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Block-avoiding point sequencings of arbitrary length in Steiner triple systems

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Abstract

An *ℓ-good sequencing* of an $\mathsf{STS}(v)$ is a permutation of the points of the design such that no ℓ consecutive points in this permutation contain a block of the design. We prove that, for every integer $\ell \geq 3$, there is an ℓ-good sequencing of any $\mathsf{STS}(v)$ provided that v is sufficiently large. We also prove some new nonexistence results for ℓ-good sequencings of $\mathsf{STS}(v)$.

1 Introduction

A *Steiner triple system of order* v is a pair $(X, \mathcal{B})$, where X is a set of v *points* and $\mathcal{B}$ is a set of 3-subsets of X (called *blocks*), such that every pair of points occur in exactly one block. We will abbreviate the phrase "Steiner triple system of order v" to $\mathsf{STS}(v)$. It is well-known that an $\mathsf{STS}(v)$ contains exactly $v(v-1)/6$ blocks, and an $\mathsf{STS}(v)$ exists if and only if $v \equiv 1, 3 \bmod 6$. The definitive reference for Steiner triple systems is the book [5] by Colbourn and Rosa.

The following problem was introduced by Kreher and Stinson in [4]. Suppose $(X, \mathcal{B})$ is an $\mathsf{STS}(v)$ and let $\ell \geq 3$ be an integer. An *ℓ-good sequencing* of $(X, \mathcal{B})$ is a permutation $\pi = [x_1\ x_2\ \cdots\ x_v]$ of X such that no ℓ consecutive points in the permutation contain a block in $\mathcal{B}$. (Some related but different sequencing problems for $\mathsf{STS}(v)$ are studied in [1] and [3].)

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Remark 1. We observe that an ℓ-good sequencing is automatically an m-good sequencing if $m < \ell$.

It is an interesting question if there exists, for a given integer $\ell \geq 3$, an ℓ-good sequencing of a specified $\mathsf{STS}(v)$, or if there exists an ℓ-good sequencing of all $\mathsf{STS}(v)$ (for sufficiently large values of v). The following results were proven in [4]:

- Any $\mathsf{STS}(v)$ with $v > 3$ has a 3-good sequencing.
- Any $\mathsf{STS}(v)$ with $v > 71$ has a 4-good sequencing.
- There is a unique $\mathsf{STS}(7)$ and a unique $\mathsf{STS}(9)$. Neither of these have a 4-good sequencing.
- All $\mathsf{STS}(13)$ and $\mathsf{STS}(15)$ have 4-good sequencings.

It was conjectured in [4], for any integer $\ell \geq 3$, that there exists an integer $n(\ell)$ such that any $\mathsf{STS}(v)$ with $v > n(\ell)$ has an ℓ-good sequencing. We prove this conjecture in Section 3 of this paper and we show that $n(\ell) \in O(\ell^6)$. We also prove a nonexistence result, in Section 2, namely, that an $\mathsf{STS}(v)$ with $v > 7$ cannot have an ℓ-good sequencing if $\ell \geq (v + 2)/3$.

We will use the following notation in the remainder of this paper. Suppose $(X, \mathcal{B})$ is an $\mathsf{STS}(v)$. Then, for any pair of points x, y, let $\mathsf{third}(x, y) = z$ if and only if $\{x, y, z\} \in \mathcal{B}$. The function third is well-defined because every pair of points occurs in a unique block in $\mathcal{B}$.

2 A counting argument

In this section, we generalize a counting argument from [4, §3.1] that was used to prove the nonexistence of 4-good sequencings of $\mathsf{STS}(7)$ and $\mathsf{STS}(9)$. Let $v \geq 7$ and $\ell \geq 3$ be integers. Suppose we take the points of an $\mathsf{STS}(v)$ to be $1, \ldots, v$. Suppose we have an ℓ-good sequencing of an $\mathsf{STS}(v)$. Without loss of generality, suppose, by relabelling points if necessary, that $[1\ 2\ 3\ \cdots\ v]$ is the ℓ-good sequencing. We say that a block B is of *type* i if $|B \cap \{1, 2, \ldots, \ell\}| = i$. Clearly, we must have $i \in \{0, 1, 2\}$.

For $i = 0, 1, 2$, let b_i denote the number of blocks of type i. Since the sequencing is ℓ-good, we know that $b_2 = \binom{\ell}{2}$. Since each point appears in $(v - 1)/2$ blocks, we have

$$b_1 = \ell \left(\frac{v-1}{2} - (\ell - 1) \right).$$

Finally, because the total number of blocks is $v(v - 1)/6$, we have

$$\begin{aligned} b_0 &= \frac{v(v-1)}{6} - \ell \left(\frac{v-1}{2} - (\ell - 1) \right) - \binom{\ell}{2} \\ &= \frac{v(v-1)}{6} - \frac{\ell(v-\ell)}{2}. \end{aligned}$$

Consider a block of type 0, say $B = \{x, y, z\}$ where $x < y < z$. We must have $x \leq v - \ell$ because otherwise $B \subseteq \{v - \ell + 1, \ldots, v - 2, v - 1, v\}$. Since B is of type 0, we also have that $x \geq \ell + 1$. For each such x, where $\ell + 1 \leq x \leq v - \ell$, we have $z \in \{x + \ell, \ldots, v - 1, v\}$, so there are $v - (x + \ell - 1)$ possible values for z. It follows that there can be at most

$$\sum_{x=\ell+1}^{v-\ell} (v - (x + \ell - 1)) = \frac{(v - 2\ell)(v - 2\ell + 1)}{2}$$

blocks of type 0. Since there are $b_0 = v(v - 1)/6 - \ell(v - \ell)/2$ blocks of type 0, we obtain

$$\frac{v(v-1)}{6} - \frac{\ell(v-\ell)}{2} \leq \frac{(v - 2\ell)(v - 2\ell + 1)}{2},$$

which simplifies to give

$$0 \leq (3\ell - 2v)(3\ell - v - 2).$$

We are assuming $v \geq 7$, so $(v + 2)/3 + 1 < 2v/3$. Hence, $\ell \leq (v + 2)/3$ or $\ell \geq 2v/3$. Therefore there does not exist a $(\lfloor (v + 2)/3 \rfloor + 1)$-good sequencing of an $\mathsf{STS}(v)$. Then, it follows from Remark 1 that we cannot have an ℓ-good sequencing with $\ell \geq 2v/3$.

Summarizing the above discussion, we have the following theorem.

Theorem 2.1. *If an* $\mathsf{STS}(v)$ *with* $v \geq 7$ *has an* ℓ*-good sequencing, then* $\ell \leq (v+2)/3$.

By analyzing the case of equality in Theorem 2.1 more carefully, we can rule out the existence of an ℓ-good sequencing of an $\mathsf{STS}(3\ell - 2)$ whenever $\ell > 3$ is odd (note that ℓ must be odd for an $\mathsf{STS}(3\ell - 2)$ to exist).

Theorem 2.2. *If* $\ell > 3$ *is an odd integer, then no* $\mathsf{STS}(3\ell - 2)$ *has an* ℓ*-good sequencing.*

Proof. Suppose, by way of contradiction, that there is an ℓ-good sequencing of an $\mathsf{STS}(3\ell - 2)$ for some $\ell > 3$. From the proof of Theorem 2.1, there are $v - 2\ell = \ell - 2$ blocks of type 0 that contain the point $\ell + 1$. Within these $\ell - 2$ blocks, the point $\ell + 1$ occurs with $2\ell - 4$ other points in the set $\{\ell + 2, \ldots, v\}$, which has cardinality $2\ell - 3$. It follows that the point $\ell + 1$ must occur in exactly one block of type 1.

Since every point occurs in exactly $(v - 1)/2$ blocks, the point $\ell + 1$ must occur in

$$\frac{v-1}{2} - (\ell - 2) - 1 = \frac{\ell - 1}{2}$$

blocks of type 2. We have assumed $\ell > 3$, so the point $\ell + 1$ must occur in at least two blocks of type 2. However, if the point $\ell + 1$ occurs in a block B of type 2, then $1 \in B$ (otherwise, the sequencing is not ℓ-good). But the pair $\{1, \ell + 1\}$ is only contained in one block, so we have a contradiction. □

Example 2.1. Consider an $\mathsf{STS}(13)$. Here, we have that $(13 + 2)/3 = 5$. Theorem 2.1 tells us that there is no 6-good sequencing of an $\mathsf{STS}(13)$, and Theorem 2.2 extends this to show that no $\mathsf{STS}(13)$ has a 5-good sequencing. Similarly, because $(19 + 2)/3 = 7$, there is no 7-good sequencing of an $\mathsf{STS}(19)$.

$\mathcal{S}_1$	$\mathcal{S}_2$	$\cdots$	$\mathcal{S}_L$	$\mathcal{G}$	last L elements

Figure 1: The overall structure of the sequencing

3 Existence of ℓ-good sequencings

For any integer $\ell \geq 3$, it was conjectured in [4] that all "sufficiently large" $\mathsf{STS}(v)$ have ℓ-good sequencings. The conjecture was proven for $\ell = 3$ and $\ell = 4$ in [4]. Here, we prove the conjecture for all $\ell \geq 3$.

We use a greedy strategy similar to the algorithms discussed in [4]. The idea is to successively choose $x_1, \ldots, x_v$ in such a way that we end up with an ℓ-good sequencing of a given $\mathsf{STS}(v)$. However, this strategy is too simple to guarantee success, so we need to incorporate some modifications that we will discuss subsequently.

In general, when we choose a value for x_i, it must be distinct from $x_1, \ldots, x_{i-1}$, of course. It is also required that

$$x_i \notin P_{i,\ell} = \{\mathsf{third}(x_j, x_k) : i - \ell + 1 \leq j < k \leq i - 1\}. \tag{1}$$

Note that $|P_{i,\ell}| \leq \binom{\ell-1}{2}$. For ease of notation in the rest of this section, we will define $L = \binom{\ell-1}{2}$.

There will be a permissible choice for x_i provided that $i - 1 + L \leq v - 1$, which is equivalent to the condition $i \leq v - L$. Thus we can define $x_1, x_2, \ldots, x_{v-L}$ in such a way that they satisfy the relevant conditions—this is what we term the "greedy strategy." Our task is then to somehow fill in the last L positions of the sequencing, after appropriate modifications, to satisfy the desired properties. We describe how to do this now, for sufficiently large values of v.

Suppose that $[x_1\ x_2\ \cdots\ x_{v-L}]$ is an *ℓ-good partial sequencing* of $X = \{1, \ldots, v\}$ (that is, there is no block contained in any ℓ consecutive points in the sequence $[x_1\ x_2\ \cdots\ x_{v-L}]$). Let

$$X \setminus \{x_1, x_2, \ldots, x_{v-L}\} = \{\alpha_1, \ldots, \alpha_L\}.$$

Suppose we temporarily define $x_{v-L+i} = \alpha_i$ for $1 \leq i \leq L$.

The overall structure of the sequencing we will construct is presented in Figure 1. We note that there are L disjoint *segments* (denoted $\mathcal{S}_i$, $1 \leq i \leq L$), followed by a *gap* (denoted by $\mathcal{G}$), followed by the last L elements. These will all be described in detail as we progress.

3.1 Segments

As mentioned above, we will construct L disjoint segments, denoted $\mathcal{S}_i$, $1 \leq i \leq L$. Each segment $\mathcal{S}_i$ will consist of

- for $i \geq 2$, a *left buffer*, $\mathcal{B}_i^L$ (however, we will not require a left buffer for the first segment),

$$\mathcal{S}_i = \begin{array}{|c|c|c|c|} \hline \mathcal{B}_i^L & \mathcal{C}_i & \mathcal{B}_i^R & \mathcal{O}_i \\ \hline \multicolumn{1}{c}{\ell - 1} & \multicolumn{1}{c}{c_i} & \multicolumn{1}{c}{\ell - 1} & \multicolumn{1}{c}{} \end{array}$$

Figure 2: A segment $\mathcal{S}_i$ and the sizes of its components

- a *core* denoted by $\mathcal{C}_i$,
- a *right buffer*, $\mathcal{B}_i^R$, and
- an *overflow*, $\mathcal{O}_i$.

The above are all *ordered* lists of points in the $\mathsf{STS}(v)$. See Figure 2.

Each buffer has size $\ell - 1$ (except that the first left buffer has size 0) and the size of the core will be denoted by c_i. We will discuss the value of c_i and the size of the the overflow a bit later. The basic strategy of our algorithm will be to (if necessary) swap each α_i with either

(a) one of $\alpha_{i+1}, \dots, \alpha_L$ (there are $L - i$ choices here), or

(b) a point from the core $\mathcal{C}_i$ (there are c_i choices for such a point).

We will perform a sequence of swaps of this type, for $i = 1, 2, \dots, L$.

When we perform a swap $\alpha_i \leftrightarrow x_j \in \mathcal{C}_i$, we need to ensure that two conditions are satisfied:

1. $x_j \notin P_{v-L+i,\ell}$ (from (1)), and
2. α_i does not lead to the formation of a new block among any ℓ consecutive points in $\mathcal{S}_i$.

3.2 The core

First, we consider how big the core $\mathcal{C}_i$ needs to be. When we are defining x_{v-L+i}, if we have $L+1$ choices, then one of them must be good (i.e., not in the set $P_{v-L+i,\ell}$). The number of choices in **(a)** or **(b)** is $c_i + L - i + 1$, so we want $c_i + L - i + 1 \geq L + 1$, or $c_i \geq i$, for $1 \leq i \leq L$. (The "+1" term on the left side of the inequality accounts for the possibility that α_i might already be a good choice, in which case no swap would be necessary.) Thus, from this point on, we will assume that $c_i = i$ for all i.

3.3 The overflow and the buffers

Define $\mathcal{T}_i = \mathcal{B}_i^L \cup \mathcal{C}_i \cup \mathcal{B}_i^R$. We need to ensure that there are no blocks contained in ℓ consecutive points of $\mathcal{S}_i$ after a point α_i is swapped for a point in $\mathcal{C}_i$. This is accomplished by considering blocks containing two points in $\mathcal{T}_i$ and placing the relevant third points "out of harm's way" in the overflow.

We only need to consider blocks contained in ℓ consecutive points in $\mathcal{T}_i$, because

- the last point in the core and the first point in the overflow are not contained in ℓ consecutive points, due to the $\ell - 1$ points in the right buffer and
- for $i \geq 2$, the first point in the core and the last point in the previous overflow are not contained in ℓ consecutive points, due to the $\ell - 1$ points in the left buffer.

For $i \geq 2$, there are $i + 2\ell - 2$ points in $\mathcal{T}_i$. Denote these points (in order) by $z_1, \ldots, z_{i+2\ell-2}$. Define $\mathcal{J}_i$ to consist of all the ordered pairs (j_1, j_2) such that

- $1 \leq j_1 < j_2 \leq i + 2\ell - 2$ and
- $j_2 - j_1 \leq \ell - 1$.

Lemma 3.1. *For $2 \leq i \leq L$, we have*

$$|\mathcal{J}_i| = (\ell - 1)\left(i + \frac{3\ell - 4}{2}\right). \tag{2}$$

Proof. First, assume $2 \leq i \leq L$. Let $1 \leq d \leq \ell - 1$. There are exactly $i + 2\ell - 2 - d$ pairs $(j_1, j_2) \in \mathcal{J}_i$ with $j_2 - j_1 = d$. Hence,

$$\begin{aligned} |\mathcal{J}_i| &= \sum_{d=1}^{\ell-1}(i + 2\ell - 2 - d) \\ &= (\ell - 1)(i + 2(\ell - 1)) - \frac{\ell(\ell - 1)}{2} \\ &= (\ell - 1)\left(i + \frac{3\ell - 4}{2}\right). \end{aligned}$$

□

Now let's look at the initial case, $i = 1$. Here, we have $\mathcal{T}_1 = \mathcal{C}_1 \cup \mathcal{B}_1^R$, so $|\mathcal{T}_1| = \ell$. We define $\mathcal{J}_1$ to consist of the ordered pairs (j_1, j_2) such that $1 \leq j_1 < j_2 \leq \ell$. Therefore,

$$|\mathcal{J}_1| = \binom{\ell}{2}. \tag{3}$$

Next, define

$$Y = \left\{\mathsf{third}(z_{j_1}, z_{j_2}) : (j_1, j_2) \in \mathcal{J}_i\right\} \setminus (\mathcal{T}_i \cup \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{i-1}). \tag{4}$$

Note that, when we define Y we omit any points $\mathsf{third}(z_{j_1}, z_{j_2})$ that have already appeared in $\mathcal{T}_i \cup \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{i-1}$. Denote the points in Y as $y_1, \ldots, y_m$. Clearly, $m \leq |\mathcal{J}_i|$.

Having already chosen the points in $\mathcal{T}_i$, we want to "pre-specify" the location of the m points $y_1, \ldots, y_m$ in the overflow $\mathcal{O}_i$. This is done according to the algorithm in Figure 3. We should explain the spacing of points $Y = \{y_1, \ldots, y_m\}$ in the overflow.

Input: the set $Y = \{y_1, \ldots, y_m\}$ and an integer i, $1 \leq i \leq L$
Insert the points in Y into $\mathcal{O}_i$ as follows:
if ℓ is even **then**
  leave an initial gap of length $\ell - 2$ and then insert y_1
  for $2 \leq i \leq m$ **do**
    leave a gap of length $(\ell - 2)/2$ between y_{i-1} and y_i
else (i.e., ℓ is even)
  leave an initial gap of length $\ell - 2$ and then insert y_1
  for $2 \leq i \leq m$ **do**
    if i is even **then**
      leave a gap of length $(\ell - 3)/2$ between y_{i-1} and y_i
    else (i.e., i is odd)
      leave a gap of length $(\ell - 1)/2$ between y_{i-1} and y_i

Figure 3: Pre-specifying elements in the overflow $\mathcal{O}_i$

We want to avoid a situation where there could be three points (within ℓ consecutive points) that might comprise a block. The initial gap of length $\ell - 2$ ensures that the last two points of $\mathcal{B}_i^R$ and y_1 are not contained in ℓ consecutive points. Also, the remaining gaps are large enough to guarantee that no three points y_i, y_{i+1} and y_{i+2} are contained in ℓ consecutive points.

We can now compute the length of an overflow.

Lemma 3.2. *For any integer i such that $1 \leq i \leq L$, we have*

$$|\mathcal{O}_i| \leq \frac{\ell(|\mathcal{J}_i| + 1)}{2} - 1.$$

Proof. First, suppose ℓ is even. Using the notation above, the overflow consists of $\ell - 2$ initial values followed by the values in Y, each separated by $(\ell - 2)/2$ points. Let $|Y| = m$. Then the overflow has length

$$\frac{(m-1)(\ell-2)}{2} + m + \ell - 2 = \frac{\ell(m+1)}{2} - 1.$$

Since $m \leq |\mathcal{J}_i|$, it follows that

$$|\mathcal{O}_i| \leq \frac{\ell(|\mathcal{J}_i| + 1)}{2} - 1.$$

Now suppose ℓ is odd. Then the overflow consists of $\ell - 2$ initial values followed by the values in Y separated by $(\ell - 3)/2$ or $(\ell - 1)/2$ points, alternating. Let $|Y| = m$. If m is odd then the overflow has length

$$\frac{(m-1)}{2} \times \frac{(\ell-3)}{2} + \frac{(m-1)}{2} \times \frac{(\ell-1)}{2} + m + \ell - 2 = \frac{\ell(m+1)}{2} - 1.$$

Otherwise, m is even so the overflow has length

$$\frac{m}{2} \times \frac{(\ell-3)}{2} + \frac{(m-2)}{2} \times \frac{(\ell-1)}{2} + m + \ell - 2 = \frac{\ell(m+1)-3}{2}.$$

Since

$$\frac{\ell(m+1)-3}{2} < \frac{\ell(m+1)}{2} - 1,$$

and $m \le |\mathcal{J}_i|$, it follows that

$$|\mathcal{O}_i| \le \frac{\ell(|\mathcal{J}_i|+1)}{2} - 1$$

for all $i \ge 1$. □

Corollary 3.3. *For any integer i such that $2 \le i \le L$, we have*

$$|\mathcal{O}_i| \le \frac{i(\ell^2-\ell)}{2} + \frac{3\ell^3 - 7\ell^2 + 6\ell - 4}{4}.$$

Also,

$$|\mathcal{O}_1| \le \frac{\ell^3 - \ell^2 + 2\ell - 4}{4}.$$

Proof. Applying equations (2) and (3) and Lemma 3.2, we obtain

$$\begin{aligned} |\mathcal{O}_i| &\le \frac{\ell\left((\ell-1)\left(i + \frac{3\ell-4}{2}\right) + 1\right)}{2} - 1 \\ &= \frac{i(\ell^2-\ell)}{2} + \frac{3\ell^3 - 7\ell^2 + 6\ell - 4}{4} \end{aligned}$$

and

$$\begin{aligned} |\mathcal{O}_1| &\le \frac{\ell\left(\binom{l}{2} + 1\right)}{2} - 1 \\ &= \frac{\ell^3 - \ell^2 + 2\ell - 4}{4}. \end{aligned}$$

□

3.4 The gap

After carrying out the operations described in Figure 3, we fill in the rest of the overflow $\mathcal{O}_i$ using what we call the "modified greedy strategy." Each time we choose a new point x_j, we make sure that $x_j \notin P_{j,\ell}$, as per (1). However, we additionally need to make sure that there is no block contained in a set of ℓ consecutive points that may include points $x_{j'}$ with $j' > j$ that have been predefined as a result of the algorithm in Figure 3. In order to ensure that this can be done, we include a *gap*, denoted $\mathcal{G}$, that follows the last overflow, $\mathcal{O}_L$. $\mathcal{G}$ will contain elements after $\mathcal{O}_L$, up to, but not including, the last L points in the sequencing. The gap will be filled using the greedy strategy.

Let's determine how big the gap needs to be. First, consider the second last element of $\mathcal{O}_L$. The last element of $\mathcal{O}_L$, say x_κ has been pre-specified to be the value y_m. Now, as we have already mentioned, $x_{\kappa-1} \notin P_{\kappa-1,\ell}$, which rules out no more than L values for $x_{\kappa-1}$. Also,

$$x_{\kappa-1} \notin \{\mathsf{third}(x_j, x_\kappa) : \kappa - \ell + 1 \leq j \leq \kappa - 2\}.$$

This rules out up to $\ell - 2$ additional values for $x_{\kappa-1}$. The number of unused values is $|\mathcal{G}| + L + 1$, since we have not yet defined $x_{\kappa-1}$, any element in the gap, or any of the last L elements. So we require $L + \ell - 2 + 1 \leq |\mathcal{G}| + L + 1$, or $|\mathcal{G}| \geq \ell - 2$, in order to ensure that $x_{\kappa-1}$ can be defined.

We should also consider the element immediately preceding $y_{m-1} = x_\kappa$. Following x_κ, there is are β undefined elements, followed by y_m, where

$$\beta \in \left\{ \frac{\ell-1}{2}, \frac{\ell-2}{2}, \frac{\ell-3}{2} \right\}.$$

Suppose we have defined all elements up to but not including $x_{\kappa-1}$. Recall that the values x_κ and $x_{\kappa+\beta+1}$ have been prespecified.

The restrictions on $x_{\kappa-1}$ are as follows:

- $x_{\kappa-1} \notin P_{\kappa-1,\ell}$ (as before, which rules out at most L values),
- $x_{\kappa-1} \notin \{\mathsf{third}(x_j, x_\kappa) : \kappa - \ell + 1 \leq j \leq \kappa - 2\}$ (as before, which rules out at most $\ell - 2$ values),
- $x_{\kappa-1} \neq \mathsf{third}(x_\kappa, x_{\kappa+\beta+1})$ (at most one value is ruled out here)
- $x_{\kappa-1} \notin \{\mathsf{third}(x_j, x_{\kappa+\beta+1}) : \kappa + \beta - \ell + 2 \leq j \leq \kappa - 2\}$ (at most $\ell - \beta - 3$ values are ruled out).

Therefore the total number of values that are ruled out is at most

$$L + \ell - 2 + 1 + \ell - \beta - 3 = L + 2\ell - \beta - 4.$$

Since the β elements between x_κ and $x_{\kappa+\beta+1}$ have not yet been defined, the number of available elements is $|\mathcal{G}| + L + \beta + 1$. Therefore we can choose a value for $x_{\kappa-1}$ provided that

$$|\mathcal{G}| + L + \beta + 1 \geq L + 2\ell - \beta - 4 + 1,$$

which simplifies to give

$$|\mathcal{G}| \geq 2(\ell - \beta - 2).$$

If ℓ is even, then $\beta = (\ell - 2)/2$ and it suffices to take

$$|\mathcal{G}| \geq 2\left(\ell - \frac{\ell-2}{2} - 2\right) = \ell - 2.$$

```
Input: an STS(v) and an integer ℓ ≥ 3
L ← binom(ℓ−1, 2)
for i ← 1 to L do
    Fill in the values in B_i^L, C_i and B_i^R using the greedy strategy
    Compute the set Y = {y_1, ..., y_m}.
    Place the elements in Y into O_i as described in Figure 3.
    Fill in the rest of O_i using the "modified greedy strategy."
Fill in the points in G using the greedy strategy.
Compute X \ {x_1, x_2, ..., x_{v−L}} = {α_1, ..., α_L}.
for i ← 1 to L do
    x_{v−L+i} ← α_i
    If required, swap x_{v−L+i} with one of α_{i+1}, ..., α_L or a point from C_i.
return (π = [x_1 x_2 ··· x_v]).
```

Figure 4: Algorithm to find an ℓ-good sequencing for an $\mathsf{STS}(v)$, $(X, \mathcal{B})$

If ℓ is odd, then we have $\beta \geq (\ell - 3)/2$ and it suffices to take

$$|\mathcal{G}| \geq 2\left(\ell - \frac{\ell-3}{2} - 2\right) = \ell - 1.$$

Thus we have proven the following.

Lemma 3.4. *If ℓ is even, then the gap $\mathcal{G}$ can have any length $\geq \ell - 2$, and if ℓ is odd, then the gap $\mathcal{G}$ can have any length $\geq \ell - 1$.*

3.5 The algorithm

Finally, the last L points may be swapped (as described above) in order to ensure that we have an ℓ-good sequencing. Putting all the pieces together, we obtain the algorithm presented in Figure 4. The following lemma establishes the correctness of the algorithm.

Lemma 3.5. *There is no block contained in ℓ consecutive points of $\mathcal{S}_i$ after a swap.*

Proof. Suppose a block B is contained in ℓ consecutive points of $\mathcal{S}_i$ after a swap. Clearly, B must contain α_i, which is the point that was "swapped in." Suppose that $\{z_{j_1}, z_{j_2}, \alpha_i\}$ is such a block, where $j_1 < j_2$. Then $(j_1, j_2) \in \mathcal{J}_i$ and $\alpha_i = \mathsf{third}(j_1, j_2)$. However, from (4), it must be the case that $\mathsf{third}(j_1, j_2) \in Y$, in which case it occurs in the overflow; or

$$\mathsf{third}(j_1, j_2) \in \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_{i-1} \cup \mathcal{T}_i.$$

In each case, $\alpha_i \neq \mathsf{third}(j_1, j_2)$, so we have a contradiction. □

3.6 Analysis

In this section, we prove our general existence result. Recall that we have various components in our sequencing:

- L segments, $\mathcal{S}_i$ $(1 \leq i \leq L)$, each consisting of
  - for $i \geq 2$, a left buffer of size $\ell - 1$,
  - a core of size i,
  - a right buffer of size $\ell - 1$, and
  - an overflow, whose size is given in Corollary 3.3.
- the gap $\mathcal{G}$ of size $\geq \ell - 1$, and
- the final L elements.

Therefore a sequencing of an $\mathsf{STS}(v)$ will exist if v is at least as big as the sum of the lengths of all the components enumerated above. We therefore obtain the following sufficient condition for an ℓ-good sequencing to exist.

$$\begin{aligned}
v &\geq \sum_{i=1}^{L}(|\mathcal{B}_i^L| + |\mathcal{C}_i| + |\mathcal{B}_i^R| + |\mathcal{O}_i|) + |\mathcal{G}| + L \\
&= |\mathcal{C}_1| + |\mathcal{B}_1^R| + |\mathcal{O}_1| + \sum_{i=2}^{L}(|\mathcal{B}_i^L| + |\mathcal{C}_i| + |\mathcal{B}_i^R| + |\mathcal{O}_i|) + |\mathcal{G}| + L \\
&= 1 + \ell - 1 + |\mathcal{O}_1| + \sum_{i=2}^{L}(\ell - 1 + i + \ell - 1 + |\mathcal{O}_i|) + \ell - 1 + L \\
&= 2\ell - 1 + L + |\mathcal{O}_1| + \sum_{i=2}^{L}(2\ell - 2 + i + |\mathcal{O}_i|) \\
&= 2\ell - 1 + \binom{\ell-1}{2} + \frac{\ell^3 - \ell^2 + 2\ell - 4}{4} \\
&\quad + \sum_{i=2}^{\binom{\ell-1}{2}} \left(2\ell - 2 + i + \frac{i(\ell^2 - \ell)}{2} + \frac{3\ell^3 - 7\ell^2 + 6\ell - 4}{4}\right).
\end{aligned}$$

After some simplification, the following is obtained.

Theorem 3.6. *An* $\mathsf{STS}(v)$ *with*

$$v \geq \frac{(\ell - 1)(\ell^5 - 9\ell^3 + 20\ell^2 - 36\ell + 16)}{16} \tag{5}$$

has an ℓ-good sequencing.

Here is a simpler bound that follows from Theorem 3.6.

Table 1: Upper bounds on $n(\ell)$

ℓ	$n(\ell) \leq$
4	119
5	556
6	1984
7	5270
8	12760
9	26400
10	52118

Corollary 3.7. *An* $\mathsf{STS}(v)$ *with* $v \geq \ell^6/16$ *has an* ℓ*-good sequencing.*

Proof. Consider the polynomial

$$9\ell^3 - 20\ell^2 + 36\ell - 16.$$

This polynomial has a single root at $\ell \approx 0.58421$. Since $\ell \geq 3$, we know that $9\ell^3 - 20\ell^2 + 36\ell - 16 > 0$, from which it follows that

$$\ell^5 - 9\ell^3 + 20\ell^2 - 36\ell + 16 < \ell^5.$$

Clearly, $\ell - 1 < \ell$, so

$$(\ell - 1)(\ell^5 - 9\ell^3 + 20\ell^2 - 36\ell + 16) < \ell^6$$

for $\ell \geq 3$. Hence, (5) holds, and the result follows from Theorem 3.6. □

For small values of ℓ, we obtain the explicit bounds on $n(\ell)$ given in Table 1. We obtain slightly stronger bounds than Theorem 3.6 by using a gap of size $\ell - 2$ when feasible (see Lemma 3.4) and a more precise bound on the size of the overflow $\mathcal{O}_i$ when ℓ is odd and $|\mathcal{J}_i|$ is even, as described in the proof of Lemma 3.2.

Note that the upper bound on $n(4)$ is not as good as the one proven in [4]. Of course, the result from [4] is obtained from an algorithm that was specially designed for the case $\ell = 4$.

4 Discussion and Conclusion

Our algorithm is based on ideas from [4], where an algorithm to find a 4-good sequencing of an $\mathsf{STS}(v)$ was developed. The algorithm from [4] also employed the "greedy strategy," "modified greedy strategy" and an overflow (although the latter term was not used in [4]) in much the same way as the present algorithm. In [4], only a single overflow and swap was needed. As a result, the algorithm presented in [4] works for smaller values of v than the general algorithm we describe in this paper. However, the approach in [4] did not seem to generalize well to larger values of ℓ, so the algorithm we have presented here employs a series of (up to) L swaps that take place in disjoint intervals. This permits the development of an algorithm for arbitrary values of ℓ.

It would of course be of interest to obtain more accurate upper and lower bounds on ℓ (as a function of v) for the existence of ℓ-good sequencings of $\mathsf{STS}(v)$. Phrased in terms of asymptotic complexity, our necessary condition is that ℓ is $O(v)$, while the sufficient condition proven in this paper is that ℓ is $\Omega(v^{1/6})$. Closing this gap is an interesting open problem.

References

[1] Brian Alspach, Donald L. Kreher and Adrián Pastine, Sequencing partial Steiner triple systems, *J. Combin. Designs* **28** (2020), 327–343.

[2] Charles J. Colbourn and Jeffrey H. Dinitz, *Handbook of Combinatorial Designs, Second Edition*, Chapman & Hall/CRC, 2006.

[3] Donald L. Kreher and Douglas R. Stinson, Nonsequenceable Steiner triple systems, *Bull. ICA* **86** (2019), 64–68.

[4] Donald L. Kreher and Douglas R. Stinson, Block-avoiding sequencings of points in Steiner triple systems, *Australas. J. Combin.* **74** (2019), 498–509.

[5] Charles J. Colbourn and Alexander Rosa, *Triple Systems*, Oxford University Press, 1999.

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