate the number of sequences described in Theorem 3.4.

Corollary 3.5. *Let $(\vec{y}; z) = ((k^n); z)$ with $k \ge 2$. Then*

$$\#\mathsf{PS}_{inv}(\vec{y}; z) = z(n+z)^{n-1}.$$

Proof. For any $\mathbf{c} = (c_1, \ldots, c_n) \in \mathsf{PS}_{inv}(\vec{y}; z)$ let $\mathsf{f}(\mathbf{c}) = \mathbf{c}'$, where $\mathbf{c}'$ is the sequence whose entries are given by

$$c_i' = \begin{cases} c_i, & \text{if } 1 \le c_i \le z \\ z + s, & \text{if } c_i = z + sk. \end{cases}$$

The condition $c_{(i)} \le z + (i-1)k$ implies $c'_{(i)} \le z + i - 1$, hence $\mathbf{c}'$ is a vector parking function associated to the vector $\vec{u} = (z, z+1, \ldots, z+n-1)$, and f is a map from $\mathsf{PS}_{inv}(\vec{y}; z)$ to $\mathsf{PF}_n(\vec{u})$. It is clear that f is a bijection since the map can be easily inverted. By [5, Corollary 5.5], the number of $\vec{u}$-parking functions is $z(z+n)^{n-1}$. ☐

Remark. Note that Theorem 3.4 and Corollary 3.5 hold trivially for $k = 1$, where the condition (ii) in Theorem 3.4 is redundant. In fact, when $k = 1$ or $n = 1$, $\mathsf{PS}_{inv}((k^n); z)$ is the same as the set $\mathsf{PS}((k^n); z)$, which is exactly the set of $\vec{u}$-parking functions associated to $\vec{u} = (z, z+1, \ldots, z+n-1)$. For $k, n \ge 2$, $\mathsf{PS}_{inv}((k^n); z)$ is a proper subset of $\mathsf{PS}((k^n); z)$.

3.3 Length vector $\vec{y} = (a, \ldots, a, b, \ldots, b)$ where $a < b$

Let $n \ge 2$ and z, a, b, r be positive integers with $a < b$ and $1 \le r < n$. In this subsection we fix $\vec{y} = (\underbrace{a, a, \ldots, a}_{r}, \underbrace{b, \ldots, b}_{n-r}) = (a^r, b^{n-r})$, i.e. the first r cars are of size a and the remaining $n - r$ cars are of size b. First we prove a couple of lemmas that characterize the set of permutation-invariant parking sequences for $(\vec{y}; z)$. In the following we will refer to any car of size a (respectively, size b) as an A-car (respectively, B-car).

Lemma 3.6. *Assume* $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$*. Then in the final parking configuration of* $\mathbf{c}$*, all* A*-cars park in* $[z, z + ra - 1]$*.*

Proof. Suppose not. Then there is some $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$ with which at least one A-car is not parked in the interval $[z, z + ra - 1]$ in the final parking configuration $\mathcal{F}$. Removing the trailer T and all the A-cars from $\mathcal{F}$, we are left with some blocks of consecutive spots occupied by B-cars. Assume the first such block L consists of lb spots, where $1 \leq l \leq n-r$. Let C_j be the last A-car in the configuration $\mathcal{F}$. Then C_j occupies some spots in $[z+ra, z+ra+(n-r)b-1]$, and no other A-car has checked the spots C_j occupied in the parking process. In addition, let C_k be the first B-car in $\mathcal{F}$. Then $j \leq r < k$ and C_k parks before C_j in $\mathcal{F}$. Let $\mathbf{c}'$ be any permutation of $\mathbf{c}$ in which the first r terms are $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_r, c_k$, and the $(r + 1)$-th term is c_j. Note that the first r terms appear in the same order in both $\mathbf{c}$ and $\mathbf{c}'$. We have the following two cases.

C1: Assume in $\mathcal{F}$ there are some other A-cars parked between C_k and C_j. First let the A-cars park according to the preference $\mathbf{c}'$. It is easy to see that these A-cars occupy the same spots as in $\mathcal{F}$ except that

(a) there is no A-car taking the same spots as C_j in $\mathcal{F}$, and

(b) In the b spots originally occupied by C_k in $\mathcal{F}$, there is now an A-car taking the first a spots, leaving $(b - a)$ of these spots unoccupied.

Hence after all the A-cars are parked, the first block of consecutive open spots has size $lb - a$, which is not a multiple of b. Thus it is impossible for all the remaining B-cars to park and hence $\mathbf{c}' \notin PS(\vec{y}; z)$.

C2: There is no A-car parked between C_k and C_j in $\mathcal{F}$. Then $\mathcal{F}$ is of the form $A \cdots AB \cdots BAB \cdots B$, where there are $r - 1$ A-cars before the first B-car C_k, and $c_j = z + (r - 1)a + lb$. Let the cars park according to $\mathbf{c}'$. Then the A-cars all park successfully, occupying the spots $[z, z + ra - 1]$, and the first B-car occupies spots $[c_j, c_j + b - 1]$. Now there are $c_j - 1 - (z - 1 + ra) = lb - a$ spots between the last A-car and the first B-car; these spots cannot be filled by other B-cars. Hence $\mathbf{c}' \notin \mathsf{PS}(\vec{y}; z)$.

In both cases we have a permutation of $\mathbf{c}$ that is not in $\mathsf{PS}(\vec{y}; z)$, contradicting the assumption that $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$. ☐

Lemma 3.7. *If* $(c_1, c_2, \ldots, c_n) \in \mathsf{PS}_{inv}(\vec{y}; z)$*, then* $c_i \leq z+(r-1)a$ *for all* $1 \leq i \leq n$*.*

Proof. Suppose not. Take any permutation of $\mathbf{c}$ starting with $\max\{c_i : i \in [n]\}$ and we contradict the conclusion of Lemma 3.6. ☐

Proposition 3.8. *For* $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$*, let* $c_{(1)} \leq c_{(2)} \leq \cdots \leq c_{(n)}$ *be the order statistics of* $\mathbf{c}$*. Then* $\mathsf{PS}_{inv}(\vec{y}; z)$ *is exactly the set of all sequences* $\mathbf{c}$ *whose order statistics satisfy* $c_{(i)} \leq z$ *for each* $1 \leq i \leq n-r+1$ *and* $c_{(n-r+j)} \in \{1, \ldots, z, z+a, z+2a, \ldots, z + (j - 1)a\}$ *for each* $2 \leq j \leq r$*.*

Proof. By Lemmas 3.6 and 3.7, if $\mathbf{c} \in \mathsf{PS}_{inv}(\vec{y}; z)$, then any r-term subsequence of $\mathbf{c}$, say $(c_{i_1}, c_{i_2}, \dots, c_{i_r})$, parks all r A-cars in $[z, z+ra-1]$ and hence $c_i \le z+(r-1)a$ for all $i = 1, \dots, n$. Furthermore, if we consider the last r terms of the order statistics of $\mathbf{c}$, this means $(c_{(n-r+1)}, c_{(n-r+2)}, \dots, c_{(n)}) \in \mathsf{PS}_{inv}((a, a, \dots, a); z)$. By Theorem 3.4, we obtain $c_{(n-r+j)} \in \{1, \dots, z, z+a, z+2a, \dots, z+(j-1)a\}$ for each $1 \le j \le r$. Finally by the order statistics, $c_{(i)} \le c_{(n-r+1)} \le z$ for each $1 \le i \le n-r$.

Conversely, we claim that any $\mathbf{c}$ satisfying the inequalities in Lemma 3.8 is in $\mathsf{PS}_{inv}(\vec{y}; z)$. To see this, consider first the A-cars with preferences $(c_1, \dots, c_r)$. We have $c_i \in \{1, 2, \dots, z, z+a, z+2a, \dots, z+(r-1)a\}$ for all $1 \le i \le r$, and the order statistics of these r terms are no more than $(z, z+a, \dots, z+(r-1)a)$ (coordinate-wise). By Theorem 3.4, $(c_1, \dots, c_r)$ is a parking sequence for $((a^r); z)$. Hence all A-cars must park on $[z, z+ra-1]$. Next, consider the B-cars. Since $c_i \le z+(r-1)a$ and all A-cars are parked without any unoccupied spots on $[z, z+ra-1]$, then all B-cars park in increasing order after the A-cars. In other words, the final parking configuration is $T, C'_1, \dots, C'_r, C_{r+1}, \dots, C_n$ where $C'_1, \dots, C'_r$ is some rearrangement of the A-cars. $\square$

Combining Lemmas 3.6, 3.7 and Proposition 3.8, we prove the following result.

Theorem 3.9. *Let $n \ge 2$ and z, a, b, r be positive integers with $a < b$ and $1 \le r < n$. Assume $\vec{y} = (a^r, b^{n-r})$. Let $\mathsf{PF}_n(\vec{u})$ be the set of $\vec{u}$-parking functions of length n where $\vec{u} = (\underbrace{z, z, \dots, z}_{n-r+1}, z+1, z+2, \dots, z+r-1)$. Then there is a bijection between the sets $\mathsf{PS}_{inv}(\vec{y}; z)$ and $\mathsf{PF}_n(\vec{u})$.*

Proof. Let $\vec{u} = (u_1, u_2, \dots, u_n) = (z, z, \dots, z, z+1, z+2, \dots, z+r-1)$, where there are $n-r+1$ copies of z in the vector $\vec{u}$. Consider the map $\gamma_a : \mathsf{PS}_{inv}(\vec{y}; z) \to \mathsf{PF}_n(\vec{u})$ defined as follows.

$$\gamma_a : (c_1, \dots, c_n) \mapsto (c'_1, \dots, c'_n) = \mathbf{c}'$$

where for all $1 \le j \le n$

$$c'_j = \begin{cases} c_j, & \text{if } c_j \le z \\ z+s, & \text{if } c_j = z+sa. \end{cases}$$

The map γ_a is well-defined since the sequence $\mathbf{c}'$ has order statistics satisfying $1 \le c'_{(i)} \le u_i$ for each $i = 1, 2, \dots, n$. Thus $\mathbf{c}' \in \mathsf{PF}_n(\vec{u})$. Clearly the map γ_a is invertible, hence γ_a is a bijection. $\square$

Corollary 3.10. *Let $\vec{y}$ and $\vec{u}$ be as in Theorem 3.9. Then*

$$\begin{aligned} \#\mathsf{PS}_{inv}(\vec{y}; z) &= \#\mathsf{PF}_n(\vec{u}) \\ &= \sum_{j=0}^{r-1} \binom{n}{j} (r-j)\, r^{j-1} z^{n-j}. \end{aligned}$$

In particular, when $r = 1$, $\#\mathsf{PS}_{inv}(a, b, b, \dots, b; z) = z^n$.

Proof. The result follows from Theorem 3.9 and [8, Theorem 3]. □

It is natural to ask what happens for the length vector $\vec{y} = (a^r, b^{n-r})$ with $a > b$. Unlike before, the number of sequences in $\mathsf{PS}_{inv}((a^r, b^{n-r}); z)$ with $a > b$ depends not only on r and z, but also on the values of a and b. We do not have a complete characterization yet, except some minor results for special cases.

4 Invariance for the Set of Car Lengths

In this section, we study another type of invariance. Given a fixed set of cars of various lengths and a one-way street whose length is equal to the sum of the car lengths and a trailer's length $z-1$, we consider the parking sequences for which all n cars can park on the street irrespective of the order in which they enter the street. Denote by $\mathfrak{S}_n$ the set of all permutations on n letters. For a vector $\vec{y}$ and σ in $\mathfrak{S}_n$, let $\sigma(\vec{y}) = (y_{\sigma(1)}, \dots, y_{\sigma(n)})$.

Definition 4.1. Let $\mathbf{c} = (c_1, ..., c_n)$ and $\vec{y} = (y_1, ..., y_n)$. Then $\mathbf{c}$ is a *strong parking sequence for* $(\vec{y}; z)$ if and only if

$$\mathbf{c} \in \bigcap_{\sigma \in \mathfrak{S}_n} \mathsf{PS}(\sigma(\vec{y}); z).$$

We will denote the set of all strong parking sequences for $(\vec{y}; z)$ by $\mathsf{SPS}\{\vec{y}; z\}$, or equivalently, $\mathsf{SPS}\{\vec{y}_{inc}; z\}$.

Example 3. For the case $n = 2$, let $a, b \in \mathbb{Z}_+$ with $a < b$. It is easy to see that

$$\begin{aligned} \mathsf{PS}((a, b); z) &= [z] \times [z + a] \cup \{(c_1, c_2) : c_1 = z + b, 1 \le c_2 \le z\}, \\ \mathsf{PS}((b, a); z) &= [z] \times [z + b] \cup \{(c_1, c_2) : c_1 = z + a, 1 \le c_2 \le z\}. \end{aligned}$$

This gives

$$\mathsf{SPS}\{(a, b); z\} = \mathsf{PS}((a, b); z) \cap \mathsf{PS}((b, a); z) = [z] \times [z + a].$$

Note that $\mathsf{SPS}\{(a, b); z\}$ is exactly the set of all preferences $\mathbf{c} \in \mathsf{PS}((a, b); z)$ that yields the final parking configuration T, C_1, C_2.

By Ehrenborg and Happ's result (3), we know that if $\vec{y} = (k^n)$, then

$$\#\mathsf{SPS}\{\vec{y}; z\} = \#\mathsf{PS}(\vec{y}; z) = z \cdot \prod_{i=1}^{n-1} (z + ik + n - i).$$

In the following we consider the case where $\vec{y}$ does not have constant entries.

Definition 4.2. We say that $\mathbf{c} \in \mathsf{PS}(\vec{y}; z)$ parks $\vec{y}$ in the *standard order* if the final parking configuration of $\mathbf{c}$ is given by $T, C_1, C_2, \dots, C_n$.

The following lemma is easily proved by induction.

Lemma 4.1. *Let* $\mathbf{c} = (c_1, c_2, \ldots, c_n) \in \mathsf{PS}(\vec{y}; z)$. *Then* $\mathbf{c}$ *parks* $\vec{y}$ *in the standard order if and only if*

$$c_k \leq z + y_1 + \cdots + y_{k-1} \text{ for all } k \in [n].$$

The following result characterizes strong parking sequences for any set of $n \geq 2$ cars with a given length vector $\{y_1, y_2, \ldots, y_n\}$ and a trailer T of length $z - 1$.

Theorem 4.2. *Let* $n \geq 2$. *Assume that* $\vec{y} = (y_1, \ldots, y_n)$ *is not a constant sequence. Then* $\mathbf{c}$ *is a strong parking sequence for* $\{\vec{y}; z\}$ *if and only if* $\mathbf{c}$ *parks* $\vec{y}_{inc} = (y_{(1)}, y_{(2)}, ..., y_{(n)})$ *in the standard order.*

Proof. Suppose $\mathbf{c}$ parks $\vec{y}_{inc}$ in the standard order. We need to check that $\mathbf{c}$ is a parking sequence for $(\sigma(\vec{y}); z)$ for every $\sigma \in \mathfrak{S}_n$. This follows from Lemma 4.1 and the fact that $y_{(1)} + y_{(2)} + \cdots + y_{(i)} \leq y_{\sigma(1)} + y_{\sigma(2)} + \cdots + y_{\sigma(i)}$ for any $\sigma \in \mathfrak{S}_n$ and $i \in [n]$.

Conversely, let $\mathbf{c}$ be a parking sequence for $(\vec{y}_{inc}; z)$ that does not park $\vec{y}_{inc}$ in the standard order. We will construct a permutation σ such that for a sequence of cars with length vector $\sigma(\vec{y}_{inc})$, $\mathbf{c} \notin \mathsf{PS}(\sigma(\vec{y}_{inc}); z)$. In the following, let C_i represent a car of length $y_{(i)}$, as listed in the table below. Let $\mathcal{F}$ be the final parking configuration of $\mathbf{c}$ when we park the cars $C_1, \ldots, C_n$. In $\mathbf{c}$, let k_1 be the minimal index k such that $c_k > z + y_{(1)} + y_{(2)} + \cdots + y_{(k-1)}$. Then in $\mathcal{F}$ the trailer is followed by $C_1, \ldots, C_{k_1-1}$ with no gap, but there is a gap between C_{k_1-1} and C_{k_1}. Let C_t be the last car that parks right before C_{k_1} in $\mathcal{F}$. Clearly $t > k_1$. We consider the following cases.

Car	C_1	C_2	$\cdots$	C_{k_1}	$\cdots$	C_t	$\cdots$	C_{n-1}	C_n
Car Length	$y_{(1)}$	$y_{(2)}$	$\cdots$	$y_{(k_1)}$	$\cdots$	$y_{(t)}$	$\cdots$	$y_{(n-1)}$	$y_{(n)}$

C1. Assume $y_{(k_1)} < y_{(t)}$. Let σ_1 be the transposition $(k_1 \longleftrightarrow t)$. For each $i \in [n]$ let D_i represent a car of length $y_{\sigma_1(i)}$, as shown below.

σ_1	Car	D_1	D_2	$\cdots$	D_{k_1}	$\cdots$	D_t	$\cdots$	D_{n-1}	D_n
	Car Length	$y_{(1)}$	$y_{(2)}$	$\cdots$	$y_{(t)}$	$\cdots$	$y_{(k_1)}$	$\cdots$	$y_{(n-1)}$	$y_{(n)}$

We park cars $D_1, \ldots, D_n$ using the preference sequence $\mathbf{c}$. If $\mathbf{c} \in \mathsf{PS}(\sigma_1(\vec{y}_{inc}); z)$, then $D_1, \ldots, D_t$ are able to park and

(a) $D_1, D_2, \ldots, D_{k_1-1}$ have the same lengths and preferences as $C_1, C_2, \ldots, C_{k_1-1}$. Hence they park in order right after the trailer with no gaps.

(b) D_{k_1} is longer than C_{k_1} and occupies spots in $[c_{k_1}, c_{k_1} + y_{(t)} - 1]$.

(c) Any car D_i for $i \in \{k_1 + 1, \ldots, t - 1\}$ has the same preference as C_i so it parks either before D_{k_1} and in the same spots as C_i in $\mathcal{F}$, or parks after D_{k_1}.

(d) D_t takes the first $y_{(k_1)}$ spots of the ones occupied by C_t in $\mathcal{F}$.

After parking $D_1, \ldots, D_t$, there are $y_{(t)} - y_{(k_1)}$ unused spots between cars D_t and D_{k_1}. Any car trying to park after D_t has length $\geq y_{(t)} > y_{(t)} - y_{(k_1)}$. So the spots between D_t and D_{k_1} cannot be filled and hence $\mathbf{c} \notin \mathsf{PS}(\sigma_1(\vec{y}_{inc}), z)$.

C2. Assume $y_{(k_1)} = y_{(t)}$. Then since $\vec{y}_{inc}$ is not a constant sequence, either $y_{(1)} < y_{(k_1)}$ or $y_{(t)} < y_{(n)}$.

C2a: Assume $y_{(t)} < y_{(n)}$. Let σ_2 be the transposition $(t \longleftrightarrow n)$ and E_i be a car of length $\sigma_2(i)$ for each $i \in [n]$.

σ_2	Car	E_1	E_2	$\cdots$	E_{k_1}	$\cdots$	E_t	$\cdots$	E_{n-1}	E_n
	Car Length	$y_{(1)}$	$y_{(2)}$	$\cdots$	$y_{(k_1)}$	$\cdots$	$y_{(n)}$	$\cdots$	$y_{(n-1)}$	$y_{(t)}$

We park cars $E_1, \ldots, E_n$ using the preference sequence $\mathbf{c}$. The cars $E_1, \ldots, E_{t-1}$ take the same spots as $C_1, \ldots, C_{t-1}$ in $\mathcal{F}$. Next, car E_t tries to park in the spots C_t occupies, at the interval $[c_{k_1} - y_{(t)}, c_{k_1} - 1]$, where the spot c_{k_1} is already occupied by E_{k_1}. But E_t has length $y_{(n)} > y_{(t)}$ and hence cannot fit. Therefore, $\mathbf{c} \notin \mathsf{PS}(\sigma_2(\vec{y}_{inc}); z)$.

C2b: If $y_{(k_1)} = \cdots = y_{(t)} = \cdots = y_{(n)} = b$, then we must have $k_1 > 1$ and $y_{(1)} < y_{(k_1)}$. In the final configuration $\mathcal{F}$, at the time car C_{k_1} is parked, the lengths of all the intervals of consecutive empty spots left are multiples of b. Let σ_3 be the transposition $(1 \longleftrightarrow k_1)$ and F_i be a car of length $\sigma_3(i)$ for each $i \in [n]$.

σ_3	Car	F_1	F_2	$\cdots$	F_{k_1}	$\cdots$	F_t	$\cdots$	F_{n-1}	F_n
	Car Length	$y_{(k_1)}$	$y_{(2)}$	$\cdots$	$y_{(1)}$	$\cdots$	$y_{(t)}$	$\cdots$	$y_{(n-1)}$	$y_{(n)}$

We park cars $F_1, \ldots, F_n$ using the preference sequence $\mathbf{c}$. The cars $F_1, \ldots, F_{k_1-1}$ will take the spaces right after the trailer. The total length of $F_1, \ldots, F_{k_1-1}$ is no more than the total length of $C_1, \ldots, C_{k_1-1}$, and C_t, since $y_{(1)} + y_{(2)} + \cdots + y_{(k_1-1)} + y_{(t)} > y_{(2)} + \cdots + y_{(k_1-1)} + y_{(k_1)}$. So F_{k_1} will park at the spot starting at c_{k_1}, just as C_{k_1}. But, as $y_{(1)} < y_{(k_1)}$, after F_{k_1} is parked, the available space after F_{k_1} is nonempty and not a multiple of b, while all the remaining cars are of length b. Hence, it is not possible to park all of them and $\mathbf{c} \notin \mathsf{PS}(\sigma_3(\vec{y}_{inc}); z)$.

□

Combining Lemma 4.1 and Theorem 4.2, we obtain the following counting formula.

Corollary 4.3. *Let $z \in \mathbb{Z}_+$ and $\vec{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{Z}_+^n$. If $\vec{y} \neq (s^n)$ for any integer s, then*

$$\#\mathsf{SPS}\{\vec{y}; z\} = z \cdot \prod_{i=1}^{n-1} (z + y_{(1)} + \cdots + y_{(i)}).$$

where $y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)}$ is the order statistics of $\vec{y}$.

5 Closing Remarks

In this paper, we have studied increasing parking sequences and their connections with lattice paths. We have also studied permutation-invariant parking sequences and length-invariant parking sequences. More precisely, we have characterized the permutation-invariant parking sequences for some special families of length vectors. While it may not be easy to find a general formula for all cases, a natural direction to go would be to investigate other special cases of car lengths. Furthermore, in the study of parking functions we encounter quite a number of other mathematical structures including trees, non-crossing partitions, hyperplane arrangements, polytopes etc. It will be interesting to investigate whether there is anything that connects other combinatorial structures to invariant parking sequences. Recently, in [1], parking sequences were extended to the case in which one or more trailers are placed anywhere on the street alongside n cars with length vector $\vec{y} = (1, 1, \ldots, 1)$. A natural generalization is to consider a similar scenario where $\vec{y}$ is any length vector.

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