AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **76(3)** (2020), Pages 346–365

Hall spectra and extending precolorings with extra colors

Victor Larsen

University of Sioux Falls
Sioux Falls, South Dakota 57105
U.S.A.
victor.larsen@usiouxfalls.edu

Jennifer Vandenbussche   Erik E. Westlund

Department of Mathematics
Kennesaw State University, Georgia 30144
U.S.A.
jvandenb@kennesaw.edu   ewestlun@kennesaw.edu

Abstract

The graph G with list assignment L satisfies *Hall's condition* ("is Hall", for short) if for each subgraph H of G, the inequality $|V(H)| \leq \sum_{\sigma\in\mathcal{C}} \alpha(H(\sigma, L))$ is satisfied, where $\mathcal{C}$ is the set of colors and $\alpha(H(\sigma, L))$ is the independence number of the subgraph of H induced by the set of vertices having color σ in their lists. This idea is a generalization of Hall's Marriage Theorem and provides a necessary (but not sufficient) condition for a graph to admit a proper list coloring. This paper affirmatively answers a question posed by Bobga et al. in 2011: If G is a graph that is not Hall k-extendible for some $k \geq \chi(G)$ but is Hall $(k+1)$-extendible, is it possible that G could fail to be Hall $(k+m)$-extendible for some $m \geq 2$? We also explore extending Hall precolorings with extra colors. We show that any Hall k-precoloring of a graph G is $(k+\chi(G)-1)$-extendible. However, we show that for each $k \geq 3$, there exists a k-colorable graph with a Hall k-precoloring that cannot be extended with $k+1$ colors.

1 Introduction

Throughout this paper, G is a finite, simple graph with vertex set $V(G)$ and edge set $E(G)$. For $U \subseteq V(G)$, we shall use $G[U]$ to denote the subgraph of G induced by U. Additionally $\alpha(G)$, $\delta(G)$, $\Delta(G)$, $\chi(G)$, shall denote the *independence number, minimum degree, maximum degree,* and *chromatic number* of G respectively. Let $\deg_G(v)$ denote the *degree* of the vertex v in the graph G. For any $U \subseteq V(G)$ and

ISSN: 2202-3518

any subgraph H of G, let $N_H(U)$ denote the set of vertices in H that are adjacent in G to at least one vertex in U. Let $[m]$ denote the set $\{1, \ldots, m\}$. We refer the reader to West [18] for any notation not defined here.

A *k-precoloring* of G is a proper k-coloring of $G[U]$ where $U \subseteq V(G)$. The coloring, say ϕ, can be *extended* (or is *extendible*) if there exists a proper k-coloring $\theta : V(G) \to [k]$ where $\theta(v) = \phi(v)$ for all $v \in U$. Conditions under which a precoloring of the vertices of a graph extends to a proper coloring have been studied extensively (see for example [1, 2, 3, 4, 13, 16]), with much of the literature focused on minimum distance requirements between precolored vertices. In this paper, we investigate conditions under which precolorings of graphs satisfying an obvious necessary condition called Hall's condition can be extended. This continues the work of Holliday et al. [11, 12] and answers the final question in a set of three raised by Bobga et al. [5].

To state these questions, we first need several definitions. If $\mathcal{C}$ is an infinite set of colors (the *palette*) and $\mathcal{L}$ is a set of finite subsets of $\mathcal{C}$, then a *list assignment* to G is a function $L : V(G) \to \mathcal{L}$. The list assignment L is a *k-assignment* to G if $|L(v)| \geq k$ for all $v \in V(G)$. Given a list assignment L of G with palette $\mathcal{C}$, an *L-coloring* of G is a function $\phi : V(G) \to \mathcal{C}$ such that $\phi(v) \in L(v)$ for every vertex v. An L-coloring ϕ is *proper* if each color class induces an independent set. If G has a proper L-coloring, we say G is *L-colorable*.

The following generalization of Philip Hall's 1935 Marriage Theorem ([8]) applied to list assignments of graphs was first introduced in a 1990 paper by Hilton and Johnson [9] (see also [5]). Suppose that ϕ is an L-coloring of G for some list assignment L with a palette $\mathcal{C}$ and let H be any subgraph of G. For each $\sigma \in \mathcal{C}$, consider $\phi^{-1}(\sigma) \cap V(H)$, the set of all vertices in H given color σ under ϕ, and let $H(\sigma, L)$ be the subgraph of H induced by all vertices of H having σ in their lists. Then $\phi^{-1}(\sigma) \cap V(H)$ is an independent set of vertices in $H(\sigma, L)$, which leads to the following observation and related definitions:

Observation 1.1. *If G is L-colorable, then for every subgraph H of G, we have*

$$|V(H)| \leq \sum_{\sigma \in C} \alpha(H(\sigma, L)). \qquad (*)$$

We will refer to $\sum_{\sigma \in C} \alpha(H(\sigma, L))$ as the *Hall sum* for (H, L) (or just H, when the list assignment is clear).

Definition 1.2 (Hilton and Johnson, 1990 [9])**.** The graph G with list assignment L satisfies *Hall's condition* if, for each subgraph H of G, the inequality $(*)$ is satisfied. For brevity, we say that (G, L) satisfies Hall's condition. When (G, L) satisfies Hall's condition, we call L a *Hall assignment*. If H is a subgraph of G, then (H, L) will denote the natural restriction of L to $V(H)$.

Note that satisfying Hall's condition is not sufficient for a graph to have a proper list coloring. A well-known example is a cycle $C_4 = v_1, v_2, v_3, v_4, v_1$ with $L(v_1) = \{1\}$,

$L(v_2) = \{1, 3\}$, $L(v_3) = \{2, 3\}$, and $L(v_4) = \{1, 2\}$. Clearly (C_4, L) satisfies Hall's condition but C_4 is not L-colorable.

In 2011, Bobga et al. [5] began investigating Hall's condition in the context of precoloring extensions using the natural relationship between precoloring extensions and list coloring.

Definition 1.3. For $V_0 \subseteq V(G)$, a k-precoloring $\phi : V_0 \to [k]$ of a graph G is a *Hall k-precoloring* if L_ϕ is a Hall assignment (meaning (G, L_ϕ) satisfies Hall's condition), where L_ϕ is the natural list assignment associated with ϕ:

$$L_\phi(x) = \begin{cases} \{\phi(x)\} & \text{if } x \in V_0 \\ [k] \setminus \{\phi(y) \colon y \in N_G(x) \cap V_0\} & \text{if } x \notin V_0. \end{cases}$$

A graph G is *Hall k-extendible* if every Hall k-precoloring is extendible. Furthermore, G is *Hall chromatic extendible* if G is Hall $\chi(G)$-extendible and G is *total Hall extendible* if G is Hall k-extendible for all $k \geq \chi(G)$.

A fundamental result is the following:

Theorem 1.4 (Bobga et al., 2011 [5])**.** *Let G be a graph.*

1. *G is Hall k-extendible for all $k \geq \Delta(G) + 1$.*
2. *G is Hall k-extendible if and only if every component of G is Hall k-extendible.*
3. *Let $\phi : V_0 \to [k]$ be a k-precoloring of G, and let $G' = G[V \setminus V_0]$. Then (G, L_ϕ) satisfies Hall's condition if and only if (G', L_ϕ) satisfies Hall's condition.*

The issue of Hall extendibility is nuanced, as shown by the following result:

Theorem 1.5 (Bobga et al., 2011 [5])**.** *All bipartite graphs are Hall chromatic extendible, but for all $k \geq 3$, there exists a bipartite graph that is not Hall k-extendible.*

Also, a graph may fail to be Hall k-extendible for $k < \chi(G)$. In Figure 1, the empty 3-precoloring is a Hall 3-precoloring, but the chromatic number of the graph is 4. This is the reason that total Hall extendibility only refers to Hall k-extendibility for $k \geq \chi(G)$.

The authors of [5] suggested three important questions for further study:

Question 1 Are there examples of graphs that are Hall k-extendible but not Hall $(k+1)$-extendible for some $k \geq 3$?

Question 2: Let G be a connected graph that is neither complete nor an odd cycle. Is it true that G is Hall $\Delta(G)$-extendible?

Question 3: If G is a graph that is not Hall k-extendible for some $k \geq \chi(G)$, but is Hall $(k+1)$-extendible, is it possible that G could fail to be Hall $(k+m)$-extendible for some $m \geq 2$?

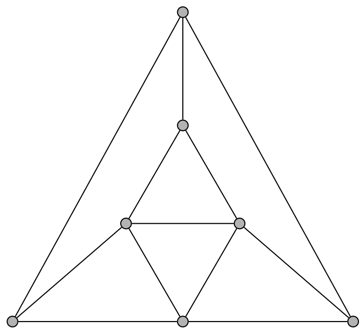

Figure 1: This graph is 4-chromatic, but the empty 3-precoloring ϕ is a Hall 3-precoloring.

Questions 1 and 2 have been answered affirmatively; see [11, 12]. In Section 2 of this paper, we prove the following theorem, which affirmatively answers Question 3.

Theorem 1.6 (Main Result 1). *For every $k \geq 3$, there exists a connected k-chromatic graph that is Hall $(k+1)$-extendible, but is neither Hall k-extendible nor Hall $(k+2)$-extendible.*

Inspired by precoloring extension results in which k-precolorings are extended with $k + r$ colors for small values of r (see for example [13]), Section 3 investigates extending Hall k-precolorings with small numbers of "extra" colors. The results in this area are primarily negative; it seems that requiring Hall's condition be satisfied has little effect on the number of extra colors that may be needed. In particular, we prove the following:

Theorem 1.7 (Main Result 2). *For each $k \geq 3$, there exists a k-colorable graph having a Hall k-precoloring that cannot be extended with $k + 1$ colors.*

2 The Hall spectrum

Since graphs may be Hall k-extendible but not Hall $(k+1)$-extendible, the following definition was introduced:

Definition 2.1 (Holliday et al., 2016 [12]). Let G be a graph such that $\chi(G) \leq \Delta(G)$. The *Hall spectrum* of G is the binary vector $\overline{h}(G) = [h_0, \ldots, h_\beta]$ where $\beta = \Delta(G) - \chi(G)$ and for each $i \in \{0, \ldots, \beta\}$,

$$h_i = \begin{cases} 1 & \text{if } G \text{ is Hall } (\chi(G) + i)\text{-extendible} \\ 0 & \text{otherwise.} \end{cases}$$

Questions 1–3 mentioned in the introduction can be rephrased in terms of Hall spectra.

Question 1: Are there examples of non-bipartite graphs whose Hall spectra contain a one followed by a zero?

Question 2: Does the Hall spectrum of every graph end in a one?

Question 3: Does there exist a graph whose Hall spectrum has non-consecutive zeros?

As mentioned previously, the answer to the first two questions was shown to be "yes". The main result in this section provides an affirmative answer to Question 3 as well; we will prove several smaller results along the way. We first present some background definitions and results which will be used throughout the paper.

Theorem 2.2 (Hilton and Johnson, 1990 [9])**.** *If L is a $\chi(G)$-assignment to G, then (G, L) is a Hall assignment.*

Theorem 2.3 (Hilton and Johnson, 1990 [9])**.** *A graph G with list assignment L satisfies Hall's condition if and only if* $(*)$ *holds for each connected induced subgraph H of G.*

Thus if (G, L) does not satisfy (i.e., fails) Hall's condition, then there exists some connected induced subgraph H of G such that (H, L) does not satisfy the inequality $(*)$.

Definition 2.4 (Hilton and Johnson, 1990 [9])**.** The *Hall number* of a graph G is the smallest positive integer k such that whenever L is a k-assignment to G and (G, L) satisfies Hall's condition, G is L-colorable. The Hall number of G is denoted $h(G)$.

In other words, $h(G)$ is the smallest positive integer such that Hall's condition on k-assignments is both necessary and sufficient for the existence of a proper L-coloring of G. (We use the notation $\overline{h}(G)$ for the Hall spectrum of G to distinguish it from the notation $h(G)$, which is the Hall number of G.) The following result characterizes graphs with Hall number equal to one.

Theorem 2.5 (Hilton and Johnson, 1990; Hilton et al., 1996 [9, 10])**.** *The following statements are equivalent:*

1. $h(G) = 1$.
2. *Every block (maximal 2-connected subgraph) of G is a clique.*
3. *G contains no induced cycle C_n, $n \geq 4$, nor an induced copy of $K_4 - e$ (that is, K_4 with an edge deleted).*

Definition 2.6 (Holliday et al., 2016 [12])**.** Let L be a Hall assignment to a graph G and let $\sigma \in \mathcal{C}$. A vertex $v \in G(\sigma, L)$ is called a *mandatory witness* for color σ for the list L if the list assignment L' created from L by removing σ from $L(v)$ is not a Hall assignment to G.

Theorem 2.7 (Holliday et al., 2016 [12]). *If any vertex is a mandatory witness for a color in a Hall assignment L to G, then $(*)$ is satisfied with equality on (G, L). Furthermore, for any Hall assignment to a graph, a vertex can be a mandatory witness for at most one color.*

We also use the following well-known result:

Theorem 2.8 (Erdős et al., 1980 [7]). *If L is any 2-assignment to a path P, then P is L-colorable.*

Our first results leading to Question 3 involve how a graph's Hall spectrum is related to that of its connected components.

Recall that if G_1 and G_2 are disjoint graphs, then the *union* of G_1 and G_2 is the graph denoted by $G_1 \cup G_2$, where $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_2) \cup E(G_2)$. When $\chi(G_1) = \chi(G_2) \leq \Delta(G_1) = \Delta(G_2)$, we shall use $\overline{h}(G_1) \circ \overline{h}(G_2)$ to denote the component-wise product (or *Hadamard product*) of the Hall spectra $\overline{h}(G_1) = [f_0, \ldots, f_\alpha]$ and $\overline{h}(G_2) = [g_0, \ldots, g_\alpha]$. That is, $\overline{h}(G_1) \circ \overline{h}(G_2) = [h_0, \ldots, h_\alpha]$, where $h_i = f_i \cdot g_i$ for all $i \in \{0, \ldots, \alpha\}$.

The following simple observation follows immediately from Definition 2.1 and Theorem 1.4 (statement 2).

Observation 2.9. *If G_1 and G_2 are graphs satisfying $\chi(G_1) = \chi(G_2) \leq \Delta(G_1) = \Delta(G_2)$, then*

$$\overline{h}(G_1 \cup G_2) = \overline{h}(G_1) \circ \overline{h}(G_2).$$

Connecting two components with a sufficiently long path preserves this property:

Definition 2.10. Let G_1 and G_2 be graphs, $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ be fixed vertices, and $P = x_0, \ldots, x_t$ be a path of length t. Let G be the graph formed from $G_1 \cup G_2 \cup P$ by adding two edges, v_1x_0 and v_2x_t. We shall say that G is formed by *tethering* G_1 and G_2 with P at v_1 and v_2. If the choice of v_1 and v_2 is irrelevant, we shall simply say that G is formed by tethering G_1 and G_2 with P.

Proposition 2.11. *Suppose G_1 and G_2 are graphs that are not regular, with $\Delta(G_1) = \Delta(G_2)$ and $\chi(G_1) = \chi(G_2) \geq 3$. Let $v_1 \in V(G_1)$ be a vertex with $deg(v_1) \neq \Delta(G_1)$ and $v_2 \in V(G_2)$ be a vertex with $deg(v_2) \neq \Delta(G_2)$. If G is a graph obtained by tethering G_1 and G_2 with a path $P = x_0, \ldots, x_t$ at v_1 and v_2 and $t \geq 1$, then $\overline{h}(G) = \overline{h}(G_1) \circ \overline{h}(G_2)$.*

Proof. It suffices to show that G is Hall k-extendible if and only if both G_1 and G_2 are Hall k-extendible for all $\chi(G_1) \leq k \leq \Delta(G_1)$.

Suppose first that both G_1 and G_2 are Hall k-extendible for some $k \geq \chi(G_1)$, and fix a Hall k-precoloring ϕ of G. Since (G_1, L_ϕ) and (G_2, L_ϕ) satisfy Hall's condition, ϕ can be extended to G_1 and to G_2. Since $t \geq 1$, each vertex of the new path P has at most two colored neighbors, and $k \geq 3$ implies that the coloring can be extended greedily to $V(P)$.

Now suppose G_1 (or symmetrically G_2) is not Hall k-extendible for some $k \geq \chi(G_1)$, and let $\phi : V_0 \to [k]$ be a Hall k-precoloring of G_1 that does not extend. Consider ϕ as a k-precoloring of G. Clearly ϕ does not extend to G. It remains to show that ϕ is a Hall precoloring of G, and we do so by verifying $(*)$ for all subgraphs of G.

Delete x_0, the vertex of P adjacent to $v_1 \in V(G_1)$. Each component of $G - x_0$ satisfies Hall's condition: G_1 by hypothesis, and $G_2 \cup P - x_0$ because all lists are $[k]$ and $G_2 \cup P$ is k-colorable. Hence any subgraph of these components satisfies $(*)$. Let H be a subgraph of G containing x_0; we show that H satisfies $(*)$. By the previous observation, ϕ is a Hall precoloring of $H - x_0$, and thus $H - x_0$ satisfies $(*)$. Further, we claim x_0 can contribute at least one more to the Hall sum for H. If v_1 is precolored, then $|L_\phi(x_0)| \geq 2$, and since by Theorem 2.7, x_1 can be a mandatory witness for at most one color when ϕ is viewed as a Hall precoloring of $H - x_0$, the vertex x_0 can contribute to the independence number of at least one color $\sigma \in L_\phi(x_0)$. If v_1 is not precolored, then $|L_\phi(x_0)| \geq 3$, and similarly, v_1 and x_1 are mandatory witnesses for at most one color when ϕ is viewed as a Hall precoloring of $H - x_0$, leaving at least one color to which x_0 can contribute. Hence, (H, L_ϕ) satisfies $(*)$ and thus (G, L_ϕ) satisfies Hall's condition. □

Note 2.12. When $\chi(G_1) \neq \chi(G_2)$ or $\Delta(G_1) \neq \Delta(G_2)$, results similar to Observation 2.9 hold for $G_1 \cup G_2$ and Proposition 2.11 for G (with some restrictions), respectively, but the length of the Hall spectrum of the resulting graph must be adjusted to accommodate its maximum degree and chromatic number. For example, suppose $\overline{h}(G_1) = [f_0, f_1, f_2]$ and $\overline{h}(G_2) = [g_0, g_1, g_2, g_3, g_4]$. If it were the case that $\chi(G_1) = \chi(G_2)$ (so $\Delta(G_1) < \Delta(G_2)$), then $\overline{h}(G_1 \cup G_2) = [f_0g_0, f_1g_1, f_2g_2, g_3, g_4]$. On the other hand, if it were the case that $\Delta(G_1) = \Delta(G_2)$ (so $\chi(G_2) < \chi(G_1)$), then $\overline{h}(G_1 \cup G_2) = [f_0g_2, f_1g_3, f_2g_4]$. We omit the details of this natural extension of the proof of Proposition 2.11.

For the remainder of this section, we will let G_k denote the graph in Figure 2, $k \geq 3$. We will show G_k is a k-chromatic graph with Hall spectrum $\overline{h}(G_k) = [h_0, 1, 0, \dots]$, where h_0 is currently unknown. (We conjecture it to be 1, but the second and third positions are the only ones of interest for our purposes, so we have not verified the conjecture.) We should mention that the results of Theorem 2.15 and Lemmas 2.13–2.22 all hold in the case $k = 2$, though some proofs require additional care, and as the focus in answering Question 3 will be on $k \geq 3$, they are omitted.

Lemma 2.13. *For each $k \geq 3$, $\chi(G_k) = k$.*

Proof. Since G_k contains a k-clique, it is not $k - 1$ colorable; we provide a proper k-coloring ϕ. Let $\phi(x_4) = \phi(z) = 3$ and $\phi(x_1) = \phi(x_2) = \phi(x_3) = \phi(u_1) = \phi(u_2) = 1$. The vertices $\{v_1, v_2, w_1, \dots, w_{k-1}, y_1, \dots, y_k\}$ form an independent set which are collectively only adjacent to vertices colored 1 or 3, so they can all be given color 2. Finally, $N_{G_k}(u_1)$ can be colored with colors 2 through k. □

Note 2.14. For the remainder of this section we define $X = \{x_1, \dots, x_{k+3}\}$. The graph family in [11] that provided an affirmative answer to Question 1 is exactly the

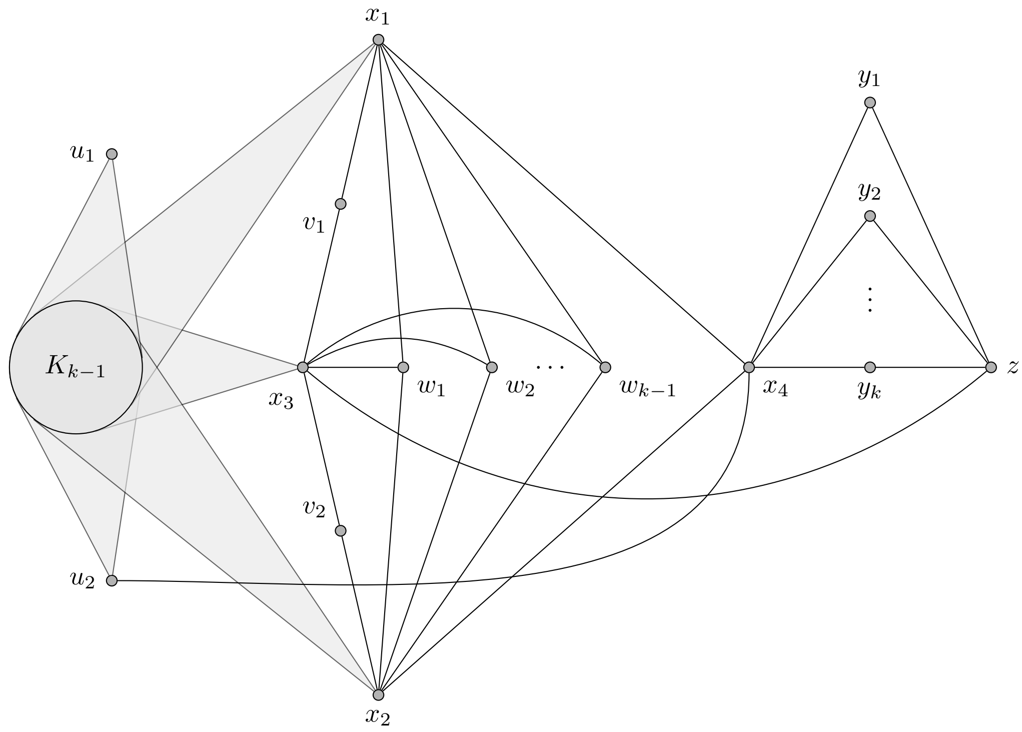

Figure 2: The graph family $\mathcal{G} = \{G_k : k \geq 3\}$. The shaded sectors indicate that each of the vertices u_1, u_2, x_1, x_2, and x_3 dominate the clique K_{k-1}. In the proofs, we use $N_{G_k}(u_1) = V(K_{k-1}) = \{x_5, \ldots, x_{k+3}\}$ and $X = \{x_1, \ldots, x_{k+3}\}$.

subgraph of G_k from Figure 2 induced by the vertices in X with lists produced by the following $(k+2)$-precoloring ψ: $\psi(u_1) = 1$, $\psi(u_2) = k+2$, $\psi(v_1) = 3$, $\psi(v_2) = 2$, $\psi(y_j) = j+1$ for each $j \in \{1, \ldots, k\}$, and $\psi(w_i) = i+3$ for each $i \in \{1, 2, \ldots, k-1\}$.

The following theorem from [11] (rephrased in context of G_k from Figure 2) and proof of Lemma 2.16 refer to the specific $(k+2)$-precoloring ψ described in Note 2.14 above.

Theorem 2.15 (Holliday et al., 2015 [11])**.** *Let H be the subgraph of G_k induced by the vertices in X and L_ψ be the corresponding list-assignment resulting from the $(k+2)$-precoloring ψ of G_k from Note 2.14. Then (H, L_ψ) satisfies Hall's condition, but $G_k - \{z, x_3\}$ is not L_ψ-colorable.*

Lemma 2.16. *For each $k \geq 3$, the graph G_k is not Hall $(k+2)$-extendible.*

Proof. Let ψ be the $(k+2)$-precoloring of G_k from Note 2.14 and L_ψ the corresponding list assignment. Let G'_k denote the subgraph of G_k induced by the set of uncolored vertices: $X \cup \{z\}$. Consider the restriction of L_ψ to G'_k. Let H be any connected induced subgraph of G'_k. If any of x_1, x_2, or x_4 is not in $V(H)$, then H is L_ψ-colorable, so (H, L_ψ) satisfies $(*)$. Hence we can assume that $x_1x_4 \in E(H)$, and let $V(H) = A \cup B$, where $A \subseteq \{x_5, \ldots, x_{k+3}\}$ and $\{x_1, x_4\} \subseteq B \subseteq \{x_1, x_2, x_3, x_4, z\}$.

Note $L_\psi(x_i) = \{2, \ldots, k+1\}$ for all $5 \le i \le k+3$ so the vertices of H in A can collectively contribute k to the Hall sum. Since $1 \in L_\psi(v)$ for all $v \in B$, $k{+}2 \in L_\psi(z)$, and $\{x_1, x_2, x_3\}$ is an independent set, one can check that the vertices of H in B can collectively contribute an additional $|B| - 1$ to the Hall sum. Hence,

$$\sum_{\sigma \in [k+2]} \alpha(H(\sigma, L_\phi)) \ge k + |B| - 1 \ge |A| + |B| = |V(H)|.$$

Therefore, (G'_k, L_ψ) and thus (G_k, L_ψ) satisfies Hall's condition. Finally, by Theorem 2.15, G'_k is not L_ψ-colorable, so ψ cannot be extended. □

Lemma 2.17. *Suppose that $k \ge 3$ and ϕ is a Hall $(k+1)$-precoloring of G_k. If ϕ extends to the subgraph of G_k induced by $X \cup \{z\}$, then ϕ extends to a proper $(k+1)$-coloring of G_k. Furthermore, it is sufficient to extend the coloring to the subgraph of G_k induced by X unless the vertices in $\{y_1, \ldots, y_k\}$ are precolored with k different colors.*

Proof. Suppose ϕ has been extended to X. Call this extension ϕ', and consider $L_{\phi'}$. To avoid considering cases, we view the lists on colored vertices as singleton sets. Since $\deg_{G_k}(z) = k+1$, z can be colored from $L_{\phi'}(z)$ unless all $k+1$ colors appear in $N_{G_k}(z)$, which implies that the vertices in $\{y_1, \ldots, y_k\}$ were all precolored different colors. The remaining vertices in $V(G_k) - (X \cup \{z\})$ form an independent set. None of their lists can be empty, since each vertex in this set has degree at most k. Therefore ϕ can be extended to a proper $(k+1)$-coloring of G_k. □

Lemma 2.18. *If every Hall $(k+1)$-precoloring of G_k that precolors z extends to a proper $(k+1)$-coloring of G_k, then every Hall $(k+1)$-precoloring of G_k extends to a proper $(k+1)$-coloring.*

Proof. Let ϕ be a Hall $(k+1)$-precoloring that does not precolor z and L_ϕ the corresponding list-assignment. By Theorem 2.7, x_3 is a mandatory witness for at most one color.

Case 1: If $|L_\phi(z)| > 1$, then color z with any color from its list for which x_3 is not a mandatory witness (say σ) and update the lists on all vertices in $N_{G_k}(z)$. Call this new coloring ϕ'. Suppose that $(G_k, L_{\phi'})$ does not satisfy $(*)$ and let H be the subgraph on the smallest number of vertices for which $(*)$ fails. Note that if H contains a vertex $y' \in \{y_1, \ldots, y_k\}$, then by minimality $H - \{y'\}$ satisfies $(*)$. As x_4 is a mandatory witness for at most one color and $|L_{\phi'}(y')| \ge 2$, $(H, L_{\phi'})$ satisfies $(*)$. This contradiction implies that $V(H) \cap N_{G_k}(z) \subseteq \{x_3\}$. Further, because x_3 is not a mandatory witness for σ, it is also not in H. But then H fails $(*)$ for the precoloring ϕ. Therefore $(G_k, L_{\phi'})$ is a Hall $(k+1)$-precoloring and thus extends to a proper coloring.

Case 2: If $|L_\phi(z)| = 1$, then without loss of generality, suppose $L_\phi(z) = \{1\}$. Therefore, all but one of the neighbors of z (we call this vertex a) is precolored. If we replace $L_\phi(a)$ with $L_\phi(a) - \{1\}$, then $\alpha(H(1, L_\phi))$ cannot change because a and z cannot both contribute to that value. Thus a is not a mandatory witness for color

1, so z can be precolored with 1, and $(*)$ is still satisfied. Therefore the coloring extends by hypothesis. □

Lemma 2.19. *Suppose ϕ is a Hall $(k+1)$-precoloring of G_k that precolors z, and $\sigma \in L_\phi(x_1) \cap L_\phi(x_2)$ (including if $\phi(x_1) = \sigma$ or $\phi(x_2) = \sigma$). If $\phi(z) = \sigma$ or u_2 is uncolored with $L_\phi(x_4) = \{\sigma\}$, then there is some color $\tau \neq \sigma$ such that either $\tau \in L_\phi(x_1) \cap L_\phi(x_2) \cap L_\phi(x_3)$, or $L_\phi(x_3) = \{\tau\}$ and $\phi(x_4) = \tau$.*

Proof. Let ϕ be as stated. We may assume $\phi(z) = \sigma$, since if $L_\phi(x_4) = \{\sigma\}$ and u_2 is not precolored, then the vertices in $\{y_1, \dots, y_k\}$ must be precolored and use all colors in $[k+1] - \{\sigma\}$, implying $\phi(z) = \sigma$. Now $\sigma \notin L_\phi(x_3)$, but since (G_k, L_ϕ) satisfies Hall's condition, $L_\phi(x_3) \neq \varnothing$. Hence there is some $\tau \neq \sigma \in L_\phi(x_3)$. (Note that perhaps $\phi(x_3) = \tau$.) Since $(N_{G_k}(x_1) \cup N_{G_k}(x_2)) - N_{G_k}(x_3) = \{x_4\}$, τ also appears on $L_\phi(x_1) \cap L_\phi(x_2)$, unless $\phi(x_4) = \tau$. Then either there is some $\tau' \neq \tau$ such that $\tau' \in L_\phi(x_1) \cap L_\phi(x_2) \cap L_\phi(x_3)$, or $L_\phi(x_3) = \{\tau\}$. □

Lemma 2.20. *If ϕ is a Hall $(k+1)$-precoloring of G_k, then there is some $\sigma \in [k+1]$ that appears in at least two of the elements in $\{L_\phi(x_1), L_\phi(x_2), L_\phi(x_3)\}$. Furthermore, if there is exactly one such σ and it appears in exactly two of the elements in $\{L_\phi(x_1), L_\phi(x_2), L_\phi(x_3)\}$ (say on $L_\phi(x_1)$ and $L_\phi(x_2)$), then all $k+1$ colors appear on the union of the lists in $X - \{x_1, x_4\}$ and $X - \{x_2, x_4\}$.*

Proof. The subgraph induced by $X - \{x_4\}$ contains $k+2$ vertices, so in order to satisfy $(*)$, $\alpha(H(\sigma, L_\phi)) \geq 2$ for some $\sigma \in [k+1]$. Since $\{x_1, x_2, x_3\}$ are the only independent vertices in $X - \{x_4\}$, σ appears in at least two of the elements in $\{L_\phi(x_1), L_\phi(x_2), L_\phi(x_3)\}$. If σ appears only in $L_\phi(x_1)$ and $L_\phi(x_2)$ and there are no other colors shared by the elements of $\{L_\phi(x_1), L_\phi(x_2), L_\phi(x_3)\}$, then $(*)$ is satisfied on the subgraphs induced by $X - \{x_1, x_4\}$ and by $X - \{x_2, x_4\}$ only if all $k+1$ colors contribute to the Hall sum. □

Lemma 2.21. *If ϕ is a Hall $(k+1)$-precoloring of G_k that precolors neither x_1 nor x_2, then $L_\phi(x_1) \cap L_\phi(x_2) \neq \varnothing$.*

Proof. If $L_\phi(x_1) \cap L_\phi(x_2) = \varnothing$, since $N_{G_k}(x_1)$ and $N_{G_k}(x_2)$ differ only at v_1 and v_2, $L_\phi(x_1) = \{\sigma\}$ and $L_\phi(x_2) = \{\sigma'\}$ for $\sigma \neq \sigma'$; further, $\phi(v_1) = \sigma'$ and $\phi(v_2) = \sigma$. But now $\{\sigma, \sigma'\} \cap L_\phi(x_3) = \varnothing$, violating Lemma 2.20. □

Lemma 2.22. *For $k \geq 3$, the graph G_k is Hall $(k+1)$-extendible.*

Proof. By Lemmas 2.17 and 2.18, we need only show that any Hall $(k+1)$-precoloring of G_k that precolors z can be extended to $G_k[X]$. Therefore, let ϕ be a Hall $(k+1)$-precoloring of G_k that precolors z and let L_ϕ be the corresponding list-assignment. Suppose that A is the (possibly empty) set of t precolored vertices in $N_{G_k}(u_1)$ (without loss of generality, suppose they are colored $1, \dots, t$) and $B = N_{G_k}(u_1) - A$. Observe that the vertices in B have identical lists. Let L_B denote this common list and let $\ell = |L_B|$. Because (G_k, L_ϕ) satisfies Hall's condition, $\ell \in \{k-1-t, k-t, k+1-t\}$.

Observation 1: If ϕ extends to $\{x_1, x_2, x_3, x_4\}$ so that at most $\ell - |B|$ colors from L_B appear on $\{x_1, x_2, x_3\}$, then ϕ extends to $G_k[X]$.

Case 1: Suppose that $\ell = k - 1 - t$, and without loss of generality, $L_B = \{t+1, \dots, k-1\}$. By Observation 1, it suffices to show that ϕ can be extended to $\{x_1, x_2, x_3, x_4\}$ so that the colors on $\{x_1, x_2, x_3\}$ are elements of the set $\{k, k+1\}$. Since L_ϕ satisfies $(*)$, each of x_1, x_2, and x_3 have color k or $k+1$ in their list (if any are precolored, the list is $\{k\}$ or $\{k+1\}$). They are independent of each other, so we simultaneously color any uncolored vertices in $\{x_1, x_2, x_3\}$ with colors from $\{k, k+1\}$, giving x_1 and x_2 the same color if possible. Let ϕ' be this extension of ϕ, and let $L_{\phi'}$ be its associated list assignment. If x_4 was precolored by ϕ, we are done. Otherwise, all that remains is to color x_4. Suppose that $L_{\phi'}(x_4) = \varnothing$. As (G_k, L_ϕ) satisfies Hall's condition, $L_\phi(x_4) \neq \varnothing$, so either $\phi'(x_1) = \phi'(x_2) = k$ and $L_\phi(x_4) = \{k\}$ (or $k+1$, symmetrically), or $\phi'(x_1) = k$, $\phi'(x_2) = k+1$ and $L_\phi(x_4) \subseteq \{k, k+1\}$. In the former case, neither the subgraph of G_k induced by $X - \{x_1, x_3\}$ nor the subgraph of G_k induced by $X - \{x_2, x_3\}$ will satisfy $(*)$ unless $k+1 \in L_\phi(x_1) \cap L_\phi(x_2)$. Therefore we can color x_1 and x_2 with color $k+1$ instead, and now x_4 can be colored k. In the latter case, without loss of generality we can assume that $k+1 \notin L_\phi(x_1)$ and $k \notin L_\phi(x_2)$, for otherwise we would have colored x_1 and x_2 the same. However, this forces $\phi(v_1) = k+1$ and $\phi(v_2) = k$ and so $\{k, k+1\} \cap L_\phi(x_3) = \varnothing$, a contradiction. Hence ϕ extends to $\{x_1, x_2, x_3, x_4\}$ in the required manner, and by Observation 1, ϕ extends to $G_k[X]$.

Case 2: Suppose that $\ell = k + 1 - t$. This implies that none of $\{u_1, u_2, x_1, x_2, x_3\}$ are precolored. By Lemma 2.21, $L_\phi(x_1) \cap L_\phi(x_2) \neq \varnothing$. Extend ϕ by coloring x_1 and x_2 with the same color (say σ), and coloring x_3 with any color from $L_\phi(x_3)$. Now either x_4 is precolored or x_4 can be colored, unless $L_\phi(x_4) = \{\sigma\}$. If so, by Lemma 2.19, there is some $\tau \in L_\phi(x_1) \cap L_\phi(x_2) \cap L_\phi(x_3)$; thus x_1, x_2, and x_3 can be recolored with τ, and x_4 can be colored with σ. In either case, by Observation 1, ϕ extends to $G_k[X]$.

Case 3: Suppose that $\ell = k - t$, and assume without loss of generality that $L_B = \{t+1, \dots, k\}$. At least one vertex in $\{u_1, u_2, x_1, x_2, x_3\}$ has been precolored, and any such vertex must have been precolored with $k+1$. Let $Y = \{x_1, x_2, x_3\}$. Since $\ell = k - t$, by Observation 1, it suffices to show that the vertices in $\{x_1, x_2, x_3, x_4\}$ can be colored in such a way that at most one color from L_B is used on Y. We consider two possibilities:

(a) The vertex x_1 (or symmetrically x_2) is precolored with $k+1$. If $k+1 \in L_\phi(x_2)$ (if $\phi(x_2) = k+1$, then $L_\phi(x_2) = \{k+1\}$), then coloring x_2 with $k+1$ leaves a color available for x_4. Coloring x_3 with any available color then ensures that at most one color from L_B is used on Y. Hence we may assume $k+1 \notin L_\phi(x_2)$. This implies $\phi(v_2) = k+1$, so $k+1 \notin L_\phi(x_3)$. By Lemma 2.20, there is some $\sigma \in L_\phi(x_2) \cap L_\phi(x_3)$. If $L_\phi(x_4) \neq \{\sigma\}$, then we can extend ϕ to $\{x_1, x_2, x_3, x_4\}$ such that σ is the only color in L_B that appears on $\{x_1, x_2, x_3\}$. If $L_\phi(x_4) = \{\sigma\}$, then at least $k-1$ vertices in $\{y_1, \dots, y_k\}$ must have been precolored, using all colors in $[k] - \{\sigma\}$, implying $\phi(z) = k+1$ (since $\phi(x_3) \neq \sigma$). Since the neighborhoods of x_2 and x_3 differ only at x_4 and z, $L_\phi(x_2) = L_\phi(x_3)$. Now there must be some $\sigma' \in L_\phi(x_3) \cap L_\phi(x_2)$, where $\sigma' \neq \sigma$, with which to color x_2 and x_3 instead, otherwise $(*)$ fails on the subgraph of G_k induced by $X - \{x_1\}$. In either case, by Observation 1, ϕ extends to $G_k[X]$.

(b) Neither x_1 nor x_2 is precolored. By Lemma 2.21, there exists $\sigma \in L_\phi(x_1) \cap L_\phi(x_2)$.

i. Suppose that $\sigma \notin L_\phi(x_3)$. Then the overlapping neighborhoods of x_1, x_2, and x_3 guarantee that $\phi(z) = \sigma$. Now Lemma 2.19 implies that for some $\tau \neq \sigma$, either $\tau \in L_\phi(x_1) \cap L_\phi(x_2) \cap L_\phi(x_3)$, or $L_\phi(x_3) = \{\tau\}$ and $\phi(x_4) = \tau$. In the former case, color the vertices of Y with τ and x_4 with σ, unless $\sigma \notin L_\phi(x_4)$, in which case we must have $\phi(u_2) = \sigma$, hence $\sigma = k+1$. Color x_1 and x_2 with $\sigma = k+1$ and x_3 and x_4 (if uncolored) with any available color. In the latter case, due to the overlapping neighborhoods of x_1, x_2, and x_3, the color σ is the only color appearing more than once in the elements of $\{L_\phi(x_1), L_\phi(x_2), L_\phi(x_3)\}$, and it appears exactly twice. Hence Lemma 2.20 implies that either $\tau = k+1$ or $\sigma = k+1$. Color x_1 and x_2 with σ and x_3 with τ. In either case, at most one color from L_B appears on Y.

ii. Suppose that $\sigma \in L_\phi(x_3)$. Then we color the vertices of Y with σ unless $L(x_4) = \{\sigma\}$. In this situation we note that $\phi(z) \neq \sigma$, and $\phi(u_2) = k+1$ (for otherwise, if u_2 is uncolored, then all vertices in $\{y_1, \ldots, y_k\}$ must be precolored different colors from $[k+1] - \{\sigma\}$, forcing $\phi(z) = \sigma$ and thereby $\sigma \notin L_\phi(x_3)$, a contradiction). Even further, $\phi(z) = k+1$ because at least $k-1$ of the vertices in $y_1, \ldots, y_k$ must be precolored and cover all colors in $[k] - \{\sigma\}$, leaving only colors σ (impossible) or $k+1$ for the precolored vertex z. If $k+1 \in L_\phi(x_1) \cap L_\phi(x_2)$, then we may color x_1 and x_2 with $k+1$ and color x_3 and x_4 with σ. If there exists $\tau \neq \sigma$ such that $\tau \in L_\phi(x_1) \cap L_\phi(x_2) \cap L_\phi(x_3)$, then we may color the vertices in Y with τ and x_4 with σ. Finally, if (without loss of generality) $k+1 \notin L_\phi(x_1)$ and $L_\phi(x_1) \cap L_\phi(x_2) \cap L_\phi(x_3) = \{\sigma\}$, then $L_\phi(x_1) = \{\sigma\}$ and so $(*)$ fails on the subgraph induced by $\{x_1, x_4\}$. To see why x_1 would have a singleton list, note that if there exists $\gamma \in L_\phi(x_1) - \{\sigma\}$, then $\gamma \in L_\phi(x_3)$ for otherwise, the overlapping neighborhoods of x_1 and x_3 would force $\phi(z) = \gamma = k+1$, a contradiction to $k+1 \notin L_\phi(x_1)$. Finally, again by the overlapping neighborhoods of x_1, x_2, and x_3 we must have $\gamma \in L_\phi(x_2)$. Hence, $\{\sigma, \gamma\} \subseteq L_\phi(x_1) \cap L_\phi(x_3) \cap L_\phi(x_2)$, a contradiction.

Therefore, in all possible cases, at most one color from L_B appears on Y.

□

Before we prove our main result in this section, we require the following theorem and generalization:

Theorem 2.23 (Holliday et al., 2016 [12]). *The Hall spectrum of the wheel graph W_n having order $n+1$ is either $\overline{h}(W_n) = [1, 1, \ldots, 1]$ if n is odd or $\overline{h}(W_n) = [0, 1, \ldots, 1]$ if n is even and $n \geq 10$.*

Recall that if G_1 and G_2 are graphs, then the *join* of G_1 and G_2 is the graph denoted by $G_1 + G_2$, where $V(G_1 + G_2) = V(G_1) \cup V(G_2)$ and

$$E(G_1 + G_2) = \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\} \cup E(G_1) \cup E(G_2).$$

Lemma 2.24. *If G_1 is a q-chromatic graph and G_2 is not Hall k-extendible, then $G_1 + G_2$ is not Hall $(q + k)$-extendible.*

Proof. Let ϕ_{G_1} be a q-coloring of G_1 using colors $\{1, \dots, q\}$, and let ϕ_{G_2} be a Hall k-precoloring of G_2 from $\{q + 1, q + 2, \dots, q + k\}$ that does not extend to G_2. Now $\phi_{G_1} \cup \phi_{G_2}$ is a Hall $(q+k)$-precoloring of G_1+G_2, since the only uncolored vertices are in G_2 and $\phi_{G_1} \cup \phi_{G_2}$ restricts to the Hall precoloring ϕ_{G_2} on G_2. Further, $\phi_{G_1} \cup \phi_{G_2}$ is not extendible, since otherwise ϕ_{G_2} would be extendible to G_2. □

Note that W_n is simply the graph join K_1+C_n. The following theorem generalizes Theorem 2.23 for the first two positions of the Hall spectrum.

Theorem 2.25. *For any $q \geq 1$ and any even $n \geq 10$, the graph join $K_q + C_n$ has Hall spectrum $\overline{h}(K_q + C_n)$ beginning $[0, 1, \dots]$.*

Proof. Let $G = K_q + C_n$; note that $\chi(G) = q + 2$. First we show that G is not Hall $(q + 2)$-extendible. Observe that G can also be represented as $G_1 + G_2$, where $G_1 = K_{q-1}$ and $G_2 = W_n$. Since G_1 has chromatic number $q - 1$ and G_2 is not Hall 3-extendible by Theorem 2.23, Lemma 2.24 implies that G is not $(q + 2)$-extendible.

Next we show that G is Hall $(q + 3)$-extendible by induction on q, with Theorem 2.23 establishing the case $q = 1$. Suppose that $q > 1$, and consider a Hall $(q + 3)$-precoloring $\phi : V_0 \to [q + 3]$ of $G = K_q + C_n$.

Case 1: ϕ precolors at least one vertex u of K_q. Without loss of generality, suppose $\phi(u) = q+3$. Then $L_\phi(v) \subseteq \{1, \dots, q+2\}$ for every $v \in V(G) \setminus \{u\}$. Since ϕ is a Hall $(q+3)$-precoloring of G, ϕ is a Hall $(q+2)$-precoloring of $G-u = K_{q-1}+C_n$, and ϕ extends to $G - u$ by induction.

Case 2: ϕ precolors no vertex of K_q. Since (K_q, L_ϕ) satisfies Hall's condition, the cardinality of the image $\phi(V_0)$ is at most three. Extend ϕ to K_q using colors $[q + 3] \setminus \phi(V_0)$. Now any uncolored vertex in C_n has at most $q + 2$ colored neighbors, with equality if and only if the entire neighborhood of the vertex is colored. Hence, what remains is a 2-assignment to a disjoint collection of paths and a 1-assignment to isolated vertices. By Theorem 2.8, ϕ may be extended to a proper $(q+3)$-coloring of G.

Hence, G is Hall $(q + 3)$-extendible. □

We now present the proof of Theorem 1.6, the main result of this section, providing an infinite family of graphs whose Hall spectra contain non-consecutive zeros.

Proof of Theorem 1.6. By Lemma 2.13, Theorem 2.15, and Lemmas 2.16-2.22, the graph G_k shown in Figure 2 has Hall spectrum $\overline{h}(G_k) = [h_0, 1, 0, \dots]$. By Theorem 2.25, when n is even and $n \geq 10$, the graph $K_{k-2} + C_n$ has Hall spectrum

$\overline{h}(K_{k-2} + C_n) = [0, 1, \ldots]$. Then, since $\chi(G_k) = \chi(K_{k-2} + C_n)$, it is immediate that the Hall spectrum of their disjoint union is $\overline{h}(G_k \cup (K_{k-2} + C_n)) = [0, 1, 0, \ldots]$. Moreover, we can form a connected graph by tethering G_k to $K_{k-2} + C_n$ with a path on 2 vertices at vertices of minimum degree in G_k and $K_{k-2} + C_n$. Because these graphs have the same chromatic number and neither are regular, we can use Proposition 2.11 to form a connected graph with Hall spectrum $[0, 1, 0, \ldots]$. This may require adding pendent vertices to one graph so that the maximum degrees match, but it is easy to verify that this does not affect the chromatic number, Hall's condition, or extendibility of any precoloring. □

3 Extending with extra colors

Definition 3.1. Given a graph G and $V_0 \subseteq V(G)$, a k-precoloring $\phi : V_0 \to [k]$ of G is *ℓ-extendible* for some $\ell \geq k$ if there exists an ℓ-coloring $\gamma : V(G) \to [\ell]$ of G such that $\gamma(v) = \phi(v)$ for all $v \in V_0$.

Recall $h(G)$ is the Hall number of G. We begin with an elementary result that allows Hall precolorings of some graphs to be extended with few extra colors. It relies on the following:

Theorem 3.2 (Hilton et al., 1996 [10]). *If H is an induced subgraph of G then $h(H) \leq h(G)$.*

Theorem 3.3. *Any Hall k-precoloring of G is $(k + h(G) - 1)$-extendible. Moreover, if G is k-colorable, then any Hall k-precoloring of G is $(k + \chi(G) - 1)$-extendible.*

Proof. Let ϕ be a Hall k-precoloring of G and let G' be the subgraph of G induced on the uncolored vertices. For each $v \in V(G')$, if we define $L(v) = L_\phi(v) \cup \{k+1, \ldots, k+h(G) - 1\}$, we obtain an $h(G)$-assignment of G' which satisfies Hall's condition, because each $L_\phi(v)$ contains at least one color. By Theorem 3.2, $h(G') \leq h(G)$, so G' has an L-coloring.

For the second statement, suppose G is k-colorable. Then $V(G')$ may be partitioned into $\chi(G) = \chi$ independent sets $V_1, \ldots, V_\chi$. As ϕ is Hall, $|L_\phi(v)| \geq 1$ for all $v \in V(G')$. Coloring each $v \in V_1$ with a color from its list, $L_\phi(v)$, and each $v \in V_i$ for $i > 1$ with color $k + i - 1$ yields an extension of ϕ with $k + \chi(G) - 1$ colors. □

It is natural to ask whether the Hall number statement or the chromatic number statement in Theorem 3.3 is stronger. In fact, the answer depends on the family of graphs under consideration. Recall from Theorem 1.5 that Hall k-precolorings of bipartite graphs do not necessarily extend with k colors when $k > 2$. The following corollary of Theorem 3.3 ensures any such colorings can be extended with only one additional color.

Corollary 3.4. *Any Hall k-precoloring of a bipartite graph is $(k + 1)$-extendible.*

Moreover, the two results below produce bipartite graphs with arbitrarily large Hall number for which Hall k-precolorings extend with only one additional color. Recall, given a graph G, the *choice number* or (*list-chromatic number*) of G, denoted $\chi_\ell(G)$, is the smallest positive integer k such that G is L-colorable for every k-assignment L to G.

Theorem 3.5 (Johnson, 2002 [14]). *If $\chi(G) < \chi_\ell(G)$, then $h(G) = \chi_\ell(G)$.*

Theorem 3.6 (Erdős et al., 1980 [7]). *If $m = \binom{2k-1}{k}$ and $k \geq 1$, then $\chi_\ell(K_{m,m}) > k$.*

For those graphs having Hall number 2, we have the following:

Corollary 3.7. *Any Hall k-precoloring of a graph with Hall number 2 is $(k+1)$-extendible.*

Graphs with Hall number 2 have been characterized (see [6, 15] for a complete description), but notable 2-connected examples are cycles with at least 4 vertices, $K_4 - e$ with one edge subdivided (which we will call $(K_4 - e)^*$), and $K_{2,3}$ with one of the vertices of degree two replaced by a path of arbitrary length. In addition, [15] describes the block structure of any graph G with $\kappa(G) = 1$ and $h(G) = 2$.

Theorem 3.8 (Johnson and Wantland, 2002 [15]). *Suppose $h(G) = 2$. For each $m \geq 0$, define $G(m)$ to be the graph obtained by tethering a clique and G with a path of length m. If $h(G(0)) = 2$, then $h(G(m)) = 2$ for all $m \geq 0$.*

The result above yields graphs with arbitrarily large chromatic number for which Hall $\chi(G)$-precolorings extend with only one additional color. As an example, let G be the graph obtained by tethering a clique of size n to one of the vertices of degree 2 in $(K_4 - e)^*$ by a path, possibly of length 0. Since $h(G) = 2$, by Corollary 3.7, any Hall k-precoloring of G is $(k+1)$-extendible.

In light of Corollary 3.4, a natural question to ask is whether Hall 3-precolorings of 3-chromatic graphs extend with 4 colors. Theorem 1.7 indicates that sometimes 5 colors are needed. We complete this section by proving several results that establish Theorem 1.7.

Definition 3.9. Let G be a graph with list assignment L and let H be a subgraph of G. The *Hall slack* of H with respect to L is

$$s(H, L) = \left(\sum_{\sigma \in [k]} \alpha(H(\sigma, L)) \right) - |V(H)|.$$

If $s(H, L) = 0$, then H is called *tight* with respect to L. G is called *loose* with respect to L if G has no nonempty subgraph that is tight with respect to L. Further, if G has a precoloring ϕ which colors vertex set V_0, then we say that G is *loose* with respect to ϕ if the graph $G - V_0$ is loose with respect to L_ϕ.

Clearly, (G, L) satisfies Hall's condition if and only if $s(H, L) \geq 0$ for every $H \leq G$, and if G is loose with respect to L, then necessarily (G, L) satisfies Hall's condition. To simplify terminology, when it is not ambiguous, we shall say a graph G is *loose* if there exists a list-assignment L such that G is loose with respect to L.

Proposition 3.10. *Suppose H_1 and H_2 are loose graphs with respect to list assignments L_1 and L_2 respectively. Then the graph G formed by adding an edge between a vertex $v_1 \in V(H_1)$ and a vertex $v_2 \in V(H_2)$ is also loose with respect to the list assignment $L = L_1 \cup L_2$.*

Proof. Let F be a subgraph of G and let F_1 and F_2 be subgraphs of F such that $V(F) = V(F_1) \cup V(F_2)$, $F_1 \subseteq H_1$, and $F_2 \subseteq H_2$. By hypothesis $s(F_i, L_i) > 0$, and by extension $s(F_i, L) > 0$, for each $i \in \{1, 2\}$. If $v_1v_2 \notin E(F)$ then clearly $s(F, L) = s(F_1, L_1) + s(F_2, L_2) > 0$. Hence we may assume $v_1v_2 \in E(F)$. Because v_1v_2 is the only edge between vertices in F_1 and vertices in F_2, we have

$$\sum_{\sigma\in[k]} \alpha(F(\sigma, L)) \geq \sum_{\sigma\in[k]} \alpha((F - v_1)(\sigma, L)) = \sum_{\sigma\in[k]} \alpha((F_1 - v_1)(\sigma, L)) + \sum_{\sigma\in[k]} \alpha(F_2(\sigma, L)).$$

Now since both $F_1 - v_1$ and F_2 are loose with respect to L,

$$\sum_{\sigma\in[k]} \alpha((F_1 - v_1)(\sigma, L)) + \sum_{\sigma\in[k]} \alpha(F_2(\sigma, L)) \geq$$

$$(|V(F_1 - v_1)| + 1) + (|V(F_2)| + 1) = |V(F)| + 1.$$

and hence F is also loose. As F was an arbitrary subgraph of G, we conclude that G is loose with respect to L. □

Lemma 3.11. *For each $k \geq 2$, there exists a k-colorable graph that is loose with respect to a non-extendible $(k + 1)$-precoloring.*

Proof. Let $k \geq 2$ and define H_k to be a graph with $V(H_k) = \{x_0, \dots, x_{3k}\} \cup \{y_0, \dots, y_{3k}\}$ and $E(H_k)$ as follows (see Figure 3): Let $N_{H_k}(x_0) = \{x_1, x_{k+1}, x_{2k+1}\}$. The vertices in the set $\{x_2, \dots, x_k\}$ form a clique X_0 and are also adjacent to x_1 and x_{k+1}; the vertices in $\{x_{k+2}, \dots, x_{2k}\}$ form a clique X_1 and are also adjacent to x_{k+1} and x_{2k+1}; the vertices in $\{x_{2k+2}, \dots, x_{3k}\}$ form a clique X_2 and are also adjacent to x_{2k+1} and x_1. Let $y_iy_j \in E(H_k)$ if and only if $x_ix_j \in E(H_k)$. Finally, let $x_{3k}y_{3k} \in E(H_k)$. It is straightforward to verify $\chi(H_k) = k$.

We now modify H_k using pendant vertices to obtain a k-colorable graph H'_k that is loose with respect to a $(k+1)$-precoloring ϕ, and we verify that ϕ is not extendible with $k + 1$ colors.

Let H'_k be the graph obtained from H_k by adding an independent set V_0 of pendant vertices (each adjacent to exactly one vertex in H_k) as follows: each vertex in $V(X_0) \cup V(X_1) \cup (V(X_2) \setminus \{x_{3k}\})$ is adjacent to one pendant; each vertex in $\{x_1, x_{k+1}, x_{2k+1}\}$ is adjacent to $k - 1$ pendant vertices; and x_0 is adjacent to $k - 2$

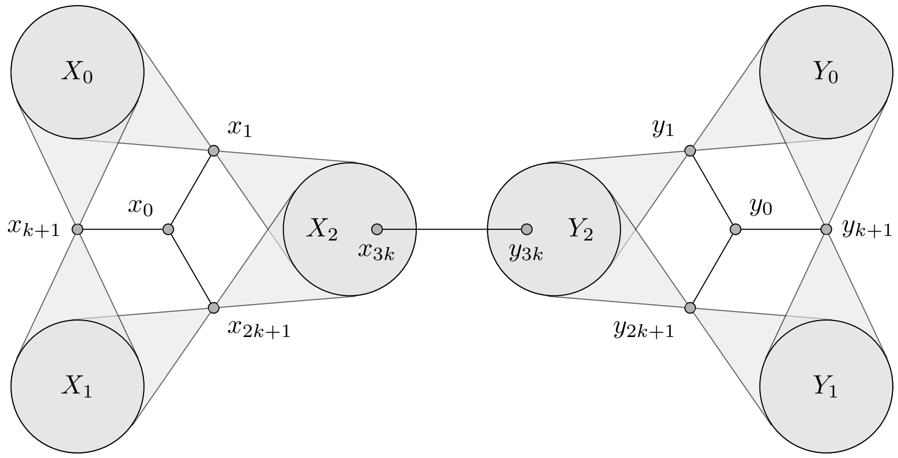

Figure 3: The k-chromatic graph H_k from Lemma 3.11. The grey circles labeled X_0, X_1, X_2, Y_0, Y_1, and Y_2 are each $(k-1)$-cliques. The shaded regions emanating from vertex x_1 indicate it dominates all vertices in X_0 and X_2 (similar for vertices $x_{k+1}, x_{2k+1}, y_1, y_{k+1}$, and y_{2k+1}). Pendant vertices can be attached to create the loose k-colorable graph H'_k.

pendant vertices. Now define a $(k+1)$-precoloring $\phi : V_0 \to \mathcal{C}$ of H'_k, where $\mathcal{C} = \{0, 1, \dots, k\}$, to produce the following lists: $L_\phi(x_0) = \{0, 1, 2\}$, $L_\phi(x_1) = \{0, 2\}$, $L_\phi(x_{k+1}) = \{0, 1\}$, $L_\phi(x_{2k+1}) = \{1, 2\}$, $L_\phi(x) = \mathcal{C} \setminus \{0\}$ for all $x \in V(X_0)$, $L_\phi(x) = \mathcal{C} \setminus \{1\}$ for all $x \in V(X_1)$, $L_\phi(x) = \mathcal{C} \setminus \{2\}$ for all $x \in V(X_2) \setminus \{x_{3k}\}$. Add pendant vertices to each y_i in a similar fashion and extend ϕ to the pendant vertices so that $L_\phi(y_i) = L_\phi(x_i)$ for all $i \in \{0, \dots, 3k\}$. Clearly $\chi(H'_k) = \chi(H_k)$.

The precoloring ϕ extends to a $(k+1)$-coloring of H'_k if and only if the subgraph H_k is L_ϕ-colorable. If x_0 is given color 0, then x_1 and x_{k+1} must be colored with 2 and 1 respectively. The $k-1$ vertices of X_0 now only have $k-2$ available colors, so ϕ cannot be extended this way. Similarly, we cannot extend ϕ to $H'_k[\{x_0, \dots, x_{3k}\}]$ by letting $\phi(x_0) = 1$. To extend ϕ to $H'_k[\{x_0, \dots, x_{3k}\}]$ we must have $\phi(x_0) = \phi(x_{3k}) = 2$. By the same argument, to extend ϕ to $H'_k[\{y_0, \dots, y_{3k}\}]$ we must have $\phi(y_0) = \phi(y_{3k}) = 2$. As $x_{3k}y_{3k} \in E(H'_k)$, ϕ cannot be extended.

It remains to verify that no subgraph of H'_k is tight with respect to L_ϕ. By Theorem 2.3 and Proposition 3.10, it suffices to show that any connected subgraph F induced by vertices that are a subset of $\{x_0, x_1, \dots, x_{3k}\}$ satisfies $s(F, L_\phi) > 0$. Let F be such a subgraph, and suppose $V(F)$ intersects r of the cliques $\{X_0, X_1, X_2\}$. Let $a = \sum_{\sigma \in \mathcal{C}} \alpha(F(\sigma, L_\phi))$; our goal is to show that $a > |V(F)|$. We consider four cases. For simplicity of argument, we will artificially remove the color 2 from $L_\phi(x_{3k})$ in cases 2 and 3 (this restriction can only decrease a). This extra color is, however, important for case 4.

Case 1: $r = 0$. Then $V(F) \subseteq \{x_0, x_1, x_{k+1}, x_{2k+1}\}$ and it is routine to check $a > |V(F)|$.

Case 2: $r = 1$. Without loss of generality, suppose $p \in V(X_0) \cap V(F)$. We seek vertices that can contribute at least $|V(F)| + 1$ to a. As $L_\phi(p) = \{1, \ldots, k\}$, the vertex p can contribute k to a. Then each vertex in $\{x_0, x_1, x_{k+1}, x_{2k+1}\} \cap V(F)$ can contribute one to a (x_1 and x_{k+1} could contribute to $\alpha(F(0, L_\phi))$, x_0 could contribute to $\alpha(F(1, L_\phi))$, and x_{2k+1} could contribute to $\alpha(F(2, L_\phi))$). Because $k > |X_0|$, it follows that $a > |V(F)|$.

Case 3: $r = 2$. Without loss of generality, we may assume that $p_1 \in V(X_1) \cap V(F)$ and $p_2 \in V(X_2) \cap V(F)$. As $L_\phi(p_1) = \mathcal{C} \setminus \{1\}$, $L_\phi(p_2) = \mathcal{C} \setminus \{2\}$, and p_1 and p_2 are not adjacent, the vertices p_1 and p_2 can each contribute k to a. If $V(F) \subseteq V(X_1) \cup V(X_2) \cup \{x_{2k+1}\}$, then $a \geq 2k > |V(F)|$. Otherwise, each vertex in $\{x_0, x_1, x_{k+1}\} \cap V(F)$ can contribute one to a (x_0 could contribute to $\alpha(F(0, L_\phi))$, x_1 could contribute to $\alpha(F(2, L_\phi))$, and x_{k+1} could contribute to $\alpha(F(1, L_\phi))$). If follows that $a > |V(F)|$.

Case 4: $r = 3$. For each $i \in \{0, 1, 2\}$ let $p_i \in V(X_i) \cap V(F)$. Each p_i contributes at least k to a (in fact $k + 1$ if $p_2 = x_{3k}$), so $a \geq 3k$. If $x_0 \in V(F)$, then because $L_\phi(x_0) = \{0, 1, 2\}$ and x_0 is not adjacent to any p_i, x_0 can contribute three to a. Thus $a \geq 3k + 3 > |V(F)|$. Thus we assume $x_0 \notin V(F)$, so $|V(F)| \leq 3k$. But now $a > |V(F)|$ unless $V(F) = \{x_1, x_2, \ldots, x_{3k}\}$. In this case, because $L_\phi(x_{3k}) = \mathcal{C}$, we can let $p_2 = x_{3k}$ which contributes $k + 1$ to a. Hence, $a = 3k + 1 > |V(F)|$. □

We now present the proof of Theorem 1.7, the main result of this section.

Proof of Theorem 1.7. By Lemma 3.11 there exists a graph, say H, that is $(k-1)$-colorable and loose with respect to a k-precoloring ϕ_H where ϕ_H is not extendible. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertices in H that are uncolored by ϕ_H. Create a graph G from H by adding n copies of H, labeled $H_1, H_2, \ldots, H_n$, and adding an edge from $v_i \in V(H)$ to each vertex of H_i for $1 \leq i \leq n$. Let ϕ_G be the k-precoloring formed by coloring H according to ϕ_H, and copying the coloring ϕ_H onto H_i for all $1 \leq i \leq n$. Finally, for all $1 \leq i \leq n$, delete any edges from v_i to a colored vertex in H_i. We must verify that G is k-colorable, that ϕ_G satisfies Hall's condition, and that ϕ_G does not extend with $k + 1$ colors.

First we show that G is k-colorable. Let $c : V(H) \to [k-1]$ be a $(k-1)$-coloring of H. For each H_i, color H_i with the colors $\{1, \ldots, k\} \setminus \{c(v_i)\}$. Since each H_i is $(k-1)$-colorable, this is a k-coloring of G.

Next we verify that (G, L_{ϕ_G}) satisfies Hall's condition. Consider any subgraph F of G. Let F_i be the subgraph of F contained in H_i for $1 \leq i \leq n$, let F_0 be the (possibly empty) subgraph of F induced by the vertices in $V(H)$ that have no neighbors in any F_i for $1 \leq i \leq n$, and let S be the set of vertices in F contained in H that have a neighbor in some F_i. Observe that $V(F) = \bigcup_{i=0}^{n} V(F_i) \cup S$. Since H is loose with respect to L_{ϕ_H} and each F_i is isomorphic to a subgraph of H, for each F_i with $1 \leq i \leq n$ we have

$$\sum_{j \in [k]} \alpha(F_i(j, L_{\phi_G})) \geq |V(F_i)| + 1.$$

Since there are no edges between F_r and F_s for $0 \le r < s \le n$ and because $|S| \le n$,

$$\sum_{j\in[k]} \alpha(F(j, L_{\phi_G})) \ge |V(F_0)| + \sum_{i=1}^{n}(|V(F_i)| + 1) \ge \left| \left(\bigcup_{i=0}^{n} V(F_i) \right) \cup S \right| = |V(F)|.$$

Since F was arbitrary, Hall's condition is satisfied.

Finally, we show that ϕ_G does not extend with $k+1$ colors. Suppose instead that ϕ_G does extend with $k+1$ colors. Since ϕ_H is not extendible with k colors, each H_i must use color $(k+1)$ on an uncolored vertex. Thus, no uncolored vertex in H has received color $(k+1)$. This implies that H was colored with only k colors, which contradicts the fact that ϕ_H is not k-extendible. Therefore ϕ_G cannot be extended to all of G using $k+1$ colors. □

4 Future Work

It is currently unknown whether all Hall 4-precolorings of 4-chromatic graphs are 6-extendible. However, we make the following conjecture:

Conjecture 4.1. For all $k \ge 3$, there exists a graph G with $\chi(G) \ge k$ which has a Hall k-precoloring that is not extendible with fewer than $k + \chi(G) - 1$ colors.

Such a result could potentially be proven by generalizing the idea of loose graphs. With regards to Hall spectra, we make the following conjecture.

Conjecture 4.2. For all $n \ge 1$, every $\{0, 1\}$-vector of length n ending in a 1 is the Hall spectrum of some graph.

References

[1] M. O. Albertson and J. P. Hutchinson, Extending colorings of locally planar graphs, *J. Graph Theory* 36(2) (2001), 105–116.

[2] M. O. Albertson and J. P. Hutchinson, Extending precolorings of subgraphs of locally planar graphs, *European J. Combin.* 25(6) (2004), 863–871.

[3] M. O. Albertson, A. V. Kostochka and D. B. West, Precoloring extensions of Brooks' theorem, *SIAM J. Discrete Math.* 18(3) (2005), 542–443.

[4] M. O. Albertson and E. H. Moore, Extending graph colorings using no extra colors, *Discrete Math.* 234 (2000), 125–132.

[5] B. B. Bobga, J. L. Goldwasser, A. J. W. Hilton and P. D. Johnson, Jr., Completing partial Latin squares: Cropper's question, *Australas. J. Combin.* 49 (2011), 127–151.

[6] C. Eslahchi and M. Johnson, Characterization of graphs with Hall number 2, *J. Graph Theory* 45 (2004), 81–100.

[7] P. Erdős, A. L. Rubin and H. Taylor, Choosability in graphs, *Proc. West Coast Conf. Combinatorics, Graph Theory and Computing, Congr. Numer.* 26 (1979), 125–157.

[8] P. Hall, On Representatives of Subsets, *J. London. Math. Soc.* 10(1) (1935), 26–30.

[9] A. J. W. Hilton and P.xi D. Johnson, Jr., Extending Hall's theorem, *Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel*, Physica-Verlag, Heidelberg, 1990, 359–371.

[10] A. J. W. Hilton, P. D. Johnson, Jr. and E. B. Wantland, The Hall number of a simple graph, *Proc. 27th Southeastern Int. Conf. Combin., Graph Theory and Computing, Congr. Numer.* 121 (1996), 161–182.

[11] S. Holliday, J. Vandenbussche and E. E. Westlund, Completing partial proper colorings using Hall's condition, *Electron. J. Combin.* 22(3) (2015), #P3.6.

[12] S. Holliday, J. Vandenbussche and E. E. Westlund, Every graph G is Hall $\Delta(G)$-extendible, *Electron. J. Combin.* 23(4) (2016), #P4.19.

[13] J. P. Hutchinson and E. H. Moore, Distance constraints in graph color extensions, *J. Combin. Theory Ser. B* 97(4) (2007), 501–517.

[14] P. D. Johnson, Jr., Hall's condition for list-coloring, and the Hall parameters: recent developments, *Discrete Math.* 249 (2002), 135–147.

[15] P. D. Johnson, Jr. and E. B. Wantland, Which graphs have Hall number two? *Ars Combin.* 65 (2002), 39–64.

[16] C. Ojima, A. Saito and K. Sano, Precoloring extension involving pairs of vertices of small distance, *Discr. Applied Math.* 166 (2013), 170–177.

[17] Zs. Tuza, Hall number for list colorings of graphs: extremal results, *Discrete Math.* 310 (2010), 461–470.

[18] D. B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996.

(Received 2 June 2018; revised 30 Dec 2019)