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Some relations on prefix-reversal generators of the symmetric and hyperoctahedral groups

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Abstract

The symmetric group S_n and the group of signed permutations B_n (also referred to as the hyperoctahedral group) can be generated by prefix-reversal permutations. A natural question is to determine the order of the "Coxeter-like" products formed by multiplying two generators, and in general, the relations satisfied by the prefix-reversal generators (also known as pancake generators or pancake flips). The order of these products is related to the length of certain cycles in the pancake and burnt pancake graphs. Using this connection, we derive a description of the order of the product of any two of these generators from a result due to Konstantinova and Medvedev. We provide a partial description of the order of the product of three generators when one of the generators is the transposition $(1, 2)$. Furthermore, we describe the order of the product of two prefix-reversal generators in the hyperoctahedral group and give connections to the length of certain cycles in the burnt pancake graph.

1 Introduction

Thinking of the symmetric group as being generated by prefix-reversals (also referred to as pancake flips and pancake generators) has been studied in several areas in mathematics and computer science. However, the purely algebraic question asking for

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the relations satisfied by said prefix reversals has not been asked directly. However, there is related work in the literature, mostly in the study of short cycles in the pancake graph.

1.1 The pancake and burnt pancake problems

The *pancake problem*, which first appeared in the Problems and Solutions section of the December 1975 *Monthly* [21] as follows.

> The chef in our place is sloppy, and when he prepares a stack of pancakes they come out all different sizes. Therefore, when I deliver them to a customer, on the way to the table I rearrange them (so that the smallest winds up on top, and so on, down to the largest on the bottom) by grabbing several from the top and flipping them over, repeating this (varying the number I flip) as many times as necessary. If there are n pancakes, what is the maximum number of flips (as a function $f(n)$ of n) that I will ever have to use to rearrange them?

The problem of determining the maximum number of flips that are ever needed to sort a stack of n pancakes is known as the *pancake problem*, and the $f(n)$ is known as the *pancake number*.

This initial posing of the problem was made by Jacob E. Goodman, under the pseudonym Harry Dweighter (a pun on "harried waiter"). In [12], as a commentary to the problem formulation in [21], Michael R. Garey, David S. Johnson, and Shen Lin gave the first upper and lower bound to the the pancake number:

$$n + 1 \leq f(n) \leq 2n - 6 \text{ for } n \geq 7.$$

Subsequent results have tightened these bounds. The first significant tightening of the bounds was described in the work of William H. Gates and Christos H. Papadimitriou [17], which incidentally is the only academic paper Gates ever wrote. The best upper and lower bound known today for the general case appeared in [7] and [19], respectively. Combined, one has that

$$15 \left\lfloor \frac{n}{14} \right\rfloor \leq f(n) \leq \frac{18n}{11} + O(1).$$

Computing the pancake number for a given n is a complicated task. To our knowledge, the exact value of $f(n)$ is only known for $1 \leq n \leq 19$ (see [2, 8, 9, 19, 28]). Furthermore, determining an optimal way of sorting a stack of pancakes utilizing pancake flips is an NP-hard problem [6], though 2-approximation algorithms exists [16].

The *burnt pancake problem* was first posed in [17]. In this variation, the pancakes to be sorted have an orientation and the goal is to sort the stack so that the pancakes are in the respective order according to size *and* orientation. In this setting, polynomial-time algorithms are known to sort optimally a stack of burnt pancakes by using *all* reversals (and not just prefix reversals), the first of which was given

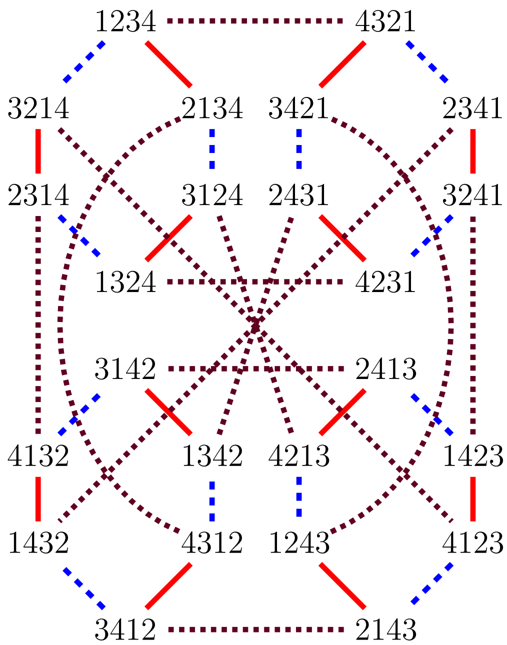

Figure 1: Pancake graph of S_4. The different colors indicate the different pancake generators.

in [18]. To our knowledge, no such exact algorithm exists to sort optimally a stack of pancakes, and no exact algorithm is known to sort a stack of burnt pancakes utilizing only prefix reversals [6].

One connection to the pancake and burnt pancake problem that has been heavily explored is to genome rearrangements. It turns out that genomes frequently evolve by reversals that transform a gene order

$$a_1a_2\cdots a_{i-1}a_ia_{i+1}\cdots a_{j-1}a_ja_{j+1}\cdots a_n$$

into

$$a_1a_2\cdots a_{i-1}a_ja_{j-1}\cdots a_{i+1}a_ia_{j+1}\cdots a_n$$

(see [15, Section 3.3] and [18]). Another connection is in the realm of parallel computing, and we discuss this in the next subsection.

1.2 Pancake graphs

The pancake problem has connections to parallel computing, in particular in the design of symmetric interconnection networks (networks used to route data between the processors in a multiprocessor computing system) where the so-called *pancake graph*, the Cayley graph of the symmetric group under prefix reversals, gives a model for processor interconnections (see [1, 32]). A pancake network is shown in Figure 1. One can also define a *burnt pancake graph* on signed permutations (See Section 2 for the necessary definitions), and we exhibit one in Figure 2.

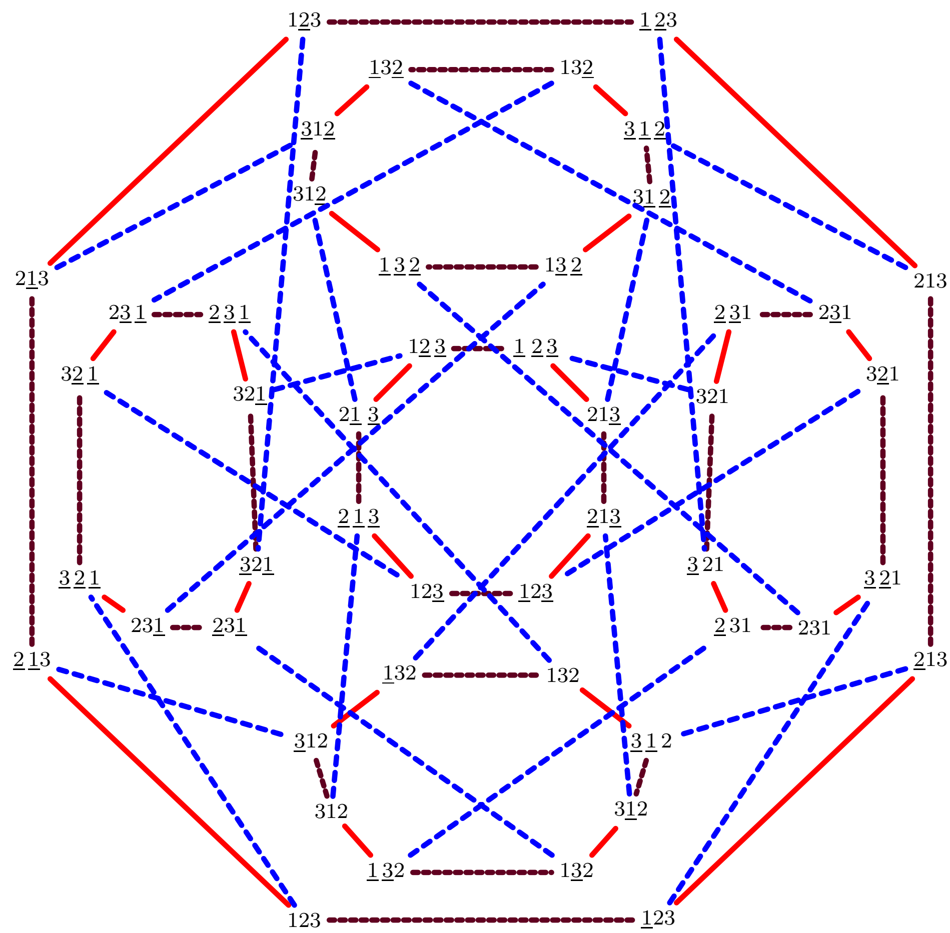

Figure 2: Pancake graph of B_3. Different edge colors indicate the different pancake generators.

Finding the diameter of the pancake graph is effectively the same as solving the pancake problem. Several properties for these graphs are known, including that they are vertex-transitive, which intuitively means that any vertex looks like any other vertex in the graph. There are some results relating to cycles of certain type that exist in the pancake graph [23, 24, 26, 27]. Since the pancake graph is vertex-transitive, any cycle C would give rise to a relation satisfied by certain generators. This connection will allow us to derive Theorem 3.1. The results for type B that we present in Section 4 are of a similar flavor, and we derive connections to the length of certain cycles in the burnt pancake graph.

It is known that the pancake graph has cycles of all lengths between its girth of six and $n!$ [20, 33]. In recent results the authors, with Akshay Patidar, have shown that the burnt pancake graph also has cycles of all lengths between its girth of eight and $2^n n!$ [4]. Furthermore, again with Akshay Patidar, we applied specific descriptions short cycles to find all permutations and signed permutations that are precisely four prefix reversals from the identity [5].

1.3 Our results

One can see that the pancake flips (also referred to as prefix reversals and pancake generators) generate both the symmetric and the hyperoctahedral group. Since both groups are Coxeter groups, then by Tits' theorem [34] it is known that they have solvable word problems regardless of the generators [14, Theorem 2.4.1]. Although the pancake flips share much similarity to the standard Coxeter generators, adjacent transpositions, of both groups these prefix reversals are *not* Coxeter generators. That is, there are more relations that are satisfied in a complete presentation of either group that are not implied by the product of two generators. So it is a natural question to ask: "what relations are satisfied by these pancake generators?" Another follow up questions is to ask: "what is a complete set of relations to give a presentation for S_n (or B_n) in terms of pancake generators?" In this paper, we provide a partial answer to the first question. Specifically, we

1. Describe the relations satisfied by any two pancake generators in S_n. This result will be derived from a result on the length of certain cycles in the pancake graph from [27]. All the details are included in Section 3.

2. Describe the relations satisfied by any two pancake generators in B_n. We also make connections to the length of certain cycles in the burnt pancake graph. All the details are included in Section 4.

3. Provide a partial result regarding the relations satisfied by three generators by describing all relations that involve the pancake generator $s_1 = (1, 2)$ in S_n (s_1 is denoted as f_1 in our notation as explained in Section 2). This is also included in Section 3.

Finding a complete presentation of a group is interesting in its own right, as in [30, 31]. The results presented here also provide progress toward answering the second question. Since we describe all relations of pairs of generators, and all of the lexicographically first relations, when ordering the indices, of three generators of S_n (those starting with the generator f_1), we have a great deal of known relations to contribute to a complete presentation of either group, and a paradigm to recovering more such relations.

In regard to the original pancake problem, having a complete presentation of the symmetric and hyperoctahedral groups using pancake generators would allow us to employ the Knuth-Bendix algorithm [22] to describe a *confluent* rewriting system of words in the generators. A system is considered confluent if there exists a unique reduced word in terms of a well-ordering of the generators. Using such a rewriting system it may be possible to implement an algorithm, like in [13], to compute pancake numbers. It would also be worth investigating whether these rules provide any combinatorial descriptions of reductions in terms of the permutation associated with a given word.

2 Terminology and Notation

Following [3, Section 1.1], if S is any set, then a *Coxeter matrix* is one whose entries $m_{s,s'} \in \mathbb{Z}^+ \cup \{\infty\}$ satisfy $m_{s,s'} = m_{s',s}$ and $m_{s,s'} = 1$ if and only if $s = s$ for every $s, s' \in S$. It is well known that, up to isomorphism, there is a one-to-one correspondence between Coxeter matrices and Coxeter systems (see [3, Theorem 1.1.2]).

The symmetric group S_n is generated by the set $S := \{s_1, \ldots, s_{n-1}\}$ of adjacent transpositions; that is, $s_i = (i, i+1)$ in cycle notation, and has the following presentation

$$S_n := \langle S \mid (s_i s_j)^{m_{i,j}} = e \rangle,$$

where e denotes the identify permutation, $m_{i,j} = 1$ if and only if $i = j$, $m_{i,j} = m_{j,i} = 2$ if and only if $|i-j| \geq 2$, and $m_{i,j} = m_{j,i} = 3$ if and only if $|i-j| = 1$ for all $i, j \in [n-1]$. It is well-known that the pair (S_n, S) is a *Coxeter system* (see [3]). In particular, the matrix $(m_{i,j})_{i,j \in [n-1]}$ is a Coxeter matrix.

Following standard notation we shall use $e = [1\ 2\ 3\ \ldots\ n]$ for the *identity permutation* in S_n. We will associate to elements of S_n permutations through left actions. That is, for example,

$$s_i \circ [1\ 2\ 3\ \ldots\ i\ (i+1)\ \ldots\ n] = [1\ 2\ 3\ \ldots\ (i+1)\ i\ \ldots\ n].$$

The pancake problem has a straight-forward interpretation in terms of permutations. A stack of n pancakes of different sizes can be thought of as an element of S_n and flipping a stack of pancakes with a spatula can be thought of as using a *prefix reversal permutation*; that is, a permutation whose only action when composed with $w \in S_n$ is to reverse the first so many characters of w, in one-line-notation. In other words, using one-line notation, a prefix reversal permutation of S_n has the form

$$\begin{aligned} f_i &= [(i+1)\ \ i\ (i-1)\ \ \ldots\ 2\ \ 1\ \ (i+2)\ \ (i+3)\ \ldots\ n] \\ &= (1, i+1)(2, i) \cdots \left(\left\lfloor \frac{i+2}{2} \right\rfloor, \left\lceil \frac{i+2}{2} \right\rceil \right), \end{aligned}$$

as a product of transpositions, for some $i \in [n-1]$. We denote the above permutation by f_i, with $1 \leq i \leq n-1$ and define $P = \{f_1, \ldots, f_{n-1}\}$. For example, in S_4 one has $f_1 = [2\ 1\ 3\ 4], f_2 = [3\ 2\ 1\ 4]$, and $f_3 = [4\ 3\ 2\ 1]$.

Considering adjacent transpositions with prefix reversal permutations, one can easily see that $s_i = f_i f_1 f_i$ and that $f_i = s_1(s_2 s_1) \cdots (s_{i-1} \cdots s_2 s_1)(s_i \cdots s_2 s_1)$. Hence, S_n is also generated by P. We refer to the elements of P as *pancake generators* (also referred to as *pancake flips* or *prefix-reversal generators*) of S_n.

Let B_n be the *hyperoctahedral group*, most commonly referred to as the group of *signed permutations* of the set $[\pm n] = \{\underline{n}, \underline{n-1}, \ldots, \underline{1}, 1, 2, \ldots, n\}$, where $\underline{i} = -i$. That is, permutations w of $[\pm n]$ satisfying $w(\underline{i}) = \overline{w(i)}$ for all $i \in [\pm n]$. We shall use *window notation* to denote $w \in B_n$; that is, we denote w by $[w(1)\ w(2)\ \ldots\ w(n)]$. The group B_n is generated by the set $\{s_0^B, s_1^B, \ldots, s_{n-1}^B\}$, where $s_0^B = [\underline{1}\ 2\ \cdots\ n]$ and for $1 \leq i \leq n-1$, $s_i^B = [1\ 2\ \cdots\ (i-1)\ (i+1)\ i\ (i+2) \cdots\ n]$ (see [3, Chapter 8]).

The burnt pancake generators affect the orientation of the entries: they are negative if they have been reversed an odd number of times and positive otherwise. We define f_i^B, $1 \le i \le n-1$ to be the signed permutation

$$\begin{aligned} f_i^B &= [\underline{i+1}\ \underline{i}\ \underline{i-1}\ \ldots \underline{2}\ \underline{1}\ (i+2)\ (i+3)\ \ldots n] \\ &= (1,\ \underline{i+1},\ \underline{1},\ i+1)(2,\ \underline{i},\ \underline{2},\ i)\ldots\left(\left\lfloor\frac{i+2}{2}\right\rfloor,\ \underline{\left\lceil\frac{i+2}{2}\right\rceil},\ \underline{\left\lfloor\frac{i+2}{2}\right\rfloor},\ \left\lceil\frac{i+2}{2}\right\rceil\right) \end{aligned}$$

in disjoint cycle form as elements of the symmetry group of $[\pm n]$,and $f_0^B = s_0^B$. Thus, for example, in B_4 we have $f_0^B = [\underline{1}\ 2\ 3\ 4], f_1^B = [\underline{2}\ \underline{1}\ 3\ 4], f_2^B = [\underline{3}\ \underline{2}\ \underline{1}\ 4]$, and $f_3^B = [\underline{4}\ \underline{3}\ \underline{2}\ \underline{1}]$. We shall define $P^B = \{f_0^B, f_1^B, \ldots, f_{n-1}^B\}$ as the set of *burnt pancake generators*, or *burnt pancake flips*. It should be noted, that these are the signed versions indicated in this paragraph.

Again, considering (signed) adjacent transpositions and prefix reversals, one can see that $s_i^B = f_i^B f_0^B f_1^B f_0^B f_i^B$ for $1 \le i \le n-1$ and $s_0^B = f_0^B$, thus B_n is also generated by P^B. Furthermore, we note that $f_i^B = s_0^B s_1^B \ldots s_0^B s_{i-1}^B \ldots s_2^B s_1^B s_0^B s_i^B \ldots s_2^B s_1^B s_0^B$.

3 S_n results

In this section we take a look at the pancake generators for S_n. In this case, the pancake matrix $M_{n-1} = (m_{i,j})_{(n-1)\times(n-1)}$ where $m_{i,j}$ is the order of $f_i f_j$ can be derived from [27, Lemma 1]. We include their result and then prove that "reflections" using the pancake generators are just the set of involutions in S_n; that is, the set of elements that have order 2. We conclude the section with some results for the order of elements of the form $f_i f_j f_k$.

The theorem below provides a description for M_{n-1}. It turns out M_{n-1} is symmetric and all its diagonal entries are 1. Most of these entries are described by rephrasing a result of Konstantinova and Medvedev [27, Lemma 1].

Theorem 3.1. *Let $m_{i-1,j-1}$ be the order of $f_{i-1}f_{j-1}$ with $1 < i < j \le n$, then*

1. $m_{i-1,i-1} = 1$,
2. $m_{i-1,j-1} = m_{j-1,i-1}$,
3. $m_{1,2} = 3$, *and*
4. *if $j \ge 4$, then*
   (a) *If $1 < i \le \lfloor \frac{j}{2} \rfloor$,then $m_{i-1,j-1} = 4$.*
   (b) *If $1 < \lfloor \frac{j}{2} \rfloor < i < j-1$, then*

$$m_{i-1,j-1} = \begin{cases} 2q(q+1), & \text{if } r \ge 2, t \ge 2, \text{ or } r = 1, t \ge 2, q \text{ is even,} \\ & \text{or } r \ge 2, t = 1, q \text{ is odd;} \\ q(q+1), & \text{if } r = 1, t \ge 2, q \text{ is odd, or } r \ge 2, t = 1, \\ & q \text{ is even, or } r = 1, t = 1; \\ 2q, & \text{if } r = 0; \end{cases}$$

$$
\begin{pmatrix}
1 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
3 & 1 & 4 & 6 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 1 & 5 & 6 & 12 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 6 & 5 & 1 & 6 & 12 & 6 & 12 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 6 & 6 & 1 & 7 & 8 & 6 & 12 & 12 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 12 & 12 & 7 & 1 & 8 & 20 & 12 & 6 & 12 & 12 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 6 & 8 & 8 & 1 & 9 & 10 & 24 & 6 & 12 & 12 & 12 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 12 & 6 & 20 & 9 & 1 & 10 & 30 & 8 & 12 & 6 & 12 & 12 & 12 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 12 & 12 & 10 & 10 & 1 & 11 & 12 & 40 & 24 & 6 & 12 & 12 & 12 & 12 & 4 \\
4 & 4 & 4 & 4 & 12 & 6 & 24 & 30 & 11 & 1 & 12 & 42 & 20 & 24 & 12 & 6 & 12 & 12 & 12 \\
4 & 4 & 4 & 4 & 4 & 12 & 6 & 8 & 12 & 12 & 1 & 13 & 14 & 10 & 8 & 24 & 6 & 12 & 12 \\
4 & 4 & 4 & 4 & 4 & 12 & 12 & 12 & 40 & 42 & 13 & 1 & 14 & 56 & 30 & 40 & 24 & 12 & 6 \\
4 & 4 & 4 & 4 & 4 & 4 & 12 & 6 & 24 & 20 & 14 & 14 & 1 & 15 & 16 & 60 & 40 & 24 & 24 \\
4 & 4 & 4 & 4 & 4 & 4 & 12 & 12 & 6 & 24 & 10 & 56 & 15 & 1 & 16 & 72 & 12 & 20 & 8 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 12 & 12 & 12 & 8 & 30 & 16 & 16 & 1 & 17 & 18 & 84 & 10 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 12 & 12 & 6 & 24 & 40 & 60 & 72 & 17 & 1 & 18 & 90 & 42 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 12 & 12 & 6 & 24 & 40 & 12 & 18 & 18 & 1 & 19 & 20 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 12 & 12 & 12 & 12 & 24 & 20 & 84 & 90 & 19 & 1 & 20 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 12 & 12 & 6 & 24 & 8 & 10 & 42 & 20 & 20 & 1
\end{pmatrix}
$$

Figure 3: Pancake Matrix M_{19} for S_{20}. The $m_{i,j}$ entry is the order of $f_i f_j$. Notice that the matrix is symmetric, the entries in the main diagonal are all 1 and the entries in the off-diagonal are the positive integers that are at least 3.

where $d = j - i$, $q = \lfloor \frac{j}{d} \rfloor$, $r = j \pmod{d}$ *and* $t = d - r$.

(c) If $i = j - 1$, *then* $m_{i-1,j-1} = j$.

Proof. For (1), since f_i is an involution, it follows that $m_{i,i} = 1$.

For (2) notice that $(f_i f_j)^{-1} = f_j f_i$, so it follows that $m_{i,j} = m_{j,i}$, and therefore M_{n-1} is symmetric.

Case (3) Follows from direct computation.

For Case (4), notice that elements in S_{n-1} can be viewed as elements in S_n leaving n fixed, the matrix M_{n-1} can be viewed as a submatrix of M_n by ignoring the last row and column of M_n. So Case (4) follows from [27, Lemma 1] by having n take different values. $\square$

Remark. We point out that in [27], the authors use r_j with $2 \leq j \leq n$ to denote the permutation that reverses the first j terms from the identity permutation $123 \ldots n$. In other words, $r_j = f_{j-1}$, for $2 \leq j \leq n$. However, our notation resembles the notation that is used for S_n viewed as a Coxeter group generated by S, the set of adjacent transpositions.

Example 3.1. In Figure 3, we depict the 19×19 Coxeter matrix M_{19} for S_{20}.

We now describe the set of so called "reflections" with respect to the generator set P, that is, the conjugates of elements of P by permutations. In Coxeter groups, if (W, S) is a Coxeter system, the set $\{wsw^{-1} \mid w \in W, s \in S\}$ plays a crucial role algebraically and geometrically (see [3]). It turns out that if one uses pancake generators, the set $\{wf_iw^{-1} \mid w \in S_n, i \in [n-1]\}$ is the set of involutions in S_n.

Theorem 3.2. *The set of conjugates of the pancake generators*

$$T = \{wf_iw^{-1} \mid i \in [n-1], w \in S_n\}$$

coincides with the set of all involutions (self-inverse permutations) in S_n.

Proof. If $f_i \in P$ and $w \in S_n$, then $(wf_iw^{-1})^2 = e$, so each element in T is an involution.

Conversely, suppose t is an arbitrary involution in S_n. Then the t can be written in disjoint cycle notation using only length two cycles. Say $t = (a_1, b_1)(a_2, b_2) \cdots (a_k, b_k)$ with $a_1 < a_2 < \cdots < a_k$ and $a_i < b_i$ for all $i \in [k]$. We know that $k \leq \left\lfloor \frac{n}{2} \right\rfloor$ thus $2k - 1 \leq 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 \leq n - 1$. Consider the flip

$$f_{2k-1} = (1, 2k)(2, 2k-1) \cdots (k, k+1)$$

which consists of k disjoint two-cycles and

$$w = [a_1\ a_2\ \ldots\ a_k\ b_k\ b_{k-1}\ \ldots\ b_2\ b_1 w_{2k+1}\ \ldots\ w_n], \text{ in one-line notation,}$$

where $w_{2k+1} \ldots w_n$ is an arbitrary permutation of $[n] \setminus \{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$.

The element t is in T if $wf_{2k-1} = tw$. Notice that

$$wf_{2k-1} = [b_1\ b_2\ \ldots\ b_{k-1}\ b_k\ a_k\ a_{k-1}\ \ldots\ a_2\ a_1\ w_{2k+1}\ \ldots\ w_n].$$

Furthermore,

$$tw = [b_1\ b_2\ \ldots\ b_{k-1}\ b_k\ a_k\ a_{k-1}\ \ldots\ a_2\ a_1\ w_{2k+1}\ \ldots\ w_n].$$

Therefore $t \in T$. □

Since $|T|$ is the same as the number of involutions in S_n, we have the following corollary.

Corollary 3.3. $|T| = \displaystyle\sum_{k=1}^{\lfloor n/2 \rfloor} \frac{n!}{2^k (n-2k)! k!}$.

Remark. In the symmetric group, every reflection, that is, every element of the form wsw^{-1}, where s is an adjacent transposition and w is a permutation, is an involution. However, there are involutions in the symmetric group that are not reflections. As Theorem 3.2 shows, if we use the pancake generators for the symmetric group, the "reflections" obtained are indeed **all** the involutions in S_n.

3.1 Order of $f_i f_j f_k$

We now discuss the order $m_{i,j,k}$ of $f_i f_j f_k$. In S_n, there would potentially be $(n-1)^3$ orders to consider for any three generators. However, all we need to consider are the orders in the case where $i \leq j \leq k$, as the order of $f_{\sigma(i)} f_{\sigma(j)} f_{\sigma(k)}$ is also $m_{i,j,k}$, as shown in the following lemma.

Lemma 3.4. *For all i, j, k with $1 \leq i, j, k \leq n$ and any permutation σ of $\{i, j, k\}$, the order of $f_i f_j f_k$ is the same as the order of $f_{\sigma(i)} f_{\sigma(j)} f_{\sigma(k)}$.*

Proof. There are two cases to consider.

Case $|\{i, j, k\}| < 3$ In this case, the order of $f_i f_j f_k$ is 2. Indeed, if $|\{i, j, k\}| = 2$ then $f_i f_j f_k$ has the form $f_a f_a f_b$ or $f_a f_b f_a$ or $f_b f_a f_a$, for $a, b \in \{i, j, k\}$, all of which have order two. Furthermore, if $|\{i, j, k\}| = 1$, then $f_i f_i f_i = f_i$, which also has order 2.

Case $|\{i, j, k\}| = 3$ Notice that $f_i f_j f_k, f_j f_k f_i$, and $f_k f_i f_j$ are in the same conjugacy class of S_n; for example, $f_k f_i f_j = f_k (f_i f_j f_k) f_k$. Therefore they all have the same order as they have the same cycle structure. Moreover, $f_k f_j f_i, f_i f_k f_j$, and $f_j f_i f_k$ are in the same conjugacy class, and so they have the same order as well. Since $f_k f_j f_i = (f_i f_j f_k)^{-1}$, the lemma follows.

□

Here are a collection of partial results on the orders of elements of the form $f_i f_j f_k$. Specifically these are all of the relations where the leftmost generator is f_1, i.e. all of the orders of $f_1 f_j f_k$.

Theorem 3.5. *Let $m_{1,j-1,k-1}$ be the order of $f_1 f_{j-1} f_{k-1}$ with $1 < j < k \leq n$, then*

1. $m_{1,1,j-1} = m_{1,j-1,j-1} = 2$,
2. *if* $j \geq 6$, *then* $m_{1,2,j-1} = 6$,
3. *if* $j = k-1$, *then* $m_{1,j-1,k-1} = k-1$,
4. *if* $j = k-2$ *and* k *is odd or* $j = k-3$ *and* $2 \neq k \pmod 3$, *then* $m_{1,j-1,k-1} = k$,
5. *if* $k \geq 5$, *then*

$$\text{(a) } m_{1,j-1,k-1} = \begin{cases} 4q, & \text{if } r = 0, d \geq 4; \\ 2q+1, & \text{if } r = 1, d = 2; \\ q(3q+1), & \text{if } r = 1, d = 4 \text{ or } r = 1, d \geq 5, q \text{ is odd}; \\ 2q(3q+1), & \text{if } r = 1, d \geq 5, q \text{ is even}. \end{cases}$$

$$\textit{(b)}\ m_{1,j-1,k-1} = \begin{cases} q(q+1), & \textit{if } r=2, d=3,\ \textit{or } r=2, d\geq 4, q \textit{ is odd}, \\ & \quad \textit{or } r=3, d=4, 0=q \pmod 3, \\ & \quad \textit{or } r=3, d\geq 5, 3=q \pmod 6, \\ & \quad \textit{or } r\geq 4, d\geq 5, 0=q \pmod 4; \\ 2q(q+1), & \textit{if } r=2, d\geq 4, q \textit{ is even}, \\ & \quad \textit{or } r=3, d\geq 5, 0=q \pmod 6, \\ & \quad \textit{or } r\geq 4, d\geq 5, 2=q \pmod 4; \\ 3q(q+1), & \textit{if } r=3, d=4, 0\neq q \pmod 3, \\ & \quad \textit{or } r=3, d\geq 5, \{1,5\}\ni q \pmod 6; \\ 4q(q+1), & \textit{if } r\geq 4, d\geq 5, q \textit{ is odd}; \\ 6q(q+1), & \textit{if } r=3, d\geq 5,\ \{2,4\}\ni q \pmod 6; \end{cases}$$

where $d = k-j, q = \left\lfloor \frac{k}{d} \right\rfloor$, *and* $r = k \pmod d$.

Proof. For Case (1), note that for any $1 \leq i \leq n-1$, $(f_i)^2 = e$. So $f_1 f_1 f_{j-1} = f_{j-1}$, which is order two, and $f_1 f_{j-1} f_{j-1} = f_1$, which is also order two.

For each of the following cases we will look at the disjoint cycle notation of the permutations to find the order of the three generators.

For Case (2), let $j \geq 6$.

$$\begin{aligned} f_1 f_2 f_{j-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & j \\ j-1 & j-2 & j & j-3 & \cdots & 1 \end{pmatrix} \\ &= (1, j-1, 2, j-2, 3, j)\,(4, j-3)\ldots\left(\left\lfloor \frac{j+1}{2} \right\rfloor, \left\lceil \frac{j+1}{2} \right\rceil\right). \end{aligned}$$

The least common multiple of these lengths is 6, which is the order of the permutation.

Since elements in S_n is a parabolic subgroup of S_{n+1}, generated by all but the largest indexed generator, then the matrix of $(m_{1,j-1,k-1})_{1\leq j,k\leq n}$ is a submatrix of the matrix $(m_{1,j-1,k-1})_{1\leq j,k\leq n+1}$ with the last row and last column removed. Thus it is sufficient to consider only the cases with $k = n$.

For Case (3), the three generators result in

$$\begin{aligned} f_1 f_{n-2} f_{n-1} &= \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 3 & 2 & 4 & \cdots & n & 1 \end{pmatrix} \\ &= (1, 3, 4, \ldots, n). \end{aligned}$$

The length of this disjoint cycle is $n-1$. Thus the order is $n-1$.

For Case (4), first consider $j = n-2$ and n is odd.

$$\begin{aligned} f_1 f_{n-3} f_{n-1} &= \begin{pmatrix} 1 & 2 & 3 & \cdots & n-2 & n-1 & n \\ 4 & 3 & 5 & \cdots & n & 2 & 1 \end{pmatrix} \\ &= (1, 4, 6, \ldots, n-1, 2, 3, 5, \ldots, n), \end{aligned}$$

whose cycle length is n. Second consider $j = n-3$ and $2 \neq n \pmod 3$. Say $n = 3\mathfrak{q} + \mathfrak{r}$.

$$f_1 f_{n-4} f_{n-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-3 & n-2 & n-1 & n \\ 5 & 4 & 6 & \cdots & n & 3 & 2 & 1 \end{pmatrix}$$

When $\mathfrak{r} = 0$,

$$f_1 f_{n-4} f_{n-1} = (1, 5, 8, \ldots, 3(\mathfrak{q}-1)+2, 2, 4, 7, \ldots, 3(\mathfrak{q}-1)+1, 3, 6, \ldots, 3\mathfrak{q}).$$

When $\mathfrak{r} = 1$,

$$f_1 f_{n-4} f_{n-1} = (1, 5, 8, \ldots, 3(\mathfrak{q}-1)+2, 3, 6, 9, \ldots, 3\mathfrak{q}, 2, 4, 7, \ldots 3\mathfrak{q}+1), \text{ when } \mathfrak{r} = 1.$$

With both possible values of $\mathfrak{r}$ the length of the disjoint cycle is n.

For Case (5) let $d = n - j$, $q = \left\lfloor \frac{n}{d} \right\rfloor$, and $r = n \pmod d$. We will consider each possible value of r followed by the possibilities of d. Since $n = qd + r$ and $d = n - j$, then $j = (q-1)d + r$. In general, three generators form the permutation

$$f_1 f_{j-1} f_{n-1} = \begin{pmatrix} 1 & 2 & 3 & \cdots & (q-1)d+r & (q-1)d+r+1 & \cdots & n \\ d+2 & d+1 & d+3 & \cdots & n & d & \cdots & 1 \end{pmatrix}$$

Say $r = 0$, then in disjoint cycle notation we have that

$$\begin{aligned} f_1 f_{j-1} f_{n-1} = {} & (1, d+2, 2d+2, \ldots, (q-1)d+2, d-1, 2d-1, \ldots, qd-1, \\ & \quad 2, d+1, 2d+1, \ldots, (q-1)d+1, d, 2d, \ldots, qd) \\ & (3, d+3, 2d+3, \ldots, (q-1)d+3, d-2, 2d-2, \ldots, qd-2) \\ & (4, d+4, 2d+4, \ldots, (q-1)d+4, d-3, 2d-3, \ldots, qd-3) \\ & \quad \vdots \\ & \left(\left\lfloor \frac{d}{2} \right\rfloor, d + \left\lfloor \frac{d}{2} \right\rfloor, \ldots, (q-1)d + \left\lfloor \frac{d}{2} \right\rfloor, \left\lceil \frac{d}{2} \right\rceil + 1, d + \left\lceil \frac{d}{2} \right\rceil + 1, \right. \\ & \quad \left. \ldots, (q-1)d + \left\lceil \frac{d}{2} \right\rceil + 1 \right) \\ & \left(\left\lfloor \frac{d}{2} \right\rfloor + 1, d + \left\lfloor \frac{d}{2} \right\rfloor + 1, \ldots, (q-1)d + \left\lfloor \frac{d}{2} \right\rfloor + 1 \right). \end{aligned}$$

The first cycle is of length $4q$, the next $\left\lfloor \frac{d}{2} \right\rfloor - 2$ cycles are of length $2q$, and the last cycle is of length q. The least common multiple of these lengths is $4q$, the order of the permutation. To verify that these are all the disjoint cycles we can see that the number of characters affected is

$$4q + 2q\left(\left\lfloor \frac{d}{2} \right\rfloor - 2 \right) + q = 4q + q(d - 1 - 4) + q = qd = n.$$

Say $r = 1$ and $d = 2$, then in disjoint cycle notation we have that

$$f_1 f_{j-1} f_{n-1} = (1, 4, 6, \ldots, 2q, 2, 3, 5, \ldots, 2q+1).$$

All of the characters are part of this cycle, since $n = 2q + 1$, and therefore the order of the permutation is $2q + 1$.

Say $r = 1$ and $d = 4$, then the disjoint cycle notation is

$$\begin{aligned} f_1 f_{j-1} f_{n-1} = {} & (1, 6, 10, \ldots, 4q - 2, 4, 8, \ldots, 4q, 2, 5, 9, \ldots, 4q + 1) \\ & (3, 7, \ldots, 4q - 1). \end{aligned}$$

The length of the first cycle is $3q + 1$ and the second cycle is of length q. The least common multiple of the cycle lengths is $q(3q + 1)$. All of the characters are part of this cycle, since $n = 4q + 1$.

Say $r = 1$ and $d \geq 5$. The disjoint cycles are

$$\begin{aligned} f_1 f_{j-1} f_{n-1} = {} & (1, d + 2, 2d + 2, \ldots, (q - 1)d + 2, d, 2d, \ldots, qd, \\ & \quad 2, d + 1, 2d + 1, \ldots, qd + 1) \\ & (3, d + 3, \ldots, (q - 1)d + 3, d - 1, 2d - 1, \ldots, qd - 1) \\ & (4, d + 4, \ldots, (q - 1)d + 4, d - 2, 2d - 2, \ldots, qd - 2) \\ & \qquad \vdots \\ & \left(\left\lfloor \frac{d+2}{2} \right\rfloor, d + \left\lfloor \frac{d+2}{2} \right\rfloor, \ldots, (q - 1)d + \left\lfloor \frac{d+2}{2} \right\rfloor, \right. \\ & \quad \left. \left\lceil \frac{d+2}{2} \right\rceil, d + \left\lceil \frac{d+2}{2} \right\rceil, \ldots, (q - 1)d + \left\lceil \frac{d+2}{2} \right\rceil \right). \end{aligned}$$

The first cycle is of length $3q + 1$ and the other $\left\lfloor \frac{d+2}{2} \right\rfloor - 2$ cycles are of length $2q$. When q is even the least common multiple of these lengths is $2q(3q + 1)$. When q is odd, $3q + 1$ is even, and thus the least common multiple of the lengths is $q(3q + 1)$. All of the disjoint cycles are accounted for since the total number of characters in the cycles is

$$3q + 1 + 2q \left(\left\lfloor \frac{d+2}{2} \right\rfloor - 2 \right) = 3q + 1 + q(d + 1 - 4) = qd + 1.$$

Say $r = 2$ and $d = 3$, then the disjoint cycles are

$$f_1 f_{j-1} f_{n-1} = (1, 5, 8, \ldots, 3q + 2)\ (2, 4, 7, \ldots, 3q + 1)\ (3, 6, \ldots, 3q).$$

The first two cycles are of length $q + 1$ and the last cycle is of length q. The least common multiple of these lengths is $q(q + 1)$. It is also clear that these are all the cycles since the sum of the cycle lengths is $3q + 2 = n$.

Say $r = 2$ and $d \geq 4$. The disjoint cycles are

$$\begin{aligned} f_1 f_{j-1} f_{n-1} = {} & (1, d+2, 2d+2, \ldots, qd+2)\ (2, d+1, 2d+1, \ldots, qd+1) \\ & (3, d+3, \ldots, (q-1)d+3, d, 2d, \ldots, qd) \\ & (4, d+4, \ldots, (q-1)d+4, d-1, 2d-1, \ldots, qd-1) \\ & \qquad \vdots \\ & \left(\left\lfloor \frac{d+3}{2} \right\rfloor, d + \left\lfloor \frac{d+3}{2} \right\rfloor, \ldots, (q-1)d + \left\lfloor \frac{d+3}{2} \right\rfloor, \right. \\ & \left. \left\lceil \frac{d+3}{2} \right\rceil, d + \left\lceil \frac{d+3}{2} \right\rceil, \ldots, (q-1)d + \left\lceil \frac{d+3}{2} \right\rceil \right). \end{aligned}$$

The first two cycles are of length $q+1$ and the last $\lfloor \frac{d+3}{2} \rfloor - 2$ cycles are of length $2q$. When q is odd then the least common multiple of the lengths is $q(q+1)$. When q is even then the least common multiple of the lengths is $2q(q+1)$. These are all of the disjoint cycles since the number of characters in them is

$$2q + 2 + 2q \left(\left\lfloor \frac{d+3}{2} \right\rfloor - 2 \right) = 2q + 2 + q(d+2-4) = qd + 2 = n.$$

Say $r = 3$ and $d = 4$. The disjoint cycles are

$$\begin{aligned} f_1 f_{j-1} f_{n-1} = {} & (1, 6, 10, \ldots, 4q+2, 2, 5, 9, \ldots, 4q+1, 3, 7, \ldots, 4q+3) \\ & (4, 8, \ldots, 4q). \end{aligned}$$

The first cycle is of length $3q+3$ and the second cycle is of length q. If q is a multiple of 3, then the least common multiple is $q(q+1)$. If q is not a multiple of 3, then the least common multiple is $3q(q+1)$. The number of characters in both cycles is $4q+3 = n$.

Say $r = 3$ and $d \geq 5$. The disjoint cycles are

$$\begin{aligned} f_1 f_{j-1} f_{n-1} = {} & (1, d+2, 2d+2, \ldots, qd+2, 2, d+1, 2d+1, \ldots, qd+1, \\ & \quad 3, d+3, \ldots, qd+3) \\ & (4, d+4, \ldots, (q-1)d+4, d, 2d, \ldots, qd) \\ & (5, d+5, \ldots, (q-1)d+5, d-1, 2d-1, \ldots, qd-1) \\ & \qquad \vdots \\ & \left(\left\lfloor \frac{d+4}{2} \right\rfloor, d + \left\lfloor \frac{d+4}{2} \right\rfloor, \ldots, (q-1)d + \left\lfloor \frac{d+4}{2} \right\rfloor, \right. \\ & \left. \left\lceil \frac{d+4}{2} \right\rceil, d + \left\lceil \frac{d+4}{2} \right\rceil, \ldots, (q-1)d + \left\lceil \frac{d+4}{2} \right\rceil \right). \end{aligned}$$

The first cycle is of length $3q+3$ and the remaining $\lfloor \frac{d+4}{2} \rfloor - 3$ cycles are of length $2q$. When q is odd and divisible by 3 the least common multiple is $q(q+1)$. When q

divisible by 6 the least common multiple is $2q(q+1)$. When q is odd and not divisible by 3 the least common multiple is $3q(q+1)$. When q is even but not divisible by 3 the least common multiple is $6q(q+1)$. The number of characters in all of the cycles is

$$3q + 3 + 2q\left(\left\lfloor\frac{d+4}{2}\right\rfloor - 3\right) = 3q + 3 + q(d+3-6) = qd + 3 = n.$$

Say $r = 4$ and $d = 5$. The disjoint cycles are

$$\begin{aligned} f_1 f_{j-1} f_{n-1} = {} & (1, 7, 12, \ldots, 5q+2, 3, 8, 13, \ldots, 5q+3, 2, 6, 11, \ldots, 5q+1, \\ & 4, 9, \ldots, 5q+4)\,(5, 10, \ldots, 5q). \end{aligned}$$

The first cycle is of length $4q+4$ and the second cycle is of length q. If q is a multiple of 4, then the least common multiple is $q(q+1)$. If q is even but not a multiple of 4, then the least common multiple is $2q(q+1)$. If q is odd, then the least common multiple is $4q(q+1)$. The number of characters in both cycles is $5q+4 = n$.

Say $r = 4$ and $d \geq 6$. The disjoint cycles are

$$\begin{aligned} f_1 f_{j-1} f_{n-1} = {} & (1, d+2, 2d+2, \ldots, qd+2, 3, d+3, \ldots, qd+3, 2, d+1, 2d+1, \\ & \quad \ldots, qd+1, 4, d+4, \ldots, qd+4) \\ & (5, d+5, \ldots, (q-1)d+5, d, 2d, \ldots, qd) \\ & (6, d+6, \ldots, (q-1)d+6, d-1, 2d-1, \ldots, qd-1) \\ & \qquad\qquad \vdots \\ & \Bigg(\left\lfloor\frac{d+5}{2}\right\rfloor, d+\left\lfloor\frac{d+5}{2}\right\rfloor, \ldots, (q-1)d+\left\lfloor\frac{d+5}{2}\right\rfloor, \\ & \quad \left\lceil\frac{d+5}{2}\right\rceil, d+\left\lceil\frac{d+5}{2}\right\rceil, \ldots, (q-1)d+\left\lceil\frac{d+5}{2}\right\rceil\Bigg). \end{aligned}$$

The first cycle is of length $4q+4$ and the remaining $\left\lfloor\frac{d+5}{2}\right\rfloor - 4$ cycles are of length $2q$. When q is even the least common multiple is $2q(q+1)$. When q is odd the least common multiple is $4q(q+1)$. The number of characters in all of the cycles is

$$4q + 4 + 2q\left(\left\lfloor\frac{d+5}{2}\right\rfloor - 4\right) = 4q + 4 + q(d+4-8) = qd + 4 = n.$$

Say $r \geq 5$ and $d \geq 6$. The disjoint cycles are

$$\begin{aligned}
f_1 f_{j-1} f_{n-1} = {} & (1, d+2, 2d+2, \ldots, qd+2, r-1, d+r-1, \ldots, qd+r-1, \\
& \quad 2, d+1, 2d+1, \ldots, qd+1, r, d+r, \ldots, qd+r) \\
& (3, d+3, \ldots, qd+3, r-2, d+r-2, \ldots, qd+r-2) \\
& (4, d+4, \ldots, qd+4, r-3, d+r-3, \ldots, qd+r-3) \\
& \qquad \vdots \\
& \left(\left\lfloor \frac{r+1}{2} \right\rfloor, d + \left\lfloor \frac{r+1}{2} \right\rfloor, \ldots, qd + \left\lfloor \frac{r+1}{2} \right\rfloor, \left\lceil \frac{r+1}{2} \right\rceil, d + \left\lceil \frac{r+1}{2} \right\rceil, \right. \\
& \quad \left. \ldots, qd + \left\lceil \frac{r+1}{2} \right\rceil \right) \\
& (r+1, d+r+1, \ldots, (q-1)d+r+1, d, 2d, \ldots, qd) \\
& (r+2, d+r+2, \ldots, (q-1)d+r+2, d-1, 2d-1, \ldots, qd-1) \\
& \qquad \vdots \\
& \left(\left\lfloor \frac{d+r+1}{2} \right\rfloor, d + \left\lfloor \frac{d+r+1}{2} \right\rfloor, \ldots, (q-1)d + \left\lfloor \frac{d+r+1}{2} \right\rfloor, \right. \\
& \quad \left. \left\lceil \frac{d+r+1}{2} \right\rceil, d + \left\lceil \frac{d+r+1}{2} \right\rceil, \ldots, (q-1)d + \left\lceil \frac{d+r+1}{2} \right\rceil \right).
\end{aligned}$$

The first cycle is of length $4q+4$, the next $\left\lfloor \frac{r+1}{2} \right\rfloor - 2$ cycles are each of length $2q+2$, and the last $\left\lfloor \frac{d+r+1}{2} \right\rfloor - r$ cycles are of length $2q$. When q is even the least common multiple is $2q(q+1)$. When q is odd the least common multiple is $4q(q+1)$. The number of characters in all of the cycles is

$$\begin{aligned}
& 4q + 4 + 2(q+1)\left(\left\lfloor \frac{r+1}{2} \right\rfloor - 2 \right) + 2q \left(\left\lfloor \frac{d+r+1}{2} \right\rfloor - r \right) \\
& \quad = 4q + 4 + (q+1)(r-4) + q(d+r-2r) = qd + r = n.
\end{aligned}$$

□

Although Theorem 3.5 and Lemma 3.4 fully characterize the order of elements of the form $f_1 f_j f_k$, we do not cover all possible combinations of three generators $f_i f_j f_k$. In order to record all such orders we propose a generalization of the "Coxeter matrix" which we refer to as the *Coxeter tensor* of rank r. In essence it would be a matrix of matrices or vectors depending on the parity of the rank. For three generators the object recording all of the orders would be the Coxeter 3-tensor, visualized as a cube. Using Lemma 3.4 we need only look at the order of the principal tetrahedron consisting of only entries with increasing indices. Theorem 3.5 would be the base of said tetrahedron.

Example 3.2. In Figure 4, we depict the 24×24 matrix m whose (j, k) entry is the order of $f_1 f_j f_k$ in S_{25}.

$$\begin{pmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 3 & 5 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\
2 & 3 & 2 & 4 & 3 & 7 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
2 & 5 & 4 & 2 & 5 & 7 & 6 & 14 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
2 & 6 & 3 & 5 & 2 & 6 & 4 & 9 & 12 & 28 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
2 & 6 & 7 & 7 & 6 & 2 & 7 & 9 & 10 & 18 & 12 & 28 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
2 & 6 & 8 & 6 & 4 & 7 & 2 & 8 & 5 & 12 & 12 & 36 & 12 & 28 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
2 & 6 & 8 & 14 & 9 & 9 & 8 & 2 & 9 & 11 & 12 & 30 & 12 & 36 & 12 & 28 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 \\
2 & 6 & 8 & 8 & 12 & 10 & 5 & 9 & 2 & 10 & 6 & 13 & 12 & 12 & 12 & 36 & 12 & 28 & 8 & 8 & 8 & 8 & 8 & 8 \\
2 & 6 & 8 & 8 & 28 & 18 & 12 & 11 & 10 & 2 & 11 & 13 & 20 & 12 & 30 & 12 & 12 & 36 & 12 & 28 & 8 & 8 & 8 & 8 \\
2 & 6 & 8 & 8 & 8 & 12 & 12 & 12 & 6 & 11 & 2 & 12 & 7 & 15 & 16 & 12 & 12 & 12 & 12 & 36 & 12 & 28 & 8 & 8 \\
2 & 6 & 8 & 8 & 8 & 28 & 36 & 30 & 13 & 13 & 12 & 2 & 13 & 15 & 16 & 52 & 12 & 30 & 12 & 12 & 12 & 36 & 12 & 28 \\
2 & 6 & 8 & 8 & 8 & 8 & 12 & 12 & 12 & 20 & 7 & 13 & 2 & 14 & 8 & 30 & 40 & 48 & 12 & 12 & 12 & 12 & 12 & 36 \\
2 & 6 & 8 & 8 & 8 & 8 & 28 & 36 & 12 & 12 & 15 & 15 & 14 & 2 & 15 & 17 & 18 & 60 & 16 & 12 & 30 & 12 & 12 & 12 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 12 & 12 & 30 & 16 & 16 & 8 & 15 & 2 & 16 & 9 & 19 & 20 & 104 & 48 & 12 & 12 & 12 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 28 & 36 & 12 & 12 & 52 & 30 & 17 & 16 & 2 & 17 & 19 & 42 & 80 & 40 & 48 & 12 & 30 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 12 & 12 & 12 & 12 & 40 & 18 & 9 & 17 & 2 & 18 & 10 & 21 & 30 & 120 & 16 & 48 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 28 & 36 & 12 & 30 & 48 & 60 & 19 & 19 & 18 & 2 & 19 & 21 & 22 & 90 & 20 & 104 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 12 & 12 & 12 & 12 & 16 & 20 & 42 & 10 & 19 & 2 & 20 & 11 & 56 & 24 & 20 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 28 & 36 & 12 & 12 & 12 & 104 & 80 & 21 & 21 & 20 & 2 & 21 & 23 & 24 & 114 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 12 & 12 & 12 & 30 & 48 & 40 & 30 & 22 & 11 & 21 & 2 & 22 & 12 & 25 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 28 & 36 & 12 & 12 & 12 & 48 & 120 & 90 & 56 & 23 & 22 & 2 & 23 & 25 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 12 & 12 & 12 & 12 & 12 & 16 & 20 & 24 & 24 & 12 & 23 & 2 & 24 \\
2 & 6 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 8 & 28 & 36 & 12 & 12 & 30 & 48 & 104 & 20 & 114 & 25 & 25 & 24 & 2
\end{pmatrix}$$

Figure 4: Three Generator Pancake Matrix with the first generator being f_1 with $n = 25$. The matrix entry $m_{j,k}$ is the order of $f_1 f_j f_k$, e.g., $m_{19,24}$ is 20, which is the order of $f_1 f_{19} f_{24}$.

In the next section, we describe the pancake matrix for B_n, and make connections to the corresponding pancake graph of B_n.

4 B_n results

We now provide a complete description for the order of $f_i^B f_j^B, 0 \leq i, j \leq n-1$ for signed permutations.

Theorem 4.1. *Let $m^B_{i-1,j-1}$ be the order of $f^B_{i-1} f^B_{j-1}$ with $1 \leq i < j \leq n$; then*

1. $m^B_{i-1,i-1} = 1$.
2. $m^B_{i-1,j-1} = m^B_{j-1,i-1}$.
3. *If* $1 < i \leq \lfloor \frac{j}{2} \rfloor$ *(with $j \geq 4$) then* $m^B_{i-1,j-1} = 4$.
4. *If* $1 \leq \lfloor \frac{j}{2} \rfloor < i < j-1$ *(with $j \geq 4$), then*

$$m^B_{i-1,j-1} = \begin{cases} 2q & \text{if } r = 0, \text{ and} \\ 2q(q+1) & \text{if } r \neq 0 \end{cases}$$

where $d = j - i$, $q = \lfloor \frac{j}{d} \rfloor$, $r = j \pmod{d}$.

5. *If* $i = j - 1$ *(with* $j \geq 3$*) then* $m^B_{i-1,j-1} = 2j$.

Proof. For Case (1), notice that $(f^B_i)^{-1} = f^B_i$, and therefore $m^B_{i,i} = 1$.

For Case (2), notice that $(f^B_i f^B_j)^{-1} = f^B_j f^B_i$, and therefore $m^B_{i,j} = m^B_{j,i}$.

Since elements in B_n can be thought of as elements in B_{n+1} that leave n fixed, M^B_n can be thought of as the $n \times n$ submatrix of M^B_{n+1} obtained by deleting the last row and the last column of M^B_{n+1}. Therefore, it is enough to prove the remaining cases when $j = n$.

For Case (3), notice that the identity permutation $e = [1\ 2\ 3\ \cdots\ n]$ becomes

$$\left[\underline{n}\ \underline{n-1}\ \cdots\ \underline{\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)}\ \cdots\ 1\ 2\ \cdots\ i\right]$$

after multiplying it by $f^B_{n-1} f^B_{i-1}$. In other words, $f^B_{n-1} f^B_{i-1}$ will reverse the last $n - i$ symbols of the identity in B_n, change their sign and place them at the beginning of the window notation. Since $i \leq \lfloor \frac{n}{2} \rfloor$, the first i characters and the last $n - i$ characters of e (seen as a string in window notation) do not overlap, and thus will behave independently after multiplying by $f^B_{n-1} f^B_{i-1}$. It takes 4 multiplications by $f^B_{n-1} f^B_{i-1}$ for the first i characters of e to return to their original position. Moreover, since $i \leq n-i$, it also takes 4 multiplications by $f^B_{n-1} f^B_{i-1}$ for the last $n-i$ characters in e to return to their original position in e. Thus $m^B_{n-1,i-1} = 4$.

For Case (4), let $d = n - j$, $q = \lfloor \frac{n}{d} \rfloor$ and $r = n \pmod{d}$. Notice that any element $w = [w_1\ w_2\ \cdots\ w_n]$ in B_n can be written in the form

$$[\beta_0 \alpha_1 \beta_1 \cdots \alpha_q \beta_q]\,,$$

where α_k and β_l are substrings of w written in window notation and satisfying $\ell(\alpha_k \beta_k) = d$, and $\ell(\beta_l) = r$ for $1 \leq k \leq q$, $0 \leq l \leq q$ (and therefore $\ell(\alpha_k) = d - r$). Here, $\ell(\cdot)$ denotes the length function on strings with brackets ignored. So, for example, $\ell([1\ 2\ 3]) = 3$ and $\ell([4\ 3\ \underline{2}\ 1]) = 4$.

Multiplying $[\beta_0 \alpha_1 \beta_1 \cdots \alpha_q \beta_q]$ by $f^B_{n-1} f^B_{j-1}$ gives

$$\left[\underline{\overline{\beta_q}}\ \underline{\overline{\alpha_q}} \beta_0 \alpha_1 \beta_1 \cdots \alpha_{q-1} \beta_{q-1}\right],$$

where if $x = x_1 x_2 \cdots x_m$, $\underline{x}$ and $\overline{x}$ denote $\underline{x_1}\ \underline{x_2}\ \cdots\ \underline{x_m}$ and $x_m x_{m-1} \cdots x_2 x_1$, respectively. Since repeated applications of $\overline{f^B_{n-1}} \overline{f^B_{j-1}}$ to w will eventually return w to itself, it follows that the α and β segments would return to their original positions. Therefore the periods of the different α and β under multiplication by $f^B_{n-1} f^B_{j-1}$ have to be the same. The effect of $f^B_{n-1} f^B_{j-1}$ reverses the last d characters of w and changes their sign. If $r = 0$ then there are q α substrings and no β substrings. The period of $[\alpha_1 \alpha_2 \cdots \alpha_q]$ under $f^B_{n-1} f^B_{j-1}$ is $2q$. Furthermore, if $r \neq 0$ then there are q α substrings and $q + 1$ β substrings, and therefore the period of $[\beta_0 \alpha_1 \beta_1 \ldots \alpha_q \beta_q]$ under $f^B_{n-1} f^B_{j-1}$ is $2q(q + 1)$. This proves Case (4).

To prove Case (5), notice that e becomes

$$[\underline{n}\ 1\ 2\ \ldots\ (n-1)]$$

$$
\begin{pmatrix}
1&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4\\
4&1&6&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4\\
4&6&1&8&12&4&4&4&4&4&4&4&4&4&4&4&4&4&4&4\\
4&4&8&1&10&6&12&4&4&4&4&4&4&4&4&4&4&4&4&4\\
4&4&12&10&1&12&24&12&12&4&4&4&4&4&4&4&4&4&4&4\\
4&4&4&6&12&1&14&8&6&12&12&4&4&4&4&4&4&4&4&4\\
4&4&4&12&24&14&1&16&40&24&12&12&12&4&4&4&4&4&4&4\\
4&4&4&4&12&8&16&1&18&10&24&6&12&12&12&4&4&4&4&4\\
4&4&4&4&12&6&40&18&1&20&60&8&24&12&12&12&12&4&4&4\\
4&4&4&4&4&12&24&10&20&1&22&12&40&24&6&12&12&12&12&4\\
4&4&4&4&4&12&12&24&60&22&1&24&84&40&24&24&12&12&12&12\\
4&4&4&4&4&4&12&6&8&12&24&1&26&14&10&8&24&6&12&12\\
4&4&4&4&4&4&12&12&24&40&84&26&1&28&112&60&40&24&24&12\\
4&4&4&4&4&4&4&12&12&24&40&14&28&1&30&16&60&40&24&24\\
4&4&4&4&4&4&4&12&12&6&24&10&112&30&1&32&144&12&40&8\\
4&4&4&4&4&4&4&4&12&12&24&8&60&16&32&1&34&18&84&10\\
4&4&4&4&4&4&4&4&12&12&12&24&40&60&144&34&1&36&180&84\\
4&4&4&4&4&4&4&4&4&12&12&6&24&40&12&18&36&1&38&20\\
4&4&4&4&4&4&4&4&4&12&12&12&24&24&40&84&180&38&1&40\\
4&4&4&4&4&4&4&4&4&4&12&12&12&24&8&10&84&20&40&1
\end{pmatrix}
$$

Figure 5: Burnt Pancake Matrix with $n = 20$. The $m^B_{i,j}$ entry is the order of $f^B_{i-1}f^B_{j-1}$. Notice that the matrix is symmetric, the entries in the main diagonal are all 1 and the entries in the off-diagonal are the even integers that are at least 4.

after multiplying it by $f^B_{n-1}f^B_{n-2}$. That is, the effect of multiplying an arbitrary signed permutation $[w(1)\ w(2)\ \cdots\ w(n)]$ by $f^B_{n-1}f^B_{n-2}$ is to place the last character into the first position and reverse its sign. Thus $2n$ applications of $f^B_{n-1}f^B_{n-2}$ are needed to return the characters of e to its original position and its original sign. Hence, $m^B_{n-1,n-2} = 2n$. □

Example 4.1. In Figure 5, we depict the 20×20 Coxeter matrix for B_{20}.

4.1 Connection with the burnt pancake graph

The Pancake graph of S_n, and in particular its cycle structure, has been extensively studied (see, for example, [2, 19, 20, 26, 28, 27, 29]). From the results from Theorem 4.1, one can derive results regarding the cycle structure of the Cayley graph corresponding to B_n generated by P^B. Figure 2 displays this graph for B_3. Indeed, the following theorem, which is a signed version of [27, Lemma 1], is obtained directly from Theorem 4.1.

Theorem 4.2. *The Cayley graph of B_n with the generators P^B (burnt pancake graph of B_n), with $n \geq 2$, contains a maximal set of $\frac{2^n n!}{\ell}$ independent ℓ-cycles of the form*

$(f_i^B f_j^B)^k$, *with* $0 \leq i < j < n$, $\ell = 2k$ *and* $k = (M_n^B)_{i+1,j+1}$, *the* $(i+1, j+1)$ *entry in* M_n^B.

Proof. The length of the cycles is given by Theorem 4.1. Furthermore, every vertex in the burnt pancake graph is incident to exactly one edge corresponding to f_i^B and one edge corresponding to f_j^B (with $0 \leq i < j < n$), these cycles are independent. Since every signed permutation is the vertex of a cycle of the form $(f_i^B f_j^B)^k$, there are $\frac{2^n n!}{\ell}$ of such independent cycles. □

To illustrate the cycle structure described in the Theorem 4.2, one can look at Figure 2 showing the burnt pancake graph of B_3. If one considers generators $f_0^B = [\underline{1}\ 2\ 3]$ and $f_1^B = [\underline{2}\ \underline{1}\ 3]$, then the order of $f_0^B f_1^B$ is 8 , and one can indeed notice that there are $\frac{2^3 \cdot 3!}{8} = 6$ independent cycles labeled with the generators f_0^B (in purple/dotted) and f_1^B (in red/solid).

It is known that the burnt pancake graph of B_n with $n \geq 2$ is an n-regular, connected graph that has no triangles nor subgraphs isomorphic to $K_{2,3}$ (see [25]). Moreover, if $g(n)$ denotes the diameter of the pancake graph of B_n, then $3n/2 \leq g(n) \leq 2n - 2$ (see [9]). Determining the diameter of the pancake graph of B_n remains an open problem, though exact values are known for $n \leq 17$ (see [8]).

We recall that a *chord* in a cycle C is an edge not belonging to a C that connects two vertices of C. Just in the case for the pancake graph of S_n (see [27]), the cycles described in Theorem 4.2 have no chords. We make this formal in the following Lemma.

Lemma 4.3. *The cycles described in Theorem 4.2 have no chords.*

To prove this lemma, we first recall that the burnt pancake graph of B_n cannot have any simple cycles of length six.

Lemma 4.4 (Theorem 10 in [10])**.** *The* girth *(length of the shortest simple cycle) of the burnt pancake graph of* B_n *is 8.*

Proof of Lemma 4.3. Let $C = (f_i f_j)^{m_{i,j}^B}$ be a cycle and suppose that C has a chord. Therefore there exists signed permutations w_1 and w_2, and $f_k^B \in P_n^B$ such that $w_2 f_k^B = w_1$, with w_1 and w_2 being vertices of C. Furthermore, either $w_1(f_i^B f_j^B)^s = w_2$ and $(f_i^B f_j^B)^s f_k^B = e$, or $w_1(f_i^B f_j^B)^s f_i^B = w_2$ and $(f_i^B f_j^B)^s f_i^B f_k^B = e$ with $s < m_{i,j}^B$. Hence, either

$$w_2 f_i^B f_j^B f_k^B = w_1 (f_i^B f_j^B)^s f_i^B f_j^B f_k^B = w_1 f_i^B f_j^B (f_i^B f_j^B)^s f_k^B = w_1 f_i^B f_j^B, \text{ or}$$

$$w_2 f_j^B f_i^B f_k^B = w_1 (f_i^B f_j^B)^s f_i^B f_j^B f_i^B f_k^B = w_1 f_i^B f_j^B (f_i^B f_j^B)^s f_i^B f_k^B = w_1 f_i^B f_j^B.$$

Therefore, there exist a 6-cycle of the form $f_i^B f_j^B f_k^B f_j^B f_i^B f_k^B$ or of the form $(f_i^B f_j^B f_k^B)^2$. This contradicts Lemma 4.4, and therefore no such cycle C exists. □

4.2 Reflections

We now describe the set of *burnt pancake reflections*

$$T_B^{\pm} = \{wf_i^B w^{-1} \mid 0 \le i \le n-1, w \in B_n\}.$$

We recall that any element in the set of reflections $T_B = \{ws_i^B w^{-1} \mid w \in B_n, 0 \le i \le n-1\}$ for signed adjacent transpositions S^B has the following form (see [3, Proposition 8.1.5]):

$$\{(i,j)(\underline{i},\underline{j}) \mid 1 \le i < |j| \le n\} \cup \{(i,\underline{i}) \mid 1 \le i \le n\}.$$

If $t \in T_B^{\pm}$ then $t = wf_i^B w^{-1}$ for some $w \in B_n$ and $0 \le i \le n-1$. If $w = [w_1\ w_2\ \cdots\ w_n]$, then from $wf_i^B w^{-1} = t$ we have

$$[\underline{w_{i+1}}\ \underline{w_i}\ \cdots\ \underline{w_1}\ w_{i+2}\ w_{i+3}\ \cdots\ w_n] = tw,$$

and so $t = (w_1, \underline{w_{i+1}})(w_2, \underline{w_i}) \cdots (w_{i+1}, \underline{w_1})$. In terms of notation, if a $w_j < 0$, $1 \le j \le n$, then $\underline{w_j} = -w_j > 0$. Therefore,

$$T_B^{\pm} = \{(w_1, \underline{w_{i+1}})(w_2, \underline{w_i}) \cdots (w_{i+1}, \underline{w_1}) \mid w_i \in [\pm n], 0 \le i < n, w_a \ne w_b \text{ if } a \ne b\} \tag{4.1}$$

In terms of comparing T_B and $T_B^{\pm}$, we notice that any permutation of the form $(i, \underline{i})$ is in both sets. However, permutations of the form $(i,j)(i,\underline{j})$ with $1 \le i < |j| \le n$ are not.

As for the number of burnt reflections, from the description in (4.1), one gets

Corollary 4.5. $\left|T_B^{\pm}\right| = \displaystyle\sum_{i=1}^{n} \binom{n}{i} 2^{\lfloor i/2 \rfloor}$.

5 Conclusion and further directions

Our main question of interest in this paper is a purely algebraic one: Can one describe all the relations satisfied by the pancake generators? Our contributions are a complete description of all relations in S_n of the form

- $(f_i f_j)^{m_{i,j}} = e$ (using a result from [27]), and
- $(f_i f_j f_k)^{m_{i,j,k}}$ where $1 \in \{i,j,k\}$.

We furthermore provide a description of all the relations of the form $f_i^B f_j^B = e$ in B_n.

There is a direct connection between relations of the form $(g_1 \cdots g_k)^{m_{1,\ldots,k}} = e$ and cycles in the pancake and burnt pancake graphs: Each relation in that form would correspond to a cycle in those graphs. Therefore, understanding these relations provides information about the cycle structure of the pancake and burnt pancake graphs, and we point out these connections in the respective sections.

Another application of having these relations is in the problem of finding reduced expressions for a permutation using generators from P or P^B. It is well-known that if (W, S) is a Coxeter system, the only relations needed in order to reduce a word are those coming from the product of two generators in S. However since neither (S_n, P) nor (B_n, P^B) are Coxeter systems, more relations other than those we provide are needed in order to reduced a word. Another interesting approach that might be worth pursuing is determining the number of pancake generators needed to generate a random permutation using ideas similar to those in [11], which in turn might provide some bounds on the original pancake problem.

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