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On existence of integral point sets and their diameter bounds*

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Abstract

A point set M in m-dimensional Euclidean space is called an integral point set if all the distances between the elements of M are integers, and M is not situated on an $(m-1)$-dimensional hyperplane. We improve the linear lower bound for the diameter of planar integral point sets. This improvement takes into account some results related to the Point Packing in a Square problem. Then for arbitrary integers $m \geq 2$, $n \geq m+1$, $d \geq 1$, we give a construction of an integral point set M of n points in m-dimensional Euclidean space, where M contains points M_1 and M_2 such that the distance between M_1 and M_2 is exactly d.

1 Introduction

Let $\mathbb{N}$ be the set of all positive integers and let $|M_1M_2|$ denote the Euclidean distance between points M_1 and M_2 in a finite-dimensional space $\mathbb{R}^m$ (and, more generally, let $|\Delta|$ denote the length of line segment Δ). An *integral point set* in m-dimensional Euclidean space, $m > 2$, is a point set M such that all the distances between the points of M are integers and M is not situated on an $(m-1)$-dimensional hyperplane. Erdős and Anning proved [1,8] that every integral point set consists of a finite number of points. Taking this into account, we denote the set of all integral point sets of n points in m-dimensional Euclidean space by $\mathfrak{M}(m,n)$ (using the notation in [4]) and denote the set of all integral point sets in m-dimensional Euclidean space by $\mathfrak{M}(m,\mathbb{N})$. The symbol $\#M$ will be used for the cardinality of M, that is the number of points in M in our case.

For every finite point set, its diameter is naturally defined as

$$\operatorname{diam} M = \max_{A,B\in M} |AB|.$$

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Another emerging question is: how does the diameter of an integral point set depend on its cardinality? One can easily see that every $M \in \mathfrak{M}(m,n)$ with $\operatorname{diam} M = h$ can be dilated to $M_p \in \mathfrak{M}(m,n)$ with $\operatorname{diam} M = ph$ for every $p \in \mathbb{N}$. So, the above question should be rephrased: how does *the least possible* diameter of an integral point set depend on its cardinality? In order to answer this question, the following function was introduced [12, 13]:

$$d(m,n) = \min_{M \in \mathfrak{M}(m,n)} \operatorname{diam} M = \min_{M \in \mathfrak{M}(m,n)} \max_{A,B \in M} |AB|.$$

We also refer to [12] for a list of known exact values of $d(m,n)$ and its bounds. In the present paper, we mostly focus on the case $m = 2$.

The most significant breakthrough on the planar case was done by Solymosi [18], who proved that $cn \le d(2,n)$ for a sufficiently small constant c. Following Solymosi's proof carefully, one can derive that the inequality holds at least for $c = 1/24$. (See [9, Exercise 2.6] for some remarks.) The constant was improved in [3] to $1/8$ for all n and in [4] to $3/8$ for sufficiently large n.

The paper [18] contains one more interesting result. Let us define a function which is "dual" to $d(m,n)$ in some sense:

$$l(m,n) = \min_{M \in \mathfrak{M}(m,n)} \min_{A,B \in M} |AB|.$$

Solymosi proved that $l(2,n) \le 2$.

In the present paper we improve Solymosi's results: first, we obtain a larger constant $c = 5/11$ in Theorem 4.5, using the combined approach with the Point Packing in a Square problem (this approach is different from Solymosi's one); second, we prove that $l(m,n) = 1$ for all possible m and n.

Sections 2–4 are devoted to the linear lower bound. Each section consists of 2 subsections: General results and Special results. In General results, we do not aim to provide tight estimates; they are discussed in Special results.

Section 5 is devoted to integral point sets with distance 1 and has its own structure.

2 Collinear subsets of integral point sets

Integral point sets with many collinear points are very dominating examples. In this section we estimate the cardinality of an intersection of an integral point set with a line segment.

2.1 General results

Assuming that the planar integral point set contains many collinear points, the following result holds.

Theorem 2.1. *[13, Theorem 4] For $\delta > 0$, $\varepsilon > 0$, and $P \in \mathfrak{M}(2,n)$ with at least n^δ collinear points there exists a $n_0(\varepsilon)$ such that for all $n \ge n_0(\varepsilon)$ we have*

$$\operatorname{diam} P \ge n^{\frac{\delta}{4 \log 2(1+\varepsilon)} \log \log n}.$$

However, the estimate is rather unsuitable for our needs, as it does not provide the values of all the constants. To obtain the required estimate, we now prove a generalization of [4, Lemma 3].

Definition 2.2. For a line segment M_1M_2 and an integer k, such that $-|M_1M_2| < k < |M_1M_2|$, we define a $\rho(k, M_1, M_2)$-curve as the set of points N for which the equality $|NM_1| - |NM_2| = k$ holds.

So, in the planar case a $\rho(k, M_1, M_2)$-curve is a branch of a hyperbola for $k \neq 0$ and the perpendicular bisector of line segment M_1M_2 for $k = 0$.

Proposition 2.3. *If points M_1, M_2, M_3, M_4 are situated on a straight line, the equality $|M_1M_2| = |M_3M_4| = h$ holds, line segments M_1M_2 and M_3M_4 do not coincide and vectors $\overrightarrow{M_1M_2}$ and $\overrightarrow{M_3M_4}$ are of equal direction, then for a fixed integer k, $|k| < h$, the $\rho(k, M_1, M_2)$-curve and the $\rho(k, M_3, M_4)$-curve do not intersect.*

Proof. For $k = 0$, the claim is obvious. Indeed, in this case the $\rho(k, M_1, M_2)$-curve and the $\rho(k, M_3, M_4)$-curve are distinct hyperplanes orthogonal to l and thus do not intersect.

For $k \neq 0$, suppose the contrary. Let N be a point in the intersection.

Consider a plane $\Pi \supset (l \cup \{N\})$ with a Cartesian coordinate system such that $M_i = (x_i, 0)$ for $1 \leq i \leq 4$. Let R_1 and R_2 be the projections of the $\rho(k, M_1, M_2)$-curve and the $\rho(k, M_3, M_4)$-curve onto Π.

Then R_1 is a branch of a hyperbola defined by the equation

$$x - \frac{x_1 + x_2}{2} = a(h,k)\frac{\sqrt{y^2 + b^2(h,k)}}{b(h,k)}$$

and R_2 is a branch of a hyperbola defined by the equation

$$x - \frac{x_3 + x_4}{2} = a(h,k)\frac{\sqrt{y^2 + b^2(h,k)}}{b(h,k)},$$

where $a(h, k)$ and $b(h, k)$ are coefficients which depend on h and k. For $N = (x_N, y_N) \in R_1 \cap R_2$ we have

$$\begin{cases} x_N - \dfrac{x_1 + x_2}{2} = a(h,k)\dfrac{\sqrt{y_N^2 + b^2(h,k)}}{b(h,k)} \\[2ex] x_N - \dfrac{x_3 + x_4}{2} = a(h,k)\dfrac{\sqrt{y_N^2 + b^2(h,k)}}{b(h,k)} \end{cases}$$

and, thus, $x_1 + x_2 = x_3 + x_4$. On the other hand, $x_2 - x_1 = x_4 - x_3 = h$, and we conclude that $x_1 = x_3$ as well as $x_2 = x_4$. So, line segments M_1M_2 and M_3M_4 coincide. The contradiction concludes the proof. □

Lemma 2.4. *Let $M \in \mathfrak{M}(2, \mathbb{N})$ and let l be a straight line. Then for every $k \in \mathbb{N}$ there are at most $2k - 1$ segments $\Delta_i \subset l$ with endpoints in M, such that $|\Delta_i| = k$.*

Proof. Consider a point $N \in M \setminus l$. Let $\Delta_i = M_i^1 M_i^2$, and let vectors $\overrightarrow{M_i^1 M_i^2}$ and $\overrightarrow{M_j^1 M_j^2}$ be of equal direction for every i and j. Then for each i there is a $\rho(n_i, M_i^1, M_i^2)$-curve containing N. Due to Proposition 2.3, all n_i are distinct; otherwise the $\rho(n_i, M_i^1, M_i^2)$-curve and the $\rho(n_i, M_j^1, M_j^2)$-curve, $j \neq i$, do not intersect. There can be only $2k-1$ distinct values for n_i, so there are at most $2k-1$ distinct segments Δ_i. □

Setting $k = 1$ in Lemma 2.4, we obtain the following result (which is exactly [4, Lemma 3]).

Corollary 2.5. *Let $M \in \mathfrak{M}(2, \mathbb{N})$ and let l be a straight line. Then there is at most one pair of points $M_1, M_2 \in M \cap l$ such that $|M_1 M_2| = 1$.*

Lemma 2.6. *Let Δ be a straight line segment, $|\Delta| = l$ and $M \in \mathfrak{M}(2, \mathbb{N})$. Let $\#(\Delta \cap M) = n^2 + 1$. Then*

$$l \geq \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{6}n.$$

Proof. Any $n^2 + 1$ points, including the endpoints of Δ, split the segment Δ into n^2 sequential segments Δ_i. Due to Lemma 2.4, there is at most one segment of length 1, at most three segments of length 2, etc. The following two expressions for sums conclude the proof:

$$1 + \sum_{k=1}^{n} (2k-1) = n^2 + 1,$$

$$\sum_{k=1}^{n} k(2k-1) = \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{6}n.$$

□

Now we will estimate the length of a line segment that intersects an integral point set by an arbitrary number of points.

Lemma 2.7. *Let Δ be a straight line segment, $|\Delta| = b$ and $M \in \mathfrak{M}(2, \mathbb{N})$. Let $\#(\Delta \cap M) = t$. Then*

$$b \geq \frac{2}{3}t^{3/2} - \frac{3}{2}t + \frac{5}{6}t^{1/2}. \tag{1}$$

Proof. Let $f(k)$ denote the minimal length of a line segment that intersects an integral point set by k points. We observe that $f(k) > f(k-1)$. Due to Lemma 2.6, $f(n^2+1) \geq \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{6}n$.

For $t \in \mathbb{N}$ the inequality $(\sqrt{t}-1)^2 + 1 \leq t$ holds, thus

$$f(t) \geq f((\sqrt{t}-1)^2+1) \geq \frac{2}{3}(\sqrt{t}-1)^3 + \frac{1}{2}(\sqrt{t}-1)^2 - \frac{1}{6}(\sqrt{t}-1) = \frac{2}{3}t^{3/2} - \frac{3}{2}t + \frac{5}{6}t^{1/2}.$$

□

2.2 Special results

In [12], thorough investigation of planar integral point sets with diameter at most 10 000 is carried out; thus, we can restrict our attention to intersections of planar integral point sets with line segments of length more than 10 000.

Lemma 2.7 leads to the following proposition.

Proposition 2.8. *Let Δ be a straight line segment, $|\Delta| = b$ and $M \in \mathfrak{M}(2, \mathbb{N})$. Let $\#(\Delta \cap M) = k$ and $b > 10000$. Then $k \le \gamma_2 b + 6$, where*

$$\gamma_2 = \frac{3846}{2593\sqrt{647} - 5823} = 0.063958...$$

Proof. We know the maximum number of points for planar integral point sets of diameters at most 10000. Thus, we are interested in integral t such that estimate (1) holds for $b > 10000$. So, let us consider $t \ge 647$ and find a coefficient γ_2, such that the inequality

$$\frac{2}{3}t^{3/2} - \frac{3}{2}t + \frac{5}{6}t^{1/2} \ge \frac{t-6}{\gamma_2} \tag{2}$$

holds for all $t \ge 647$. For such t, the left-hand side of (2) obviously grows faster than the right-hand side. Turning (2) into the same equality and solving it for $t = 647$, we obtain the required estimate. □

To conclude, we remark that Corollary 2.5 and Proposition 2.8 are the only results of this section that are used in the proofs below; all the others are auxiliary.

3 Planar integral point set in a square

In this section, we develop some methods that employ the fact that every planar integral point set is situated in a square. Then we establish a connection between distances in integral point sets and the Point Packing in the Square problem.

3.1 General results

Let us introduce the problem, basic notions and results.

Problem 3.1 (Point Packing in a Square (PPS) [7, 14])**.** Given an integer $k > 1$, place k points in the unit square $U = [0, 1]^2$ such that their minimum pairwise distance m is maximal.

Definition 3.2. For each $k > 1$, the corresponding maximal distance m from Problem 3.1 is called the k-th PPS coefficient and denoted by φ_k.

So, it is impossible to place k points in a unit square in such a way that each pairwise distance of the points is greater than φ_k.

Theorem 3.3. *[7] For every $k \geq 2$ the following inequality holds:*

$$\sqrt{\frac{2}{k\sqrt{3}}} \leq \varphi_k \leq \frac{1}{k-1} + \sqrt{\frac{1}{(k-1)^2} + \frac{2}{(k-1)\sqrt{3}}}$$

Now we will estimate the side length of a square which contains a given planar integral points set.

Lemma 3.4. *[4, Lemma 4] Let $M \in \mathfrak{M}(2,n)$, $\operatorname{diam} M = d$. Then M is situated in a square of side length d.*

Proof. Consider $M_1, M_2 \in M$ such that $|M_1 - M_2| = d$ and a Cartesian coordinate system such that $M_1 = (-d/2; 0)$, $M_2 = (d/2; 0)$.

Then for $M_i = (x_i, y_i) \in M$, $i > 2$, the inequality $|x_i| < d/2$ holds (otherwise we get $|M_i - M_1| > d$ or $|M_i - M_2| > d$). Moreover, $|y_i| < d$ (otherwise we get $|M_i - M_1| > d$ or $|M_i - M_2| > d$).

Set $y_{\max} = \max_i y_i$, $y_{\min} = \min_i y_i$, $M_{\max} = (x_{\max}, y_{\max})$, $M_{\min} = (x_{\min}, y_{\min})$ (in case of multiple points providing the maximum or the minimum, the choice is arbitrary). Then

$$\begin{aligned} d \geq |M_{\max} - M_{\min}| = \sqrt{(x_{\max} - x_{\min})^2 + (y_{\max} - y_{\min})^2} \\ \geq \sqrt{(y_{\max} - y_{\min})^2} = y_{\max} - y_{\min} \end{aligned}$$

Thus, the sides of the square that contains M are $x = \pm d/2$, $y = y_{\max}$ and $y = y_{\max} - d$. □

3.2 Special results

It is clear that for $n > 123$ we have $d(2,n) \geq d(2,123)$. Based on the exact values of $d(2,n)$ for $3 \leq n \leq 122$ and the estimate $d(2,123) > 10000$ [12], we derive the following proposition.

Proposition 3.5. *The inequality*

$$d(2,n) \geq 3^{1/4} \cdot 2^{-3/2} \cdot n$$

holds for $4 \leq n \leq 21491$.

Therefore, we will focus on planar integral points sets of more than 21491 points.

Now we will use the upper bound of Theorem 3.3 to estimate φ_n with $\beta/\sqrt{n-2}$.

Proposition 3.6. *For $n \geq 21492$ we have*

$$\varphi_n \leq \varphi_{n-1} \leq \frac{\beta}{\sqrt{n-2}},$$

where

$$\beta = \frac{1}{\sqrt{21490}} + \sqrt{\frac{2}{\sqrt{3}} + \frac{1}{21490}} < 1.07464.$$

Proof. Indeed, for $n \geq 21492$ we have

$$
\begin{aligned}
\varphi_n \leq \varphi_{n-1} \leq \frac{1}{n-2}+\sqrt{\frac{1}{(n-2)^2}+\frac{2}{(n-2)\sqrt{3}}} \\
&= \frac{1}{\sqrt{n-2}} \cdot \frac{1}{\sqrt{n-2}} + \sqrt{\frac{1}{(n-2)^2}+\frac{2}{(n-2)\sqrt{3}}} \\
&\leq \frac{1}{\sqrt{n-2}} \cdot \frac{1}{\sqrt{21490}} + \sqrt{\frac{1}{(n-2)^2}+\frac{2}{(n-2)\sqrt{3}}} \\
&= \frac{1}{\sqrt{n-2}} \left(\frac{1}{\sqrt{21490}} + \sqrt{\frac{1}{n-2}+\frac{2}{\sqrt{3}}} \right) \\
&\leq \frac{1}{\sqrt{n-2}} \left(\frac{1}{\sqrt{21490}} + \sqrt{\frac{1}{21490}+\frac{2}{\sqrt{3}}} \right).
\end{aligned}
$$

□

4 Lower bound for the diameter

In this section, we improve the lower bound for the minimum diameter of planar integral point sets employing the results of two previous sections.

4.1 General results

Definition 4.1. A *cross* for points M_1 and M_2, denoted by $cr(M_1, M_2)$, is the union of two straight lines: the line through M_1 and M_2, and the perpendicular bisector of line segment M_1M_2.

Lemma 4.2. *If open line segments M_1M_2 and M_3M_4 do not intersect, then the set $cr(M_1, M_2) \cap cr(M_3, M_4)$ is either a straight line or contains 2 or 4 points.*

Proof. Consider straight lines l_1^1, l_2^1, l_1^2, l_2^2 such that $cr(M_1, M_2) = l_1^1 \cup l_2^1$ and $cr(M_3, M_4) = l_1^2 \cup l_2^2$.

The cases $l_1^1 = l_1^2$, $l_2^1 = l_2^2$ and $l_1^1 = l_2^2$, $l_2^1 = l_1^2$ are impossible, because the centers of open line segments M_1M_2 and M_3M_4 do not coincide as far as the open line segments do not intersect. So, all the possible cases (up to enumeration of the straight lines) are:

- $l_1^1 = l_1^2$, $l_2^1 \neq l_2^2$. The straight lines l_2^1 and l_2^2 are both orthogonal to l_1^1 and thus do not intersect; thus, $cr(M_1, M_2) \cap cr(M_3, M_4) = l_1^1 = l_1^2$.

- l_1^1 is parallel to l_1^2 and l_2^1 is parallel to l_2^2. Then

$$cr(M_1, M_2) \cap cr(M_3, M_4) = (l_1^1 \cup l_2^1) \cap (l_1^2 \cup l_2^2) = (l_1^1 \cap l_2^2) \cup (l_2^1 \cap l_1^2),$$

  that is two points.

- l_1^1 is neither parallel nor orthogonal to l_1^2. Then the intersection contains 4 points.

□

Remark 4.3. It is important that we consider the intersection of open line segments in the claim of Lemma 4.2, so e.g. the cases $M_1 = M_3$ and $M_3 \in M_1M_2$ satisfy the conditions of Lemma 4.2.

Lemma 4.4. *Let $ABCD$ be a convex quadrilateral on the plane. Then*

$$\max\{AC, BD\} > \min\{AB, BC, CD, DA\},$$

that is, at least one diagonal is greater than at least one side.

Proof. Suppose the contrary. Then AC in the minimal side of the triangle ABC and, thus, the angle ABC is acute. So are also angles BCD, CDA and DAB. We conclude that all angles of a convex quadrilateral $ABCD$ are acute, which is impossible. The contradiction concludes the proof. □

4.2 Special results

Now we are ready to prove the main theorem of the section.

Theorem 4.5. *If $n \geq 4$, then $d(2,n) \geq \gamma(n-2)$, where*

$$\begin{aligned} 0.46530... &= 3^{1/4} \cdot 2^{-3/2} \\ &> \gamma = \frac{\sqrt{16\left(\sqrt{\frac{2}{\sqrt{3}} + \frac{1}{21490}} + \frac{1}{\sqrt{21490}}\right)^2 + \frac{14791716}{\left(2593\sqrt{647}-5823\right)^2}} - \frac{3846}{2593\sqrt{647}-5823}}{8\left(\sqrt{\frac{2}{\sqrt{3}} + \frac{1}{21490}} + \frac{1}{\sqrt{21490}}\right)^2} \\ &> 0.45557 > \frac{5}{11}. \end{aligned}$$

Proof. For $4 \leq n \leq 21491$, the assertion of the theorem follows immediately from Proposition 3.5. Let us consider $M \in \mathfrak{M}(2,n)$, $n \geq 21492$, $\operatorname{diam} M = b$. Lemma 3.4 yields that M is situated in a square of side length b. Let M_1, M_2, M_3, M_4 be points of M such that the distances $|M_1M_2|$ and $|M_3M_4|$ are minimal in M (M_2 and M_3 may coincide). Then $|M_1M_2| \leq b\varphi_{n-1}$ and $|M_3M_4| \leq b\varphi_{n-1}$. Due to Lemma 4.4, open line segments M_1M_2 and M_3M_4 do not intersect (otherwise they are not minimal).

Let $C = cr(M_1M_2) \cap cr(M_3M_4)$. Each point $N \in M$ satisfies one of the following conditions:

a) N belongs to C; overall at most $\gamma_2 b + 6$ points by Lemma 4.2 and Proposition 2.8;

b) N belongs to the intersection of one of $|M_1M_2| - 1$ hyperbolas with one of $|M_3M_4| - 1$ hyperbolas; overall at most $4(|M_1M_2| - 1)(|M_3M_4| - 1)$ points;

c) N belongs to the intersection of one of $|M_1M_2| - 1$ hyperbolas with $cr(M_3M_4)$; overall at most $4(|M_1M_2| - 1)$ points;

d) N belongs to the intersection of one of $|M_3M_4|-1$ hyperbolas with $cr(M_1M_2)$; overall at most $4(|M_3M_4|-1)$ points;
(see [8] for details). Summing up the above cases, we obtain the following estimate:

$$n \leq 4b^2\varphi_{n-1}^2 - 4 + \gamma_2 b + 6. \tag{3}$$

Proposition 3.6 turns estimate (3) into the following:

$$n - 2 \leq 4b^2\frac{\beta^2}{n-2} + \gamma_2 b,$$

which obviously leads to inequality

$$1 \leq 4\beta^2\left(\frac{b}{n-2}\right)^2 + \gamma_2\frac{b}{n-2}.$$

Let us denote $\lambda = b/(n-2)$ and solve the following quadratic inequality for λ:

$$4\beta^2\lambda^2 + \gamma_2\lambda - 1 \geq 0.$$

The discriminant is $\gamma_2^2 + 16 \cdot \beta^2$, so we obtain the following estimate:

$$\frac{b}{n-2} = \lambda \geq \frac{-\gamma_2 + \sqrt{\gamma_2^2 + 16 \cdot \beta^2}}{8\beta^2}.$$

Calculating the expression above for our β and γ_2, we conclude the proof. □

Corollary 4.6. *If $n \geq 4$, then $d(2,n) > \frac{5}{11}n$.*

Proof. For $4 \leq n \leq 21491$, the claim follows from Proposition 3.5 immediately. For $n > 21491$, the inequality $0.45557(n-2) > \frac{5}{11}n$ holds. □

We can improve the constant in Theorem 4.5, if some new exact values of $d(2,n)$ are found; however, the obtained constant is bounded by $3^{1/4} \cdot 2^{-3/2}$ from above. More precisely, we have the following theorem.

Theorem 4.7. *For every $\varepsilon > 0$ there exists a number n_0 such that the inequality*

$$d(2,n) \geq n \cdot (3^{1/4} \cdot 2^{-3/2} - \varepsilon)$$

holds for every $n > n_0$, that is

$$\lim_{n\to\infty} \frac{d(2,n)}{n} \geq 3^{1/4} \cdot 2^{-3/2}.$$

5 Integral point sets with distance 1

5.1 Constructing planar integral point sets with distance 1

Definition 5.1. A set $M \in \mathfrak{M}(2,n)$ is called *facher* if M consists of $n-1$ points on a straight line and one point out of the line.

Definition 5.2. A set $M \in \mathfrak{M}(m,n)$ is called *optimal* if $\operatorname{diam} M = d(m,n)$.

Definition 5.3. [11] A squarefree number q is called the *characteristic* of $M \in \mathfrak{M}(2,\mathbb{N})$, if for any points $M_1, M_2, M_3 \in M$ the area of triangle $M_1M_2M_3$ is $p_{1,2,3}\sqrt{q}$ for some rational $p_{1,2,3}$.

For a given $M \in \mathfrak{M}(2,\mathbb{N})$, the characteristic is determined uniquely.

Facher sets are the simplest planar integral point sets. It is known that for $9 \le n \le 122$ all the optimal sets are facher [13]. For every cardinality n and every squarefree number q there exists a facher set $M \in \mathfrak{M}(2,n)$ with characteristic q [4, Theorem 5]. In [2], the facher sets of characteristic 1 were investigated; they were called *semi-crabs*.

For every integer $n \ge 3$ Solymosi presented [18] a construction of a facher integral point set $M \in \mathfrak{M}(2,n)$ such that equality $|M_1M_2| = 2$ holds for some $M_1, M_2 \in M$. The constructed set has both odd and even distances.

Now we improve Solymosi's result.

Construction 5.4. Let us choose a positive integer $k > 1$ and set

$$a = 2^{2^k} - 1.$$

Then

$$a \equiv 3 \mod 4 \tag{4}$$

and, moreover,

$$\begin{aligned} a = \left(2^{2^{k-1}}\right)^2 - 1 &= \left(2^{2^{k-1}} + 1\right)\left(2^{2^{k-1}} - 1\right) \\ &= \left(2^{2^{k-1}} + 1\right)\left(2^{2^{k-2}} + 1\right) \cdot \ldots \cdot \left(2^{2^1} + 1\right)\left(2^2 - 1\right). \end{aligned}$$

We set $d_j = 2^{2^j} + 1$ for $1 \le j \le k-1$. Then $d_j \equiv 1 \mod 4$.

Let $c_J = \prod_{j \in J} d_j$ for every subset of indices $J \subset I = \{1, 2, ..., k-1\}$ (and $c_\varnothing = 1$). We obtain

$$c_J \equiv 1 \mod 4 \tag{5}$$

and, moreover, a is divisible by c_J.

Statements (4) and (5) yield $a/c_J \equiv 3 \mod 4$. Let $b_J = (c_J - a/c_J)/2$, then $b_J \equiv 1 \mod 2$. Let us further take $g_J = (c_J + a/c_J)/2$, then $g_J \equiv 0 \mod 2$.

Next we define the coordinates of the points as following:

$$M_{J\pm} = \left(\pm\frac{b_J}{2}, 0\right), \quad N = \left(0, \frac{\sqrt{a}}{2}\right).$$

Then the distances are:

$$|NM_{J\pm}| = \left(\frac{b_J^2}{4} + \frac{a}{4}\right)^{1/2} = \frac{1}{2}\left(\left(\frac{c_J - a/c_J}{2}\right)^2 + a\right)^{1/2}$$

$$= \frac{1}{2}\left(\left(\frac{c_J}{2}\right)^2 - \frac{a}{2} + \left(\frac{a}{2c_J}\right)^2 + a\right)^{1/2} = \frac{1}{2}\left(\left(\frac{c_J}{2}\right)^2 + \frac{a}{2} + \left(\frac{a}{2c_J}\right)^2\right)^{1/2}$$

$$= \frac{1}{2}\left(\left(\frac{c_J + a/c_J}{2}\right)^2\right)^{1/2} = \frac{1}{2}\left(\frac{c_J + a/c_J}{2}\right) = \frac{g_J}{2} \in \mathbb{N},$$

$$|M_{J_1\pm}M_{J_2\pm}| = \left|\frac{b_{J_1}}{2} \pm \frac{b_{J_2}}{2}\right| = \left|\frac{b_{J_1} \pm b_{J_2}}{2}\right| \in \mathbb{N}.$$

In particular, for $H = \{k-1\}$ we obtain $C_H = 2^{2^{k-1}} + 1$ and

$$b_H = \left(2^{2^{k-1}} + 1 - \frac{a}{2^{2^{k-1}} + 1}\right)/2 = \left(2^{2^{k-1}} + 1 - \left(2^{2^{k-1}} - 1\right)\right)/2 = 1.$$

Thus, one of the distances is

$$|M_{H+}M_{H-}| = \left|\frac{b_H}{2} - \frac{-b_H}{2}\right| = \left|\frac{1}{2} - \frac{-1}{2}\right| = 1.$$

Note that all the points $M_{J\pm}$ are distinct: the equality $b_J = b_K$ implies $J = K$; the equality $b_J = -b_K$ implies $c_J = -c_K$ or $c_J = a/c_K$. The first case is impossible because both c_J and c_K are positive; the second case contradicts (4) and (5).

So, $M = \{M_{J\pm}, N\}$ is indeed a planar integral point set of $2^k + 1$ points, and distance 1 occurs in M. Since k can be taken arbitrary large, it follows that we can construct a planar integral point set of arbitrary large cardinality so that the distance 1 occurs in that set.

Remark 5.5. Applying Construction 5.4 to $k = 1$ ad litteram, we get $a = 3$, $I = \varnothing$, $b_\varnothing = -1$ and then obtain an equilateral triangle of side length 1. This triangle is obviously the optimal set in $\mathfrak{M}(2,3)$.

Remark 5.6. For $k = 2$ in Construction 5.4, we obtain the optimal set in $\mathfrak{M}(2,5)$ presented in [10, Fig. 1]. If we remove one point from it, then we get one of the two optimal sets in $\mathfrak{M}(2,4)$.

5.2 Description of planar integral point sets with distance 1

Definition 5.7. [2] A set $M \in \mathfrak{M}(m,n)$ is *maximal*, if there is no set $M' \in \mathfrak{M}(m,n+1)$ such that $M \subsetneq M'$.

In order to describe all the sets $M \in \mathfrak{M}(2,n)$ with distance 1, we need [4, Proposition 6]. For the reader's convenience, we will state a slightly rephrased version of it and provide the proof.

Lemma 5.8. *Let $M = \{M_1, M_2, M_3, M_4\} \in \mathfrak{M}(2,4)$ such that $|M_1M_2| = 1$. If l is the straight line through points M_1 and M_2, then one of the points M_3 or M_4 belongs to l.*

Proof. Let m denote the perpendicular bisector of line segment M_1M_2. Due to the triangle inequality, there are two possibilities to place each of points M_i, $i = 3, 4$:

a) M_i belongs to l and $|M_iM_1| - |M_iM_2| = \pm 1$;

b) M_i belongs to m and $|M_iM_1| = |M_iM_2|$.

If (a) is true for both points, then $M \subset l$ and thus M is not an integral point set. If (a) is true for one point and (b) is true for another, then the claim of the lemma follows. So, suppose to the contrary, that both points M_3 and M_4 belong to m.

The area of the triangle $M_1M_3M_4$ is rational because $|M_3 - M_4| \in \mathbb{Z}$. Thus, the characteristic of M is 1, and there is a Cartesian coordinate system such that $M_1 = (-1/2, 0)$, $M_2 = (1/2, 0)$, $M_3 = (0, a/2)$ (see [4, Theorem 4]). It is clear that $a \neq 0$. We set $b = |M_1 - M_3|$ and $O = (0, 0)$. Applying the Pythagorean theorem to triangle OM_1M_3, we obtain the following Diophantine equation:

$$\frac{1}{4} + \frac{a^2}{4} = b^2,$$

or, equivalently,

$$1 + a^2 = (2b)^2.$$

This equation has no integral solutions. This contradiction concludes the proof. □

Using Construction 5.4, Lemma 5.8 and Corollary 2.5 together with results of [2, Section 6], we obtain the following theorem.

Theorem 5.9. *For every $n \geq 3$ there is a planar integral set M of n points such that for some $M_1, M_2 \in M$ equality $|M_1M_2| = 1$ holds. This set consists of $n - 1$ points, including M_1 and M_2, on a straight line and one point out of the line.*

And vice versa, if M is a planar integral point set of n points such that for some $M_1, M_2 \in M$ equality $|M_1M_2| = 1$ holds, then M consists of $n - 1$ points, including M_1 and M_2, on a straight line, and one point out of the line, on the perpendicular bisector of line segment M_1M_2. There is only one maximal integral point set $M' \supseteq M$, and the perpendicular bisector is the axis of symmetry for M'. Moreover, if $n > 3$, then the distance 1 occurs in M (and M') only once.

5.3 Higher dimensions

To build integral points sets with distance 1 in higher dimensions, we need the "blowing up" procedure described in [12, Theorem 1.3].

Construction 5.10. Let $M \in \mathfrak{M}(2, n)$ have many points on a straight line l_1 and $M \setminus l_1 \subset l_2$ for a straight line l_2. Moreover, let l_1 be parallel to l_2.

Construct $M' \subset \mathbb{R}^m$ by replacing all the points $M_1, ..., M_n \in M \cap l_2$ by equal $(m-2)$–dimensional simplices $S_1, ..., S_n$ in such a way that

$$\forall (j = 1, ..., n) \forall (N_i \in M \cap l_1) \forall \left(M_k^j, M_h^j \in S_j\right) \left[|N_iM_k^j| = |N_iM_h^j|\right].$$

Then under certain conditions (in particular, if $\#(M \cap l_2) = 1$ that we actually need) we get $M' \in \mathfrak{M}(m, \mathbb{N})$.

For further examples we also refer the reader to [5].

Now we can construct an integral point set M in m-dimensional Euclidean space, $m \geq 3$, with distance 1 occurring in it. If we have an integral point set with distance 1 occurring, then we can easily dilate it to turn 1 into the desired distance. These facts, together with Theorem 5.9, give the following theorem.

Theorem 5.11. *For arbitrary integers $m \geq 2$, $n \geq m+1$, $d \geq 1$ there exists $M \in \mathfrak{M}(m, n)$, such that for some $M_1, M_2 \in M$ equality $|M_1M_2| = d$ holds.*

Definition 5.12. [15] We will call an integral point set M *prime*, if the greatest common divisor of all the distances occurring in M is 1.

If an integral point set is prime, then it cannot be squashed to an integral point set of the same power and structure but smaller diameter.

Theorem 5.13. *For every $m \geq 3$, $n \geq m+1$, $d \geq 1$ there exists a* ***prime*** *integral point set $M \in \mathfrak{M}(m, n)$ which contains points M_1 and M_2 such that distance between M_1 and M_2 is exactly d.*

Proof. Take a facher integral point set according to Construction 5.4 with sufficiently large height and then apply the "blowing up" procedure described in Construction 5.10, replacing the point that is not situated on the line, with an $(m-2)$-dimensional simplex of proper side length. That simplex will consist of $m-1$ points; we can exclude practically all M_J points, except the two points with distance 1. So, the set consists of $m+1$ or more points and has distance 1 occurring in it. As distance 1 occurs in the obtained set, it is prime. □

A slightly stronger result can also be claimed.

Theorem 5.14. *For arbitrary integers $m > 2$, $n \geq m+1$, $d \geq 1$ there exists a prime set $M \in \mathfrak{M}(m, n)$, such that for some $M_1, M_2 \in M$ equality $|M_1M_2| = d$ holds, the distance d is minimal and occurs in M only once.*

Proof. We apply Construction 5.4 to obtain $M' \in \mathfrak{M}(2, n-m+2)$ with $M_1', M_2' \in M'$ and $|M_1'M_2'| = 1$. Dilate M' to M'' in such a way that M_1' and M_2' turn into M_1'' and M_2'' resp. and $|M_1''M_2''| = d$. Then "blow up" M'' to $M \in \mathfrak{M}(m, n)$ using a simplex of side length $d+1$. The greatest common divisor of d and $d+1$ is 1, so M is prime. □

6 Final remarks and open problems

Remark 6.1. The bound of Theorem 4.5 is not tight. The derivation of tight bounds for the minimum diameter $d(2, n)$ is still a challenging task for the forthcoming investigation [13, Section 7].

The known upper bound is $d(2, n) \leq 2^{c \log n \log \log n}$ [10]. However, we hope that the present article can provide a framework for possible better estimates.

Conjecture 6.2. *The approach of Theorem 4.5 can be generalized to higher dimensions.*

For the overview of current lower bounds for higher dimensions, we refer the reader to [16].

There are also several ways to improve the current bound, based on Lemma 3.4.

Problem 6.3. Is there a number $\delta < 1$ such that any $M \in \mathfrak{M}(2, \mathbb{N})$ with $\operatorname{diam} M = d$ is situated in a square of side length d^δ?

If the answer for Problem 6.3 is affirmative, then we can obtain a bound better than linear.

Problem 6.4. What are the minimal numbers $\delta \leq 1$ and $h \leq 0$ such that any $M \in \mathfrak{M}(2, \mathbb{N})$ with $\operatorname{diam} M = d$ is situated in a square of side length $\delta d + h$?

If the answer for Problem 6.4 is not $\delta = 1$ and $h = 0$, then the linear bound can be slightly improved.

However, Problems 6.3 and 6.4 do not seem to show the way for improving the bound due to the following well-known theorem, which can be proved by an application of circle inversion [19].

Theorem 6.5. *There exists a dense subset P of the unit circle such that for any $P_1, P_2 \in P$ the distance $|P_1P_2|$ is rational.*

We conjecture that thorough investigation of existing examples of circular integral point sets [6, 10, 17] will lead to the negative answers for the Problems 6.3 and 6.4.

Another possibility to increase the constant in our lower bound is introduced by the following problem.

Problem 6.6. What is the minimal shape $\mathcal{S}$ such that any $M \in \mathfrak{M}(2, \mathbb{N})$ with $\operatorname{diam} M = d$ is situated in $\mathcal{S}$?

This problem is another generalization of Lemma 3.4. We know that $\mathcal{S}$ is not a circle of diameter d (the counterexample is an equilateral triangle of side length d) and we conjecture that $\mathcal{S}$ is not a Reuleaux triangle of width d (the possible counterexample belongs to a circle of diameter d, see Theorem 6.5).

The solution to Problem 6.6 (if ever found) will lead to a sophisticated packing problem, something like "How to put n points in a convex hull of concentric circle and Reuleaux triangle?" Obviously, Problem 6.6 can be generalized to higher dimensions.

For $n = 3, 4, 5$, there are optimal sets in $\mathfrak{M}(2, n)$ that contain distance 1 (see Remarks 5.5 and 5.6). However, we can suggest the following conjecture.

Conjecture 6.7. *Every set $M \in \mathfrak{M}(2, n)$, $n \geq 6$, such that for some M_1, M_2 equality $|M_1M_2| = 1$ holds, is not optimal.*

The motivation for Conjecture 6.7 is based on the results of [13, Section 5], especially the following theorem.

Theorem 6.8. *For every* $12 \leq n \leq 122$*, there exist a facher optimal sets* $M_n \in \mathfrak{M}(2, n)$ *that consist of* $n-1$ *points* $\{(b_1, 0), ..., (b_{n-1}, 0)\}$ *and the point* $(0, \sqrt{A_n})$*, where* $b_1, ..., b_{n-1}, A_n$ *are integer and* $\sqrt{A_n} \notin \mathbb{N}$*.*

Such a set cannot contain distance 1: due to Theorem 5.9, the points with distance 1 should be $(\pm 1/2, 0)$.

For planar integral point sets containing distance 1, we can emphasize the following problem.

Problem 6.9. For given $n \geq 3$ and squarefree number $q \neq 1$, is there $M \in \mathfrak{M}(2, n)$, such that characteristic of M is q and distance 1 occurs in M?

The consideration of integral point sets containing distance 1 in higher dimensions gives a rise to another problem.

Problem 6.10. How many times can the distance 1 occur in a set $M \in \mathfrak{M}(m, n)$?

The answer for $m = 2$ is given by Theorem 5.9. For greater m, we will be bold enough to suggest the following conjecture.

Conjecture 6.11. *Consider* $M \in \mathfrak{M}(m, n)$*,* $M_1, M_2 \in M$*,* $|M_1 M_2| = 1$*. Then either* $n = m + 1$ *or* M *is a "blowup" of a facher set and distance 1 occurs in* M *no more than* $1 + \frac{(m-1)^2 - (m-1)}{2}$ *times.*

Such an estimate is based on the number of edges in an $(m-2)$-dimensional simplex.

We want to conclude the list of conjectures with the one which appears the easiest to deal with.

Conjecture 6.12. *For every* $n \geq 3$*,* $d \geq 1$ *there exists a* ***prime*** *set* $M \in \mathfrak{M}(2, n)$ *which contains points* M_1 *and* M_2 *such that distance between* M_1 *and* M_2 *is exactly* d*.*

This conjecture gives the same claim as Theorem 5.13 does, but for $m = 2$. Obviously, Theorem 5.14 can be generalized (for $m = 2$) into a similar conjecture.

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