Some constraints on the missing Moore graph

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Abstract

We derive some constraints on the structure of the missing Moore graph.

1 Moore graphs

A Moore graph $\Gamma_k$ is a regular graph of degree $k$ with the property that every pair of adjacent vertices has no common neighbor, and every pair of non-adjacent vertices has precisely one common neighbor. Equivalently, the graph has diameter two and girth five. By simple counting, it follows that the adjacency matrix $X_k$ of a Moore graph is a $v \times v$ matrix with $v = k^2 + 1$ satisfying

$$X_k^2 + X_k = (k-1)I_v + J_v,$$

where $I_v$ (respectively, $J_v$) is the $v \times v$ identity matrix (respectively, all-ones matrix). Using spectral techniques, Hoffman and Singleton showed [9] that the only possibilities are $k = 2$ (the 5-cycle), $k = 3$ (the Petersen graph), $k = 7$ (the Hoffman-Singleton graph), and $k = 57$ (called the ‘missing Moore graph’, because no one knows whether it exists or not).\footnote{For more about Moore graphs, see, e.g., Godsil and Royle [7], Miller and Širáň [12], and Dalfó [5].}

Despite much effort on the problem the existence question remains undecided. Higman (unpublished, see [4], Proposition 5.4 or [3], Proposition 11.5.2), extending earlier work of Aschbacher [1], showed that, if $\Gamma_{57}$ exists, it cannot be vertex-transitive. Mačaj and Širáň [10] and Makhnev and Paduchikh [11] have put limits on the possible size of the automorphism group of $\Gamma_{57}$. In this note we exhibit some constraints on the possible structure of the adjacency matrix of $\Gamma_{57}$.

\footnote{The Smith form of the Laplacian matrix of $\Gamma_{57}$ was almost completely determined by Ducey [6].}
2 The Moore graphs for $k \in \{2, 3, 7\}$

Before investigating what can be said about $X_{57}$, we recall the “canonical constructions” for the adjacency matrices of the other Moore graphs. It turns out that the adjacency matrices of all the known Moore graphs can be built from the cyclic permutation matrix of size five:

$$P := \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}. $$

2.1 The 5-cycle

The adjacency matrix of the 5-cycle is just $X_2 := P + P^{-1}$. To see that this satisfies (1), we first note that $P^0 + P^1 + P^2 + P^3 + P^4 = J_5$, where $P^0 = I_5$. Thus, $X_2^2 + X_2 = P^2 + P^{-2} + 2I_5 + P + P^{-1} = I_5 + J_5$.

2.2 The Petersen graph

The adjacency matrix of the Petersen graph is $X_3 := \begin{pmatrix} P + P^{-1} & I_5 \\ I_5 & P^2 + P^{-2} \end{pmatrix}$. To see this, observe that

$$\left( \begin{array}{cc}
P + P^{-1} & I_5 \\
I_5 & P^2 + P^{-2}
\end{array} \right) \left( \begin{array}{cc}
P + P^{-1} & I_5 \\
I_5 & P^2 + P^{-2}
\end{array} \right) = \left( \begin{array}{cc}
P^2 + P^{-2} + 3I_5 & J_5 - I_5 \\
J_5 - I_5 & P + P^{-1} + 3I_5
\end{array} \right),$$

Hence,

$$X_3^2 + X_3 = \begin{pmatrix} J_5 + 2I_5 & J_5 \\ J_5 & J_5 + 2I_5 \end{pmatrix} = 2I_{10} + J_{10},$$

and once again, (1) is satisfied.

2.3 The Hoffman-Singleton graph

Based on unpublished work of Robertson, Berlekamp, van Lint, and Seidel [2] constructed an adjacency matrix for the Hoffman-Singleton graph, as follows. Define

$$B := \begin{pmatrix}
p^0 & p^0 & p^0 & p^0 & p^0 \\
p^0 & p^1 & p^2 & p^3 & p^4 \\
p^0 & p^2 & p^4 & p^6 & p^8 \\
p^0 & p^3 & p^6 & p^9 & p^{12} \\
p^0 & p^4 & p^8 & p^{12} & p^{16}
\end{pmatrix} = \begin{pmatrix}
p^0 & p^0 & p^0 & p^0 & p^0 \\
p^0 & p^1 & p^2 & p^3 & p^4 \\
p^0 & p^2 & p^4 & p^1 & p^3 \\
p^0 & p^3 & p^1 & p^4 & p^2 \\
p^0 & p^4 & p^3 & p^2 & p^1
\end{pmatrix}. $$
Note that

\[
BB^T = \begin{pmatrix}
5I_5 & J_5 & J_5 & J_5 & J_5 \\
J_5 & 5I_5 & J_5 & J_5 & J_5 \\
J_5 & J_5 & 5I_5 & J_5 & J_5 \\
J_5 & J_5 & J_5 & 5I_5 & J_5 \\
J_5 & J_5 & J_5 & J_5 & 5I_5 \\
\end{pmatrix} = 5I_{25} + J_{25} - (I_5 \otimes J_5),
\]

where \( M \otimes N \) is the Kronecker product of \( M \) and \( N \). Also, observe that, although \( B \) is not symmetric, it is normal (i.e., \( BB^T = B^T B \)) and it has the property that

\[
(I_5 \otimes (P + P^{-1}))B + B(I_5 \otimes (P^2 + P^{-2})) = J_{25} - B.
\]

Now define

\[
X_7 := \begin{pmatrix}
I_5 \otimes (P + P^{-1}) & B \\
B^T & I_5 \otimes (P^2 + P^{-2})
\end{pmatrix}.
\]

Then

\[
X_7^2 = \begin{pmatrix}
I_5 \otimes (P^2 + P^{-2} + 2I_5) + BB^T & J_{25} - B \\
J_{25} - B^T & B^T B + I_5 \otimes (P^1 + P^{-1} + 2I_5)
\end{pmatrix}.
\]

Hence,

\[
X_7^2 + X_7 = \begin{pmatrix}
6I_{25} + J_{25} & J_{25} \\
J_{25} & 6I_{25} + J_{25}
\end{pmatrix} = 6I_{50} + J_{50},
\]

and once again, (1) holds.

### 3 The pattern ends

The obvious question is whether or not the adjacency matrix of the missing Moore graph can be written in a similar form (assuming it exists). More precisely, we may ask if \( X_{57} \) has the following form:

\[
\begin{pmatrix}
I_{325} \otimes (P + P^{-1}) & B \\
B^T & I_{325} \otimes (P^2 + P^{-2})
\end{pmatrix},
\]

where \( B \) is some \( 1625 \times 1625 \) 0-1 matrix. In this section we show that the answer is ‘no’. Already we can see where things might go wrong. The row (and column) sums of \( X_{57} \) must be 57. The block matrices in the upper left and bottom right corners have row sums equal to 2, which means that \( B \) must have row sum 55. Each matrix of the form \( P^k \) has row sum equal to unity, so we can only have 55 copies of various powers of \( P \) as the blocks of the rows of \( B \). These would have to be augmented by 270 \( 5 \times 5 \) zero matrices. But then one suspects intuitively that there are not enough 1’s to satisfy (1). The following result confirms our intuitions.
Before stating our main result, we need some preliminaries. Suppose the adjacency matrix of $\Gamma_k$ is written in the form
\[ X_k = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}, \]
where $A$ and $D$ are square symmetric matrices of sizes $v_1$ and $v_2$, respectively. The matrices $A$ and $D$ may be viewed as the adjacency matrices of induced subgraphs $\Gamma_A$ and $\Gamma_D$, respectively, of $\Gamma_k$, and these subgraphs partition the vertices of $\Gamma_k$. Hence,
\[ v_1 + v_2 = v = k^2 + 1. \]

Denote the row sums of $A$ by $(a_1, \ldots, a_{v_1})$, the row sums of $B$ by $(b_1, \ldots, b_{v_1})$, the column sums of $B$ by $(c_1, \ldots, c_{v_2})$, and the row sums of $D$ by $(d_1, \ldots, d_{v_2})$. By the degree constraint,
\[ a_i + b_i = c_j + d_j, \quad (1 \leq i \leq v_1, 1 \leq j \leq v_2). \]
Note that the row sums of $A$ and $D$ are just the degrees of the vertices of the corresponding induced subgraphs $\Gamma_A$ and $\Gamma_D$. If $\Gamma_A$ is regular of degree $\alpha$ and $\Gamma_D$ is regular of degree $\delta$, then we say that $\{A, D\}$ is a biregular bipartition of $\Gamma_k$ of bidegree $(\alpha, \delta)$.

**Theorem 3.1.** If $k = 57$, then the only possible biregular bipartition of $\Gamma_k$ has equal size parts and bidegree $(32, 32)$.

**Corollary 3.1.** The adjacency matrix of $\Gamma_{57}$ cannot be of the form (2) (which corresponds to a biregular bipartition of bidegree $(2, 2)$).

**Proof.** Suppose that the adjacency matrix of a Moore graph $\Gamma_k$ is written in the block form (3). By the Moore graph condition (1), we have
\[
\begin{align*}
A^2 + A + BB^T &= (k - 1)I_{v_1} + J_{v_1}, \\
AB + BD + B &= J_{v_1, v_2}, \\
B^T B + D^2 + D &= (k - 1)I_{v_2} + J_{v_2},
\end{align*}
\]
where $J_{m,n}$ is the $m \times n$ all-ones matrix.

For any matrix $M$, set
\[ |M| := \sum_{i,j} M_{ij}. \]
Then (6), (7), and (8) imply
\[
\begin{align*}
|A^2| + |A| + |BB^T| &= (k - 1)v_1 + v_1^2, \\
|AB| + |BD| + |B| &= v_1v_2, \\
|B^T B| + |D^2| + |D| &= (k - 1)v_2 + v_2^2.
\end{align*}
\]
Suppose \( \{A, D\} \) is a biregular bipartition of \( \Gamma_k \) of bidegree \((\alpha, \delta)\). Then, for \( 0 \leq i \leq v_1 \) and \( 0 \leq j \leq v_2 \), we have \( a_i = \alpha, b_i = k - \alpha, c_j = k - \delta, \) and \( d_j = \delta \). Hence, (9) implies
\[
\alpha^2 + \alpha + (k - \delta)^2 = (k - 1) + v_1.
\]
But \(|B| = |B^T|\), so
\[
(k - \alpha)v_1 = (k - \delta)v_2,
\]
whence we obtain
\[
\alpha^2 + \alpha + (k - \alpha)^2 \left(\frac{v_1}{v_2}\right)^2 = (k - 1) + v_1.
\]
Substituting \( k = 57 \) into this equation and expanding (using (4)) gives
\[
v_1^3 - (2 \alpha^2 - 113 \alpha + 9693) v_1^2
+ (6500 \alpha^2 + 6500 \alpha + 10198500) v_1
- 10562500 (\alpha + 8) (\alpha - 7) = 0.
\]
A computer check for \( 1 \leq \alpha \leq 56 \) reveals that there is only one nontrivial integral solution for \( v_1 \), namely \( v_1 = 1625 \), corresponding to \( \alpha = \delta = 32 \).

Observe that, if \( B \) were normal, (6) and (8) would imply \( A^2 + A = D^2 + D \). Even without the assumption of normality, we may still deduce a relationship between \( A^2 + A \) and \( D^2 + D \), as long as \( \Gamma_{57} \) admits a biregular bipartition. In what follows, \([M, N] := MN - NM\) denotes the commutator of \( M \) and \( N \).

**Theorem 3.2.** Suppose \( \{A, D\} \) is a biregular bipartition of \( \Gamma_{57} \). Then \( A^2 + A \) and \( D^2 + D \) are cospectral.

**Proof.** First, we observe that \( A, D, BB^T \), and \( B^T B \) are all real symmetric matrices, hence diagonalizable by orthogonal transformations. Moreover, as \( B \) is square, \( BB^T \) and \( B^T B \) are cospectral (see, e.g., [13], Section 2.5).

Let \( j \) be the all-ones vector of size 1625, and let
\[
U := \{u \in \mathbb{R}^{1625} : (u, j) = 0\}
\]
be the orthogonal complement of the subspace spanned by \( j \). (Here \((\cdot, \cdot)\) denotes the ordinary Euclidean inner product.) By Theorem 3.1, we have \( Aj = Dj = \alpha j \) (where \( \alpha = 32 \)). Hence, \( A^2 + A \) and \( D^2 + D \) both share \( \alpha^2 + \alpha \) as an eigenvalue. As \( A \) and \( D \) are symmetric, their remaining eigenvectors can be taken to lie in \( U \).

Define \( M := (k - 1)I_{1625} + J_{1625} \). As \( A \) and \( D \) have constant row sums, \([A, J] = [D, J] = 0\), and therefore \([A^2 + A, J] = [D^2 + D, J] = 0\). Hence, \([A^2 + A, M] = [D^2 + D, M] = 0\). From (6) and (8) we may conclude that \([A^2 + A, BB^T] = [D^2 + D, B^T B] = 0\). It follows that \( A^2 + A \) and \( BB^T \) are simultaneously diagonalizable, as are, respectively, \( D^2 + D \) and \( B^T B \).
We have
\[(\alpha^2 + \alpha)j = (A^2 + A)j = Mj - BB^T j = (k + 1624)j - BB^T j.\]

If \(Au = \mu u\) with \(u \in U\), then
\[(\mu^2 + \mu)u = (A^2 + A)u = Mu - BB^T u = (k - 1)u - BB^T u,\]
whence we conclude (plugging in \(k = 57\) and \(\alpha = 32\)) that
\[\text{spec} (BB^T) = \{625, 56 - \mu^2 - \mu\},\]

where \(\mu\) runs over the eigenvalues of \(A\) associated to the eigenspaces orthogonal to \(j\). A similar argument shows that
\[\text{spec} (B^T B) = \{625, 56 - \nu^2 - \nu\},\]

where \(\nu\) runs over the eigenvalues of \(D\) associated to the eigenspaces orthogonal to \(j\). The theorem now follows.

\[\square\]

**Remark.** The equation \(\mu(\mu + 1) = \nu(\nu + 1)\) has two solutions, namely, \(\mu = \nu\) and \(\mu = -(\nu + 1)\), so, although it could be true, we cannot conclude from Theorem 3.2 that \(A\) and \(D\) are cospectral.

Theorem 3.2 relates the eigenvalues of \(A^2 + A\) and \(D^2 + D\). The next theorem allows us to relate some of their eigenvectors.

**Theorem 3.3.** Suppose \(\{A, D\}\) is a biregular bipartition of \(\Gamma_{57}\). Then
\[(A^2 + A)B = B(D^2 + D).\]  \hspace{1cm} (13)

**Proof.** By Theorem 3.1, \(JD = AJ = \alpha J\). Now multiply (7) on the left by \(A\) and on the right by \(D\) and subtract the two resulting equations. \[\square\]

Suppose \(Du = \nu u\). Then (13) implies
\[(A^2 + A)Bu = (\nu^2 + \nu)Bu.\]

Hence, if \(u \notin \ker B\), then \(Bu\) is an eigenvector of \(A^2 + A\) with eigenvalue \(\nu^2 + \nu\). Similarly, if \(Au = \mu u\) and \(u \notin \ker B^T\), then the transpose of (13) implies that \(B^T u\) is an eigenvector of \(D^2 + D\) with eigenvalue \(\mu^2 + \mu\). (The kernel of \(B\) is nontrivial; see Theorem 4.1 below.)

### 4 More constraints

In this section we see what can be said about general bipartitions \(\{A, D\}\) of \(\Gamma_{57}\) into equal size parts (i.e., \(v_1 = v_2 = 1625\)), with or without the assumption of biregularity. The primary tool used in this section is interlacing (see, e.g., [3], Section 2.5 or [7], Chapter 9). For brevity, in what follows we write \(X\) for \(X_{57}\) and \(J\) for \(J_{1625}\).
4.1 Rank constraints

Let \( \text{rk}(M) \) denote the rank of \( M \).

**Theorem 4.1.** Suppose that \( X \) is written in the form (3) with \( v_1 = v_2 = 1625 \). Then
\[
\text{rk}(B) \leq 1522,
\]
and \( \text{rk}(A) = \text{rk}(D) = 1625 \). In particular, \( B \) is singular, while \( A \) and \( D \) are both invertible.

**Proof.** By ([3], Theorem 2.5.1 or [7], Theorem 9.1.1), the eigenvalues of \( A \) and \( D \) interlace those of \( X \). The spectrum of \( X \) is well-known (e.g., ([7], Section 10.2) or [9]) to be \( (57, 7^{1729}, (−8)^{1520}) \) (where the exponents denote the multiplicities). Let the spectrum of \( A \) be denoted \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{1625} \). In this case, the interlacing inequalities read
\[
\lambda_i \geq \mu_i \geq \lambda_{1625+i}, \quad 1 \leq i \leq 1625.
\]
Hence,
\[
57 \geq \mu_1 \geq 7, \quad \mu_2 = \cdots = \mu_{105} = 7, \quad \text{and} \quad 7 \geq \mu_j \geq −8 (j \geq 106), \quad (14)
\]
and similarly for the eigenvalues of \( D \).

Let \( E \) denote the eigenspace of \( A \) corresponding to \( \mu = 7 \). Then
\[
\dim E \geq 104.
\]
Let \( U \) be the orthogonal complement of the all-ones vector \( j \) in \( \mathbb{R}^{1625} \). By the modular law of subspaces,
\[
\dim(E \cap U) = \dim E + \dim U - \dim(E + U) \geq 103.
\]
Let \( u \in E \cap U \) be normalized to unity. Then
\[
Au = 7u, \quad (u,u) = 1, \quad \text{and} \quad (u,j) = 0.
\]
From (6), we get
\[
(u, A^2u) + (u, Au) + (u, BB^T u) = k - 1,
\]
which gives
\[
49 + 7 + (u, BB^T u) = 56 \quad \Rightarrow \quad |B^T u|^2 = 0 \quad \Rightarrow \quad B^T u = 0.
\]
It follows that \( \dim \ker B^T \geq 103 \), so \( \text{rk}(B) = \text{rk}(B^T) \leq 1522 \).

Now, \( X \) has no zero eigenvalues and so is full rank. In particular, for every nonzero vector \( w \in \mathbb{R}^{3250} \), \( Xw \neq 0 \). Write \( w = \begin{pmatrix} x \\ y \end{pmatrix} \), where \( x, y \in \mathbb{R}^{1625} \). Then, if \( x \neq 0 \) or \( y \neq 0 \), we must have
\[
Ax + By \neq 0.
\]
Choose \( y \in \ker B \). Then \( Ax \neq 0 \) for every \( x \in \mathbb{R}^{1625} \), which shows that \( A \) is full rank.
(A similar argument shows that \( D \) is full rank.)
Corollary 4.1. Suppose that \( \{A, D\} \) is a biregular bipartition of \( \Gamma_{57} \). Then \( \Gamma_A \) and \( \Gamma_D \) are connected subgraphs.

Proof. By Theorem 3.1, \( \Gamma_A \) and \( \Gamma_D \) are regular of degree 32. By (14), the largest eigenvalue of \( A \) (and \( D \)) is 32 and the second largest eigenvalue of \( A \) (and \( D \)) is 7. Now apply Proposition 1.3.8 in [3]. \( \Box \)

4.2 Inertia and spectral constraints

As \( A \) and \( D \) are invertible, their Schur complements exist:

\[
X/A := D - B^T A^{-1} B \quad \text{and} \quad X/D := A - BD^{-1} B^T.
\]

We have

\[
\det X = \det A \cdot \det(X/A) = \det D \cdot \det(X/D).
\]

As \( A \) and \( D \) are integer matrices, their determinants are nonzero integers. This implies that their Schur complements have nonzero rational determinants. In particular, \( X/A \) and \( X/D \) are both full rank.

For any matrix \( M \), define the ordered triple

\[
\text{Inert}(M) = (n_+(M), n_-(M), n_0(M)),
\]

where \( n_+(M) \), \( n_-(M) \), and \( n_0(M) \) are the numbers of positive, negative, and zero eigenvalues of \( M \), respectively. By the Haynsworth inertia additivity formula [8],

\[
\text{Inert}(X) = \text{Inert}(A) + \text{Inert}(X/A) = \text{Inert}(D) + \text{Inert}(X/D).
\]

Theorem 4.2. Suppose that \( \{A, D\} \) is a biregular bipartition of \( \Gamma_{57} \). Then

\[
[D, B^T A^{-1} B] = [A, BD^{-1} B^T] = 0. \tag{15}
\]

Proof. If \( \{A, D\} \) is biregular then \( v_1 = v_2 \), so by (12), \( AJ = JA = JD = DJ = \alpha J \). Also, if \( \beta := k - \alpha \) then \( BJ = JB = B^T J = J B^T = \beta J \). (By Theorem 3.1, \( \beta = 25 \).)

Multiplying (7) on the left by \( B^T A^{-1} \) gives

\[
B^T B + B^T A^{-1} BD + B^T A^{-1} B = \alpha^{-1} \beta J.
\]

Similarly, multiplying the transpose of (7) on the right by \( A^{-1} B \) gives

\[
B^T B + DB^T A^{-1} B + B^T A^{-1} B = \alpha^{-1} \beta J.
\]

Subtracting these two equations yields the first equation in (15). The other equation in (15) follows similarly. \( \Box \)

Corollary 4.2. Suppose that \( \{A, D\} \) is a biregular bipartition of \( \Gamma_{57} \). Then

\[
\text{spec}(X/A) = \text{spec}(D) - \text{spec}(B^T A^{-1} B)
\]

\[
\text{spec}(X/D) = \text{spec}(A) - \text{spec}(BD^{-1} B^T),
\]

where the notation means that the eigenvalues of the matrices on the left are differences of the eigenvalues of the matrices on the right.
4.3 Average degree constraints

For any vector \((x_1, \ldots, x_{v/2})\), define
\[
\langle x \rangle = \frac{1}{v/2} \sum_{i=1}^{v/2} x_i.
\]

Then, as \(|B| = |B^T|\), we have
\[
\langle b \rangle = \langle c \rangle, \quad (16)
\]
which implies
\[
\langle a \rangle = \langle d \rangle. \quad (17)
\]

That is, the average degree of \(\Gamma_A\) must equal the average degree of \(\Gamma_D\).

**Theorem 4.3.** Suppose \(X\) is written in the form (3) with \(v_1 = v_2 = 1625\). Then
\[
24.5 \leq \langle a \rangle \leq 32. \quad (18)
\]

Similarly, \(25 \leq \langle b \rangle \leq 32.5\), \(25 \leq \langle c \rangle \leq 32.5\), and \(24.5 \leq \langle d \rangle \leq 32\).

**Proof.** Define
\[
Y := \begin{pmatrix} \langle a \rangle & \langle b \rangle \\ \langle c \rangle & \langle d \rangle \end{pmatrix} = \begin{pmatrix} \langle a \rangle & \langle b \rangle \\ \langle b \rangle & \langle a \rangle \end{pmatrix}.
\]

By ([3], Corollary 2.5.4), the eigenvalues of \(Y\) must interlace the eigenvalues of \(X\). The eigenvalues of \(Y\) are easily seen to be \(\langle a \rangle + \langle b \rangle = k\) and \(\langle a \rangle - \langle b \rangle = 2\langle a \rangle - k\).

Define
\[
y_1 := \max\{k, 2\langle a \rangle - k\} \quad \text{and} \quad y_2 := \min\{k, 2\langle a \rangle - k\}.
\]
Then the interlacing inequalities \((\lambda_i \geq y_i \geq \lambda_{v-2+i})\) give
\[
57 \geq y_1 \geq -8 \quad \text{and} \quad 7 \geq y_2 \geq -8.
\]

This forces \(y_1 = 57\) and
\[
7 \geq 2\langle a \rangle - 57 \geq -8,
\]
from which (18) follows. The other claims follow from Equations (16) and (17).

We conclude that the average degrees of \(\Gamma_A\) and \(\Gamma_D\) are both approximately half of the degree of \(\Gamma\). Observe that this is consistent with Theorem 3.1.

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