

# Weak Lefschetz property and stellar subdivisions of Gorenstein complexes\*

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## Abstract

Assume  $\sigma$  is a face of a Gorenstein\* simplicial complex  $D$ , and  $k$  is an infinite field. We investigate the question of whether the Weak Lefschetz Property of the Stanley–Reisner ring  $k[D]$  is equivalent to the same property of the Stanley–Reisner ring  $k[D_\sigma]$  of the stellar subdivision  $D_\sigma$ . We prove that this is the case if the dimension of  $\sigma$  is big compared to the codimension.

## 1 Introduction

An important open question in algebraic combinatorics is whether for a simplicial sphere, or more generally for a Gorenstein\* simplicial complex  $D$ , the Stanley–Reisner ring  $k[D]$  of  $D$  over an infinite field  $k$  satisfies the Weak Lefschetz Property (WLP for short). This implies, in particular, that the  $f$ -vector of  $D$  satisfies McMullen’s  $g$ -conjecture. For details see, for example, [25], [26] or [7, Section 5.6]. Actually, Stanley [23] proved, using the theory of toric varieties from algebraic geometry, that in the special case that  $D$  is the boundary complex of a convex simplicial

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\* This work has been supported by the German Research Foundation (DFG) through SPP 1489 and TRR 195. For parts of the project J.B. was supported by DFG through Grant BO3330/1-1. For parts of the project S.P. was financially supported by FCT, Portugal and RIMS, Kyoto University, Japan.

polytope it holds that  $\mathbb{R}[D]$  satisfies the stronger Strong Lefschetz Property (SLP for short).

Babson and Nevo [4] proved that if  $k$  is an infinite field of characteristic 0,  $D$  is a homology sphere with  $k[D]$  satisfying SLP and  $\sigma$  is a face of  $D$  with  $k[L]$  satisfying SLP (where  $L$  denotes the link of  $\sigma$  in  $D$ ), then it follows that  $k[D_\sigma]$  has the SLP, where  $D_\sigma$  denotes the stellar subdivision of  $D$  with respect to  $\sigma$ . We investigate similar questions for  $D$  a Gorenstein\* simplicial complex and SLP replaced by WLP.

The Stanley–Reisner rings  $k[D]$  and  $k[D_\sigma]$  are Gorenstein (see Lemma 3.4). Unfortunately, there is no structure theorem known for Gorenstein rings in codimension  $\geq 4$ . Unprojection theory, which originated in commutative algebra [13] and algebraic geometry [20, 21] aims to (partially) fill this gap. In the present paper we use constructions motivated by the interpretation of stellar subdivision in terms of Kustin–Miller unprojections established in [1].

To state our main results we need the following terminology, which will be explained in more detail in Section 2 and Subsection 3.1. Assume  $k$  is an infinite field,  $D$  is a Gorenstein\* simplicial complex of dimension  $d - 1$  and  $\sigma$  is a  $q$ -face of  $D$ . Denote by  $L$  the link of  $\sigma$  in  $D$ . We define  $p_1$  to be the integral part of  $(d - 1)/2$  and  $p_2$  to be the integral part of  $d/2$ . We say that  $k[L]$  satisfies the  $M_{q,p_1}$  property if there exists a pair  $(B, \omega)$ , where  $B = \bigoplus_{i \geq 0} B_i$  is an Artinian reduction of  $k[L]$  and  $\omega \in B_1$ , with the property that the map  $B_{p_1-q} \rightarrow B_{p_1}$  given by multiplication with  $\omega^q$  is injective. We now state our two main results:

**Theorem 1** *Assume that  $k[D]$  has the WLP. Then  $k[D_\sigma]$  has the WLP if any of the following conditions hold: i)  $q > p_1$ ; ii)  $q = p_1$  and the field  $k$  has characteristic 0 or  $> d - q - 1$ ; iii)  $k[L]$  satisfies the SLP; iv)  $k[L]$  satisfies property  $M_{q,p_1}$ .*

**Theorem 2** *Assume that  $k[D_\sigma]$  has the WLP. If  $q > p_2$ , then  $k[D]$  has the WLP.*

The proof of Theorem 1 is given in Subsection 4.1, while the proof of Theorem 2 is given in Subsection 4.2. An interesting corollary of the above theorems is the following result, which will be proven in Subsection 4.3.

**Corollary 3** *Assume  $k$  is an infinite field,  $D$  is a Gorenstein\* simplicial complex and  $\sigma \in D$  is a face with  $2(\dim \sigma) > \dim D + 1$ . Then the Stanley–Reisner ring  $k[D]$  has the WLP if and only if the Stanley–Reisner ring  $k[D_\sigma]$  of the stellar subdivision  $D_\sigma$  has the WLP.*

If this equivalence could be proven without any assumptions on the dimension of  $\sigma$ , it would then have as corollary that the Stanley–Reisner ring  $k[D]$  has the WLP for all PL-spheres  $D$ . This would imply, in particular, the  $g$ -conjecture for the class of PL-spheres, cf. [4, Remark 1.3.2]. For some further considerations concerning this question, see [2, Section 5]. Also note that a generalization of Corollary 3 has been proven in [19] for the class of homology manifolds.

Section 2 introduces the basic notation. Section 3 studies a number of important intermediate rings between  $k[D]$  and  $k[D_\sigma]$  and their Artinian reductions. Section 4 contains proofs of our main results. In Appendix A we collect a number of general lemmas we use, while Appendix B contains a lemma that states that certain Artinian reductions of a WLP  $k$ -algebra inherit the WLP property.

## 2 Notation

### 2.1 Hilbert functions and Lefschetz properties

In the following  $k$  denotes an infinite field of arbitrary characteristic. All graded  $k$ -algebras will be commutative, Noetherian and of the form  $G = \bigoplus_{i \geq 0} G_i$  with  $G_0 = k$  and  $\dim_k G_i < \infty$  for all  $i$ . The *Hilbert function* of  $G$  is the function  $\text{HF}(G) : \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $m \mapsto \dim_k G_m$ . For  $m \in \mathbb{Z}$  we set  $\text{HF}(m, G) = \dim_k G_m$ .

The  $k$ -algebra  $G$  is called standard graded if it is generated, as a  $k$ -algebra, by  $G_1$ . An element  $a \in G$  is called a linear form if  $a \in G_1$ . For a polynomial ring we use the notions of monomial order, initial term, initial ideal and reverse lexicographic order as defined in [6, Section 15].

Assume  $G$  is a standard graded Cohen–Macaulay  $k$ -algebra with Krull dimension  $\dim G = d$ . In the present paper by Artinian reduction of  $G$  we mean a quotient  $G/(f_1, \dots, f_d)$ , where  $f_1, \dots, f_d \in G_1$  is a  $G$ -regular sequence. Assume  $t \geq 1$ , and  $H$  is a graded  $k$ -subalgebra of  $G$ . Generalizing the discussion in [3, before Theorem 4.2.12], we say that for general  $f_1, \dots, f_t \in H_1$  a property  $\mathcal{P}$  holds if there exists a non-empty Zariski open subset  $U$  of the irreducible affine  $k$ -variety  $(H_1)^t$  such that the property  $\mathcal{P}$  holds for all  $f \in U$ .

We say that an Artinian standard graded algebra  $F$  has the *Weak Lefschetz Property* (WLP for short) if for general  $\omega \in F_1$  and all  $i$  the multiplication by  $\omega$  map  $F_i \rightarrow F_{i+1}$  is of maximal rank, which means that it is injective or surjective (or both). It is well-known (see, for example, [4, Lemma 4.1]) that  $F$  has the WLP if and only if there exists  $a \in F_1$  such that for all  $i$  the multiplication by  $a$  map  $F_i \rightarrow F_{i+1}$  is of maximal rank.

We say that a standard graded  $k$ -algebra  $G$  with  $\dim G \geq 1$  has the WLP if it is Cohen–Macaulay and for general linear forms  $f_1, \dots, f_{\dim G}$  of  $G$  we have that the algebra  $G/(f_1, \dots, f_{\dim G})$ , which is Artinian by Lemma A.2, has the WLP. Good general references for the Weak and Strong Lefschetz Properties are [9, 16]. By [9, Proposition 3.2], if  $F$  is an Artinian standard graded  $k$ -algebra with the WLP it follows that  $\text{HF}(F)$  is unimodal, which means that there is no triple  $j_1 < j_2 < j_3$  such that  $\text{HF}(j_1, F) > \text{HF}(j_2, F)$  and  $\text{HF}(j_3, F) > \text{HF}(j_2, F)$ .

We say that an Artinian standard graded algebra  $F = \bigoplus_{i=0}^r F_i$  with  $F_r \neq 0$  has the *Strong Lefschetz Property* (SLP for short) if  $\dim F_i = \dim F_{r-i}$  for all  $i$  with  $0 \leq i \leq r$  and for a general linear form  $\omega$  of  $F$  and all  $i$  with  $0 \leq 2i \leq r$ , the multiplication by  $\omega^{r-2i}$  map  $F_i \rightarrow F_{r-i}$  is bijective. We say that a standard graded  $k$ -algebra  $G$  with  $\dim G \geq 1$  has the SLP if it is Cohen–Macaulay and for general linear forms  $f_1, \dots, f_{\dim G}$  of  $G$  we have that the algebra  $G/(f_1, \dots, f_{\dim G})$ , which is Artinian by Lemma A.2, has the SLP. If  $J \subset R$  is an ideal, we say that  $J$  has the WLP (resp. the SLP) if  $R/J$  has the WLP (resp. the SLP).

Assume  $a, b$  are integers. We say that a standard graded Cohen–Macaulay  $k$ -algebra  $G$  satisfies the property  $M_{a,b}$  if there exists a pair  $(B, \omega)$ , where  $B$  is an Artinian reduction of  $G$  and  $\omega \in B_1$ , with the property that the map  $B_{b-a} \rightarrow B_b$  given by multiplication with  $\omega^a$  is injective.

For a function  $h : \mathbb{Z} \rightarrow \mathbb{Z}$  we define

$$\Delta(h) : \mathbb{Z} \rightarrow \mathbb{Z}, \quad m \mapsto h(m) - h(m - 1).$$

For  $q > 0$  we inductively define  $\Delta^q(h) : \mathbb{Z} \rightarrow \mathbb{Z}$  by  $\Delta^1(h) = \Delta(h)$  and  $\Delta^q(h) = \Delta^{q-1}(\Delta(h))$  for  $q > 1$ . Assume  $G$  is a standard graded  $k$ -algebra and  $a_1, a_2, \dots, a_p$  is a regular sequence in  $G$  consisting of linear forms, then  $\text{HF}(G/(a_1, \dots, a_p)) = \Delta^p(\text{HF}(G))$ .

We also define

$$\Delta^+(h) : \mathbb{Z} \rightarrow \mathbb{Z}, \quad m \mapsto \max(0, h(m) - h(m - 1)).$$

Assume  $F$  is a standard graded Artinian  $k$ -algebra. Then  $F$  has the WLP if and only if for general  $\omega \in F_1$  we have  $\text{HF}(F/(\omega)) = \Delta^+(\text{HF}(F))$ .

Assume that  $h : \mathbb{Z} \rightarrow \mathbb{Z}$  has the property that there exists  $m_0 \in \mathbb{Z}$  such that  $h(m) = 0$  when  $m < m_0$ . We define

$$\Gamma(h) : \mathbb{Z} \rightarrow \mathbb{Z}, \quad m \mapsto \sum_{i=-\infty}^m h(i).$$

For  $q > 0$  we inductively define  $\Gamma^q(h) : \mathbb{Z} \rightarrow \mathbb{Z}$  by  $\Gamma^1(h) = \Gamma(h)$  and  $\Gamma^q(h) = \Gamma^{q-1}(\Gamma(h))$  for  $q > 1$ . Assume  $G$  is a standard graded  $k$ -algebra and  $T_1, \dots, T_p$  are new variables of degree 1; then  $\text{HF}(G[T_1, \dots, T_p]) = \Gamma^p(\text{HF}(G))$ .

For a graded  $k$ -algebra  $G$  we denote by  $\text{depth } G$  the depth of  $G$ . By [3, Theorem 1.2.8]

$$\text{depth } G = \min\{i : \text{Ext}_G^i(k, G) \neq 0\}, \tag{1}$$

where  $k$  is considered as a  $G$ -module via  $k = G/(\oplus_{i \geq 1} G_i)$ . For an ideal  $I$  of a ring  $R$  and  $u \in R$  we write  $(I : u) = \{r \in R \mid ru \in I\}$  for the ideal quotient.

### 2.2 Simplicial complexes

Assume  $A$  is a finite set. We set  $2^A$  to be the simplex with vertex set  $A$ ; by definition it is the set of all subsets of  $A$ . A simplicial subcomplex  $D \subset 2^A$  is a subset with the property that if  $\tau \in D$  and  $\sigma \subset \tau$  then  $\sigma \in D$ . The elements of  $D$  are also called faces of  $D$ , and the dimension of a face  $\tau$  of  $D$  is one less than the cardinality of  $\tau$ . A facet of  $D$  is a maximal face of  $D$  with respect to (set-theoretic) inclusion. The dimension of  $D$  is the maximum dimension of a facet of  $D$ . We define the support of  $D$  by

$$\text{supp } D = \{i \in A \mid \{i\} \in D\}.$$

We denote by  $R_A$  the polynomial ring  $k[x_a \mid a \in A]$  with the degrees of all variables  $x_a$  equal to 1. For a simplicial subcomplex  $D \subset 2^A$  we define the *Stanley–Reisner ideal*  $I_{D,A} \subset R_A$  to be the ideal generated by the square-free monomials  $\prod_{t=1}^p x_{i_t}$  where  $\{i_1, i_2, \dots, i_p\}$  is not a face of  $D$ . In particular,  $I_{D,A}$  contains linear polynomials if and only if  $\text{supp } D \neq A$ . The *Stanley–Reisner ring*  $k[D, A]$  is defined by  $k[D, A] = R_A/I_{D,A}$ . For a non-empty face  $\sigma$  of  $D$  we set  $x_\sigma = \prod_{i \in \sigma} x_i \in R_A$ . For

a non-empty finite set  $A$ , we set  $\partial A = 2^A \setminus \{A\} \subset 2^A$  to be the boundary complex of the simplex  $2^A$ . In the following, when the set  $A$  is clear we will simplify the notation  $k[D, A]$  to  $k[D]$ .

Assume that, for  $i = 1, 2$ ,  $D_i \subset 2^{A_i}$  is a subcomplex and the finite sets  $A_1, A_2$  are disjoint. By the join  $D_1 * D_2$  of  $D_1$  and  $D_2$  we mean the subcomplex  $D_1 * D_2 \subset 2^{A_1 \cup A_2}$  defined by

$$D_1 * D_2 = \{\alpha_1 \cup \alpha_2 \mid \alpha_1 \in D_1, \alpha_2 \in D_2\}.$$

If  $\sigma$  is a face of  $D \subset 2^A$  we define the link of  $\sigma$  in  $D$  to be the subcomplex

$$\text{lk}_D \sigma = \{\alpha \in D \mid \alpha \cap \sigma = \emptyset \text{ and } \alpha \cup \sigma \in D\} \subset 2^{A \setminus \sigma}.$$

It is clear that the Stanley–Reisner ideal of  $\text{lk}_D \sigma$  is equal to the intersection of the ideal  $(I_{D,A} : x_\sigma)$  with the subring  $R_{A \setminus \sigma}$  of  $R_A$ . In other words, it is the ideal of  $R_{A \setminus \sigma}$  generated by the minimal monomial generating set of  $(I_{D,A} : x_\sigma)$ . Furthermore, we define the (open) star of  $\sigma$  in  $D$  to be the subset

$$\text{star}_D \sigma = \{\alpha \in D \mid \alpha \cup \sigma \in D\} \subset 2^A.$$

If  $\sigma$  is a non-empty face of  $D \subset 2^A$  and  $j \notin A$ , we define the *stellar subdivision*  $D_\sigma$  with new vertex  $j$  to be the subcomplex

$$D_\sigma = (D \setminus \text{star}_D \sigma) \cup (2^{\{j\}} * \text{lk}_D \sigma * \partial \sigma) \subset 2^{A \cup \{j\}}.$$

Following [25, p. 67], we say that a subcomplex  $D \subset 2^A$  is *Gorenstein\* over  $k$*  if  $A = \text{supp } D$ ,  $k[D]$  is Gorenstein, and for every  $i \in A$  there exists  $\sigma \in D$  with  $\sigma \cup \{i\}$  not a face of  $D$ . The last condition combinatorially means that  $D$  is not a join of the form  $2^{\{i\}} * E$ , and algebraically that  $x_i$  divides at least one element of the minimal monomial generating set of  $I_{D,A}$ .

Assume  $D \subset 2^A$  is a Gorenstein\* simplicial complex and  $\sigma$  is a face of  $D$ . Set  $L = \text{lk}_D \sigma$ . It is well known (cf. [25, Section II.5]) that the subcomplex  $L \subset 2^{\text{supp } L}$  is Gorenstein\* with  $\dim L = \dim D - \dim \sigma - 1$ .

### 3 Weak Lefschetz Property and stellar subdivisions

In this section we define intermediate rings relating the Stanley–Reisner rings  $k[D]$  and  $k[D_\sigma]$  of a simplicial complex  $D$  and its stellar subdivision  $D_\sigma$ . We describe how the Weak Lefschetz Property behaves under certain Artinian reductions of these rings. We also study the reductions with regard to the Weak Lefschetz Property.

#### 3.1 The main players

We fix an infinite field  $k$  of arbitrary characteristic and a pair  $(D, \sigma)$ , where  $D$  is a Gorenstein\* simplicial complex with vertex set  $\{1, \dots, n\}$ , and  $\sigma = \{1, 2, \dots, q+1\}$  is a  $q$ -face of  $D$  with  $q \geq 1$ . We set  $d = \dim D + 1$ ,  $R = k[x_1, \dots, x_n]$  with the degrees of all variables equal to 1,  $x_\sigma = \prod_{i=1}^{q+1} x_i \in R$ . By  $I \subset R$  we denote the Stanley–Reisner

ideal of  $D$ , hence  $k[D] = R/I$  and  $\dim R/I = d$ . We set  $I_L = (I : x_\sigma) \subset R$ . We denote by  $f_1, \dots, f_d$  a sequence of  $d$  linear forms of  $R$ .

Moreover, we denote by  $T$  a new variable of degree 1 and  $J_{st} = (I, x_\sigma, TI_L) \subset R[T]$  denotes the Stanley–Reisner ideal of the stellar subdivision  $D_\sigma$ , hence  $k[D_\sigma] = R[T]/J_{st}$ . We set

$$\begin{aligned} I_C &= (I, T^{q+1}, TI_L) \subset R[T], & I_G &= (I, T^{q+1} - x_\sigma, TI_L) \subset R[T], \\ A &= R/(I + (f_1, \dots, f_d)), & B &= R/(I_L + (f_1, \dots, f_d)), \\ C &= R[T]/(I_C + (f_1, \dots, f_d)), & G &= R[T]/(I_G + (f_1, \dots, f_d)). \end{aligned}$$

The rings  $R[T]/I_C$  and  $R[T]/I_G$  are closely related to the Kustin–Miller unprojection ring  $S$  appearing in [1, Theorem 1.1] and we view them as intermediate rings connecting  $k[D_\sigma]$  and  $k[D]$ .

We will see that for general  $f_1, \dots, f_d \in R_1$ , the rings  $A, B, C$  and  $G$  are Artinian reductions of  $k[D], R/I_L, R[T]/I_C$  and  $R[T]/I_G$  respectively. Note that the linear forms  $f_1, \dots, f_d$  in the reductions only involve the variables  $x_i$ , that is, they do not involve the variable  $T$ . The basic properties of  $A, B$  and  $R/I_L$  are contained in Lemma 3.4, of  $C$  are contained in Lemma 3.5, of  $R[T]/I_G$  and  $G$  are contained in Lemma 3.8 and of  $R[T]/I_C$  are contained in Lemma 3.9.

We denote by  $L \subset 2^{\{q+2, q+3, \dots, n\}}$  the link of  $\sigma$  in  $D$  and set  $k[L] = k[L, \{q + 2, q + 3, \dots, n\}]$ . Using that

$$R/I_L \cong (k[L])[x_1, \dots, x_{q+1}] \tag{2}$$

it follows that for general  $f_1, \dots, f_d \in R_1$  the  $k$ -algebra  $B$  is isomorphic to an Artinian reduction of  $k[L]$ .

**Remark 3.1** We will use the well-known fact, see for example [14, Proposition 3.3] or [9, Theorem 2.79], that if  $F = \bigoplus_{i=0}^r F_i$  with  $F_r \neq 0$  is a standard graded Gorenstein Artinian  $k$ -algebra, then  $F_r$  is 1-dimensional, and for all  $i$  with  $0 \leq i \leq r$  the multiplication map  $F_i \times F_{r-i} \rightarrow F_r \cong k$  is a perfect pairing. We will refer to  $F_{r-i}$  as the Poincaré dual of  $F_i$ . As a consequence, given  $i, j$  with  $0 \leq i \leq j \leq r$  and  $0 \neq e \in F_i$  there exists  $e' \in F_{j-i}$  such that  $ee' \neq 0$  in  $F_j$ .

Recall from the Introduction that if  $d$  is even we have  $p_1 = d/2 - 1, p_2 = d/2$ , while if  $d$  is odd we have  $p_1 = p_2 = (d - 1)/2$ .

**Remark 3.2** For general  $f_1, \dots, f_d \in R_1$  the following hold. Since, by Lemma 3.4,  $B$  is Artinian Gorenstein with  $B_i = 0$  for  $i \geq d - q$  and  $B_{d-q-1} \neq 0$ , by Gorenstein duality  $k[L]$  has the property  $M_{q,p_1}$  if and only if for a general element  $\omega \in B_1$  the multiplication map by  $\omega^q : B_{d-q-1-p_1} \rightarrow B_{d-q-1-(p_1-q)}$  is surjective. If  $d$  is even, then  $d = 2p_1 + 2$ , hence  $d - q - 1 - p_1 = p_1 + 1 - q = p_2 - q$ , while, if  $d$  is odd, then  $d = 2p_1 + 1$ , hence  $d - q - 1 - p_1 = p_2 - q$ . Hence no matter if  $d$  is even or odd,  $k[L]$  has the property  $M_{q,p_1}$  if and only if for a general element  $\omega \in B_1$  the multiplication by  $\omega^q$  map  $B_{p_2-q} \rightarrow B_{p_2}$  is surjective.

**Remark 3.3** For general  $f_1, \dots, f_d \in R_1$  the following hold. Assume  $d$  is odd. Since  $d = 2p_1 + 1$ , we have  $d - q - 1 - (p_1 - q) = d - p_1 - 1 = p_1$ , hence the multiplication by  $\omega^q$  map in the definition of property  $M_{q,p_1}$  is between Poincaré dual graded components of  $B$ . Assume  $d$  is even. We have  $d = 2p_1 + 2$  and  $d - q - 1 - (p_1 - q) = d - p_1 - 1 = p_1 + 1$ , hence the multiplication by  $\omega^q$  map in the definition of the property  $M_{q,p_1}$  factors as  $f \circ g$ , where  $g$  is the multiplication by  $\omega$  map  $B_{p_1 - q} \rightarrow B_{p_1 - q + 1}$  and  $f$  is multiplication by  $\omega^{q-1}$  map between the Poincaré duals  $B_{p_1 - q + 1}$  and  $B_{p_1}$ . As a consequence, no matter if  $d$  is even or odd, if  $B$  has the SLP, then it follows that property  $M_{q,p_1}$  holds for  $B$ .

### 3.2 Weak Lefschetz Property under Artinian reduction

In this subsection, we prove that our assumptions  $D$  Gorenstein\* and  $k$  infinite imply that the rings  $k[D]$ ,  $R/I_L$ ,  $R[T]/I_G$  are Gorenstein, and the ring  $R[T]/I_C$  is Cohen–Macaulay. Consequently, for general  $f_1, \dots, f_d \in R_1$  the corresponding Artinian reductions  $A$ ,  $B$ ,  $G$  are Gorenstein. We also derive relations between the Hilbert functions of these rings. We prove that  $k[D]$  (respectively  $R/I_L$ ,  $R[T]/I_C$ ,  $R[T]/I_G$ ) has the Weak Lefschetz Property if and only if for general  $f_1, \dots, f_d \in R_1$  the  $k$ -algebra  $A$  (respectively  $B$ ,  $C$ ,  $G$ ) does.

**Lemma 3.4** *For general  $f_1, \dots, f_d \in R_1$  the following hold. The rings  $k[D]$ ,  $k[D_\sigma]$ ,  $A$  and  $B$  are Gorenstein. We have  $A = \bigoplus_{i=0}^d A_i$  with  $A_d$  1-dimensional. The ring  $R/I_L$  is Gorenstein with  $\dim R/I_L = d$ . We have  $B = \bigoplus_{i=0}^{d-q-1} B_i$  and  $B_{d-q-1}$  is 1-dimensional. Moreover, for all  $m \geq 0$  we have*

$$\text{HF}(m, k[D_\sigma]) = \text{HF}(m, k[D]) + \sum_{i=1}^q \text{HF}(m - i, R/I_L).$$

**Proof:** Since, by assumption,  $D$  is Gorenstein\*, it follows that  $k[D]$  is Gorenstein. By definition,  $A$  is an Artinian reduction of  $k[D]$ , hence it is also Gorenstein. Since  $\dim k[D] = d$ , by Lemma A.6, we get  $A = \bigoplus_{i=0}^d A_i$  with  $A_d$  1-dimensional.

By [24, p. 188], Gorenstein\* is a topological property. Hence a stellar subdivision of a Gorenstein\* simplicial complex is again Gorenstein\*. As a consequence  $k[D_\sigma]$  is Gorenstein.

Recall that  $L$  is Gorenstein\*, with  $\dim k[L] = d - q - 1$ , hence by Equation (2),  $R/I_L$  is Gorenstein of dimension  $d$ . Since  $B$  is isomorphic to an Artinian reduction of  $k[L]$ , it follows that  $B$  is Gorenstein. Moreover, by Lemma A.6,  $B = \bigoplus_{i=0}^{d-q-1} B_i$  and  $B_{d-q-1}$  is 1-dimensional.

The equation between the Hilbert functions follows from [1, Remark 5]. □

**Lemma 3.5** *(Recall  $\sigma$  is a  $q$ -face, with  $q \geq 1$ ) For general  $f_1, \dots, f_d \in R_1$  the following hold. There is a well-defined bijective  $k$ -linear map  $\phi$  of vector spaces*

$$\phi : A \oplus B^q \rightarrow C, \quad ([a], [b_1], \dots, [b_q]) \mapsto [a + \sum_{i=1}^q b_i T^i]$$

for  $a, b_i \in R$ . As a corollary,  $C$  is Artinian and for all  $m \geq 0$

$$\text{HF}(m, C) = \text{HF}(m, A) + \sum_{i=1}^q \text{HF}(m - i, B). \tag{3}$$

Hence,  $\text{HF}(C)$  is equal to the Hilbert function of an Artinian reduction of  $k[D_\sigma]$ . In particular,  $C_d$  is 1-dimensional,  $C_i = 0$  for  $i \geq d + 1$  and  $\dim C_{d-i} = \dim C_i$  for all  $0 \leq i \leq d$ .

**Proof:**  $\phi$  is well-defined: Assume  $a, a', b_i, b'_i \in R$  with  $[a] = [a']$  in  $A$  and  $[b_i] = [b'_i]$  in  $B$  for all  $i$ . Then  $a - a' \in (I, f_1, \dots, f_d)$  and  $b_i - b'_i \in (I_L, f_1, \dots, f_d)$ , hence  $T(b_i - b'_i) \in (TI_L, f_1, \dots, f_d)$ . As a consequence

$$\left[ a + \sum_{i=1}^q b_i T^i \right] = \left[ a' + \sum_{i=1}^q b'_i T^i \right]$$

in  $C$ .

$\phi$  is surjective: Obvious from the definitions of  $\phi$  and  $C$ .

$\phi$  is injective: Assume  $a, b_i \in R$  with  $[a + \sum_{i=1}^q b_i T^i] = 0$  in  $C$ . This implies that there exist  $e_{a,1}, \dots, e_{a,r_1} \in I, e_{b,1}, \dots, e_{b,r_2} \in I_L, g_{a,1}, \dots, g_{a,r_1} \in R[T], g_{b,1}, \dots, g_{b,r_2} \in R[T], g_c \in R[T], g_{e,1}, \dots, g_{e,d} \in R[T]$  such that

$$a + \sum_{i=1}^q b_i T^i = \sum_{j=1}^{r_1} g_{a,j} e_{a,j} + T \sum_{j=1}^{r_2} g_{b,j} e_{b,j} + g_c T^{q+1} + \sum_{j=1}^d g_{e,j} f_j$$

with equality in  $R[T]$ . Looking at the coefficients of the powers of  $T$  we get  $a \in (I, f_1, \dots, f_d)$  and  $b_i \in (I_L, f_1, \dots, f_d)$  for all  $1 \leq i \leq q$ . Hence  $\phi$  is injective.

Since  $A, B$  are Artinian, they are finite dimensional  $k$ -vector spaces. Since  $\phi$  is surjective  $C$  is finite dimensional as a  $k$ -vector space which implies that  $C$  is Artinian.

Equation 3 immediately follows from the bijectivity of  $\phi$ .

Using [1, Remark 5] and Lemma 3.4, it follows that  $\text{HF}(C)$  is equal to the Hilbert function of an Artinian reduction of  $k[D_\sigma]$ . As a consequence, the statements  $C_d$  is 1-dimensional,  $C_i = 0$  for  $i \geq d + 1$  and  $\dim C_{d-i} = \dim C_i$  for all  $0 \leq i \leq d$ , follow from Lemma A.6 applied to the Gorenstein\* simplicial complex  $D_\sigma$ . □

**Remark 3.6** For general  $f_1, \dots, f_d \in R_1$  the following hold. Taking graded components, Lemma 3.5 immediately implies that, for  $i \geq 0$ , there exists a  $k$ -vector space decomposition

$$C_i = A_i \oplus \left( \bigoplus_{j=1}^q T^j B_{i-j} \right) \tag{4}$$

Hence, if  $c \in C_i$ , there exist unique  $a \in A_i$  and  $b_j \in B_j$  such that

$$c = a + b_{i-1}T + b_{i-2}T^2 + \dots + b_{i-q}T^q.$$

**Corollary 3.7** *For general  $f_1, \dots, f_d \in R_1$  and for general  $h_1, \dots, h_d \in (R[T])_1$  the following hold. Assume that the Hilbert function of a general linear section of  $C$  is equal to the Hilbert function of a general linear section of  $k[D_\sigma]/(h_1, \dots, h_d)$ . Then  $C$  has the WLP if and only if  $k[D_\sigma]$  has the WLP.*

**Proof:** By Lemma 3.5,  $\text{HF}(C)$  is equal to the Hilbert function of an Artinian reduction of  $k[D_\sigma]$ . Hence  $\Delta^+(\text{HF}(C)) = \Delta^+(\text{HF}(k[D_\sigma]/(h_1, \dots, h_d)))$ , and the result follows. □

Recall  $I_G = (I, T^{q+1} - x_\sigma, TI_L)$  and  $G = R[T]/(I_G + (f_1, \dots, f_d))$ .

**Lemma 3.8** *For general  $f_1, \dots, f_d \in R_1$  the following hold.*

- i) The  $k$ -algebra  $R[T]/I_G$  is Gorenstein,  $\dim R[T]/I_G = d$  and  $\text{HF}(R[T]/I_G) = \text{HF}(k[D_\sigma])$ .*
- ii) The ring  $G$  is Artinian Gorenstein, and  $\text{HF}(G) = \text{HF}(C)$ , which, by Lemma 3.5, is equal to the Hilbert Function of an Artinian reduction of  $k[D_\sigma]$ .*
- iii) Assume  $k[D_\sigma]$  has the WLP. Then both  $R[T]/I_G$  and  $G$  have the WLP.*

**Proof:** Let  $z, z'$  be two new variables and  $c \in k$ . We set  $I_V = (I, TI_L, Tz - x_\sigma) \subset R[T, z]$  where  $\deg T = 1, \deg z = q$ . We set  $\mathcal{M} = R[T, z, z']/(I_V)$ , where  $(I_V)$  is the ideal of  $R[T, z, z']$  generated by  $I_V$  and  $\deg T = \deg z' = 1, \deg z = q$ . We also set  $\mathcal{Q} = \mathcal{M}/(z - T^{q-1}z')$  and

$$H_c = \mathcal{Q}/(z' - c^{q+1}T) \cong R[T]/(I, TI_L, c^{q+1}T^{q+1} - x_\sigma).$$

By [1, Theorem 1.1],  $R[T, z]/I_V$  is Gorenstein and  $\dim R[T, z]/I_V = d + 1$ . It follows that  $\mathcal{M}$  is Gorenstein and  $\dim \mathcal{M} = d + 2$ . Hence  $\dim \mathcal{Q} \geq d + 1$ . Since  $\mathcal{Q}/(z') = k[D_\sigma]$  which has dimension  $d$ , it follows that  $\dim \mathcal{Q} \leq d + 1$ . Hence  $\dim \mathcal{Q} = d + 1$ . Using that  $\mathcal{M}$  is Gorenstein, hence Cohen–Macaulay, it follows that  $z - T^{q-1}z'$  is an  $\mathcal{M}$ -regular element, hence  $\mathcal{Q}$  is Gorenstein.

Clearly

$$\mathcal{Q} = R[T, z']/(I, TI_L, T^q z' - x_\sigma),$$

hence  $\mathcal{Q}$  is standard graded. We have  $H_0 = k[D_\sigma]$ , hence  $\dim \mathcal{Q}/(z') = \dim \mathcal{Q} - 1$ . Since  $\mathcal{Q}$  is Gorenstein, hence Cohen–Macaulay, it follows that  $z'$  is a  $\mathcal{Q}$ -regular element.

Hence Lemma A.4 implies that for general  $c \in k$  we have that  $z' - c^{q+1}T$  is a  $\mathcal{Q}$ -regular element, since the property is true for  $c = 0$ . As a consequence, for general  $c \in k$  the ring  $H_c$  is Gorenstein of dimension  $d$  and  $\text{HF}(H_c) = \text{HF}(k[D_\sigma])$ . For nonzero  $c$ , using the linear change of coordinates  $T \mapsto cT$ , we have  $H_c \cong H_1 \cong R[T]/I_G$ . It follows that  $R[T]/I_G$  is Gorenstein,  $\dim R[T]/I_G = d$  and  $\text{HF}(R[T]/I_G) = \text{HF}(k[D_\sigma])$ .

We now prove ii) We first prove that  $G$  is Artinian. The arguments used in the proof of Lemma 3.5 also give that there exists a well-defined surjective  $k$ -linear map of vector spaces

$$\psi : A \oplus B^q \rightarrow G, \quad ([a], [b_1], \dots, [b_q]) \mapsto [a + \sum_{i=1}^q b_i T^i]$$

for  $a, b_i \in R$ , hence  $G$  is Artinian. Since we proved that  $R[T]/I_G$  is Gorenstein, hence Cohen–Macaulay, and of dimension  $d$ , and since  $G$  is Artinian it follows that  $f_1, \dots, f_d$  is a regular sequence for  $R[T]/I_G$ . Since  $\text{HF}(R[T]/I_G) = \text{HF}(k[D_\sigma])$ , it follows that

$$\text{HF}(G) = \Delta^d(\text{HF}(R[T]/I_G)) = \Delta^d(\text{HF}(k[D_\sigma])) = \text{HF}(C)$$

with the last equality by Lemma 3.5. Hence  $\dim_k G = \dim_k C$ . By Lemma 3.5,  $\dim_k C = (\dim_k A) + q(\dim_k B)$ , hence  $\dim_k G = (\dim_k A) + q(\dim_k B)$ . Since  $\psi$  is surjective it follows that  $\psi$  is bijective.

We now prove iii). Assume  $k[D_\sigma]$  has the WLP. By Lemma A.4, we have that for general  $c \in k$  the algebra  $H_c$  has the WLP, since the property is true for  $c = 0$ . For nonzero  $c$ , using the linear change of coordinates  $T \mapsto cT$ , we have  $H_c \cong H_1 \cong R[T]/I_G$ . Hence  $R[T]/I_G$  has the WLP. Since, by ii),  $G$  is Artinian, Lemma B.1 implies that  $G$  has the WLP. □

**Lemma 3.9** *The ring  $R[T]/I_C$  is Cohen–Macaulay with the properties  $\dim R[T]/I_C = d$  and  $\text{HF}(R[T]/I_C) = \text{HF}(k[D_\sigma])$ . Moreover,  $I_C$  is the initial ideal of  $I_G$  with respect to any monomial order in  $R[T]$  such that  $T > x_i$  for all  $1 \leq i \leq n$ .*

For the proof of Lemma 3.9 we need the following proposition.

**Proposition 3.10** *Set  $\mathcal{H} = k[x_1, \dots, x_n, T, z]/(I, Tz, TI_L)$ . Then  $\mathcal{H}$  is Cohen–Macaulay and  $\dim \mathcal{H} = d + 1$ .*

**Proof:** Recall that  $*$  denotes the join of simplicial complexes and for a finite set  $S$  we denote by  $2^S$  the simplex with vertex set  $S$ . We set  $\deg x_i = \deg T = \deg z = 1$ . By definition,  $\mathcal{H}$  is the quotient of the polynomial ring  $k[x_1, \dots, x_n, T, z]$  by a square-free monomial ideal. We denote by  $D_{\mathcal{H}}$  the simplicial complex on the vertex set  $\{1, 2, \dots, n, T, z\}$  that corresponds to the monomial ideal. The set of facets of  $D_{\mathcal{H}}$  is equal to the union

$$\{\{z, u\} : u \text{ facet of } D\} \cup \{\{T, 1, 2, \dots, q + 1, w\} : w \text{ facet of } L\}.$$

As a consequence, we have the following decomposition

$$D_{\mathcal{H}} = E_1 \cup E_2$$

where  $E_1 = 2^{\{z\}} * D$ ,  $E_2 = 2^{\{T, 1, 2, \dots, q+1\}} * L$ .

Since  $D$  is Cohen–Macaulay over  $k$  with dimension  $d - 1$  we have that  $E_1$  is Cohen–Macaulay over  $k$  with dimension equal to  $d$ . Since  $L$  is Cohen–Macaulay over  $k$  with dimension  $d - q - 2$  we have that  $E_2$  is Cohen–Macaulay over  $k$  with dimension equal to  $d$ .

Moreover,  $E_1 \cap E_2 = 2^{\{1, 2, \dots, q+1\}} * L$  is also Cohen–Macaulay over  $k$  with dimension equal to  $d - 1$ . Hence, using [10, Lemma 23.6], it follows that  $D_{\mathcal{H}}$  is Cohen–Macaulay over  $k$  with dimension  $d$ . Hence  $\mathcal{H}$  is a Cohen–Macaulay ring with dimension equal to  $d + 1$ . □

We now prove Lemma 3.9:

**Proof:** We denote by  $\mathcal{H}_a$  the ring  $\mathcal{H}$  but with  $\deg x_i = \deg T = 1$  and  $\deg z = q$ . Since the dimension and the Cohen–Macaulay property is independent of the grading, Proposition 3.10 implies that  $\mathcal{H}_a$  is Cohen–Macaulay and  $\dim \mathcal{H}_a = d + 1$ .

The element  $z - T^q \in \mathcal{H}_a$  is homogeneous and  $\mathcal{H}_a/(z - T^q) \cong R[T]/I_C$  as graded  $k$ -algebras. Hence  $\dim R[T]/I_C \geq d + 1 - 1 = d$ . By Lemma 3.5,  $C$  is Artinian. Since  $C = R[T]/(I_C, f_1, \dots, f_d)$  it follows that  $\dim R[T]/I_C \leq d$ . As a consequence  $\dim R[T]/I_C = d$ . Since  $z - T^q$  is homogeneous,  $\mathcal{H}_a$  is Cohen–Macaulay and  $\dim \mathcal{H}_a/(z - T^q) = \dim \mathcal{H}_a - 1$ , it follows that  $z - T^q$  is  $\mathcal{H}_a$ -regular. Hence  $R[T]/I_C$  is Cohen–Macaulay. As a consequence,  $f_1, \dots, f_d$  is an  $R[T]/I_C$ -regular sequence. Hence  $\text{HF}(R[T]/I_C) = \Gamma^d(\text{HF}(C))$ .

Since  $I_G = (I, TI_L, T^{q+1} - x_\sigma)$ , it is clear that  $I_C$  is a subset of the initial ideal of the  $I_G$  with respect to any monomial order in  $R[T]$  such that  $T > x_i$  for all  $1 \leq i \leq n$ . Since, by Lemma 3.5,  $\text{HF}(C)$  is equal to the Hilbert function of an Artinian reduction of  $k[D_\sigma]$  and since, by Lemma 3.4,  $k[D_\sigma]$  is Gorenstein of dimension  $d$ , it follows that

$$\text{HF}(R[T]/I_C) = \Gamma^d(\text{HF}(C)) = \text{HF}(k[D_\sigma]) = \text{HF}(R[T]/I_G),$$

with the last equality by Lemma 3.8. Consequently,  $I_C$  is the initial ideal of the  $I_G$ .  $\square$

**Remark 3.11** In general, the ring  $R[T]/I_C$  is not Gorenstein. For example, assume  $D$  is the boundary complex of the 3-gon and  $\sigma = \{1, 2\} \in D$ . We have that  $I_C = (x_1x_2x_3, Tx_3, T^2) \subset k[x_1, x_2, x_3, T]$  which is a codimension 2 ideal that needs 3 generators. Hence  $R[T]/I_C$  is not Gorenstein by [6, Corollary 21.20].

**Lemma 3.12** *For general  $f_1, \dots, f_d \in R_1$  the following hold. The  $k$ -algebra  $k[D]$  (respectively  $k[L]$ ,  $R[T]/I_C$ ) has the WLP if and only if  $A$  (respectively  $B$ ,  $C$ ) has the WLP.*

**Proof:** Since  $f_1, \dots, f_d \in R_1$  are general it is immediate that  $A$  has the WLP if and only if  $k[D]$  has the WLP. Since  $B$  is an Artinian reduction of the Gorenstein  $k[L]$  and  $f_1, \dots, f_d \in R_1$  are general it is immediate that  $B$  has the WLP if and only if  $k[L]$  has the WLP.

By Lemma 3.9,  $R[T]/I_C$  is Cohen–Macaulay of dimension  $d$ . Since, by Lemma 3.5,  $C$  is Artinian, it follows that  $f_1, \dots, f_d$  is an  $R[T]/I_C$ -regular sequence. Hence the WLP holds for  $C$  implies the WLP holds for  $R[T]/I_C$ . Conversely, assume that  $R[T]/I_C$  has the WLP. The result that  $C$  has the WLP follows from Lemma B.1.  $\square$

### 3.3 Weak Lefschetz Property relations between the Artinian reductions

The main result of this subsection is that for general  $f_1, \dots, f_d \in R_1$  the  $k$ -algebra  $C$  has the WLP if and only if  $A$  has the WLP and property  $M_{q,p_1}$  holds for  $B$ .

The following lemma will be used in the proof of Lemma 3.15. Note that, since, by Remark 3.11,  $C$  is not always Gorenstein, the lemma does not follow from Remark 3.1.

**Lemma 3.13** *For general  $f_1, \dots, f_d \in R_1$  the following hold. Assume  $1 \leq i < p_1$  and  $0 \neq c \in C_i$ . Then there exists  $c' \in C_{p_1-i}$  such that  $cc' \neq 0$  in  $C_{p_1}$ . As a corollary, assume  $\omega \in C_1$  is any element. If the multiplication by  $\omega$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective, it follows that for all  $r$  with  $1 \leq r \leq p_1$  the multiplication by  $\omega$  map  $C_r \rightarrow C_{r+1}$  is injective.*

**Proof:** Using Remark 3.6, write

$$c = a_i + \sum_{j=1}^q b_{i-j}T^j$$

(equality in  $C$ ) with  $a_i \in A_i$  and  $b_{i-j} \in B_{i-j}$  for all  $1 \leq j \leq q$ . Since  $c \neq 0$ , we have  $a_i \neq 0$  or  $b_{i-e} \neq 0$  for some  $e$  with  $1 \leq e \leq \min\{i, q\}$ .

Assume first that  $a_i \neq 0$ . By Lemma 3.4,  $A$  is Artinian Gorenstein with  $A_d \neq 0$ . Hence, by Remark 3.1, there exists  $a \in A_{p_1-i}$  such that  $a_i a \neq 0 \in A_{p_1}$ . Hence  $cc' \neq 0$  in  $C_{p_1}$ , where  $c' = a$ .

For the rest of the argument we assume that there exists  $e$  with  $1 \leq e \leq \min\{i, q\}$  such that  $b_{i-e} \neq 0$  in  $B$ . We have two cases:

First Case: We assume  $p_1 \leq d - q - 1$ . By Lemma 3.4,  $B$  is Artinian Gorenstein with  $B_{d-q-1} \neq 0$ . By Remark 3.1, there exists  $b \in R_{p_1-i}$  such that  $b_{i-e} b \neq 0$  in  $B_{p_1-e}$ . Hence  $cc' \neq 0$  in  $C_{p_1}$ , where  $c' = b$ .

Second Case: We assume  $d - q - 1 < p_1$ . Hence  $d - p_1 - 1 < q$ . If  $d$  is even, since  $d = 2p_1 + 2$  we get  $p_1 + 1 < q$ . If  $d$  is odd, since  $d = 2p_1 + 1$  we get  $p_1 < q$ . Hence no matter if  $d$  is even or odd we have  $p_1 < q$ . Since  $e \leq i$  and  $i < p_1$  we get  $0 < p_1 - i < q$  and  $0 < p_1 + e - i < q$ . As a consequence  $0 \neq b_{i-e} T^e T^{p_1-i}$  in  $C_{p_1}$ . Hence  $cc' \neq 0$  in  $C_{p_1}$ , where  $c' = T^{p_1-i}$ .

We now prove the corollary. We assume  $1 \leq r < p_1$ , that the multiplication by  $\omega$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective and that the multiplication by  $\omega$  map  $C_r \rightarrow C_{r+1}$  is not injective and we will get a contradiction. By the assumptions, there exists  $0 \neq c \in C_r$  such that  $\omega c = 0$  in  $C_{r+1}$ . By the first part of the present lemma, there exists  $c' \in C_{p_1-r}$  such that  $cc' \neq 0$  in  $C_{p_1}$ . Hence by the assumptions,  $\omega cc' \neq 0$  in  $C_{p_1+1}$ , which contradicts  $\omega c = 0$  in  $C_{r+1}$ . □

The ring  $C$  is not always Gorenstein, hence we cannot use Lemma A.3. The following lemma is a substitute.

**Lemma 3.14** *For general  $f_1, \dots, f_d \in R_1$  the following are equivalent:*

- i)  $C$  has the WLP.*
- ii) For general  $\omega \in R_1$  the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective and the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is surjective.*

**Proof:** Assume that i) holds. Since for nonzero  $c \in k$  the map  $C \rightarrow C$ , with  $x_i \mapsto x_i$  and  $T \mapsto cT$  is well-defined and an automorphism, it follows that for general  $\omega \in R_1$  we have that  $\omega + T$  is a general element of  $C_1$ . Since  $C$  is assumed to have the WLP

to prove ii) it is enough to prove  $\dim C_{p_1} \leq \dim C_{p_1+1}$  and  $\dim C_{p_2+1} \leq \dim C_{p_2}$ . We assume it is not the case and we will get a contradiction.

By [9, Proposition 3.2],  $\text{HF}(C)$  is unimodal. By Lemma 3.5,  $\dim C_i = \dim C_{d-i}$  for all  $i \in \mathbb{Z}$ . First we assume that  $\dim C_{p_1} > \dim C_{p_1+1}$ . If  $d$  is odd, then  $\dim C_{p_1} = \dim C_{p_1+1}$  which is a contradiction. Hence  $d$  is even. Since  $d - p_1 = p_1 + 2$  it follows that  $\dim C_{p_1+2} = \dim C_{p_1} > \dim C_{p_1+1}$ , which contradicts the unimodality of  $\text{HF}(C)$ .

We now assume that  $\dim C_{p_2+1} > \dim C_{p_2}$ . If  $d$  is odd, then  $d - p_2 = p_2 + 1$ , and hence  $\dim C_{p_2} = \dim C_{p_2+1}$ , which is a contradiction. If  $d$  is even, then  $p_2 = p_1 + 1$  and  $d - (p_2 + 1) = p_1$ . Hence  $\dim C_{p_1} > \dim C_{p_1+1}$  which we proved above that cannot happen.

Conversely assume that ii) holds. Then, by Lemma 3.13, for all  $0 \leq i \leq p_1$  the multiplication by  $\omega + T$  map  $C_i \rightarrow C_{i+1}$  is injective. Since  $C$  is standard graded, the assumption that the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is surjective implies that for all  $j \geq p_2$  the multiplication by  $\omega + T$  map  $C_j \rightarrow C_{j+1}$  is surjective.

Indeed, assume  $h \in C_{j+1}$ . Since  $C$  is standard graded, there exist  $N > 0$  and, for  $1 \leq t \leq N$ ,  $a_t \in C_{p_2+1}, b_t \in C_{j-p_2}$  such that  $h = \sum_{t=1}^N (a_t b_t)$ . Hence, for  $1 \leq t \leq N$ , there exists  $e_t \in C_{p_2}$  such that  $a_t = (\omega + T)e_t$ . Consequently,  $h = (\omega + T) \sum_{t=1}^N (b_t e_t)$ . For an alternative argument, see [15, Proposition 2.1].

Since  $p_2 = p_1$  if  $d$  is odd, and  $p_2 = p_1 + 1$  if  $d$  is even, we get that for all  $j \in \mathbb{Z}$  multiplication by  $\omega + T$  as a map  $C_j \rightarrow C_{j+1}$  is injective or surjective (or both), hence  $C$  has the WLP. □

**Lemma 3.15** *For general  $f_1, \dots, f_d \in R_1$  the following are equivalent:*

- i)  $C$  has the WLP.
- ii)  $A$  has the WLP and property  $M_{q,p_1}$  holds for  $B$ .

**Proof:** By Lemma 3.4,  $A = \bigoplus_{i=0}^d A_i$  with  $A_d$  1-dimensional and  $B = \bigoplus_{i=0}^{d-q-1} B_i$  with  $B_{d-q-1}$  1-dimensional. By Lemma 3.5,  $C_d$  is 1-dimensional,  $C_i = 0$  for  $i \geq d + 1$  and  $\dim C_{d-i} = \dim C_i$  for all  $0 \leq i \leq d$ .

Let  $\omega \in R_1$  be a general linear form. By Lemma 3.14,  $C$  has the WLP if and only if the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective and the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is surjective.

We assume that  $A$  has the WLP and  $B$  satisfies property  $M_{q,p_1}$  and we will show that  $C$  has the WLP. For that we first show that the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective. Assume it is not. Then there exists  $0 \neq c \in C_{p_1}$  such that

$$(\omega + T)c = 0 \tag{5}$$

in  $C_{p_1+1}$ . By Equation (4), there exist (unique)  $a_{p_1} \in A_{p_1}$  and, for  $1 \leq j \leq q$ ,  $b_{p_1-j} \in B_{p_1-j}$  such that

$$c = a_{p_1} + \sum_{j=1}^q b_{p_1-j} T^j.$$

Since  $A$  is assumed to have the WLP, if  $a_{p_1} \neq 0$ , we have  $\omega a_{p_1} \neq 0$  which implies  $(\omega + T)c \neq 0$ , which contradicts Equation (5). Hence we have  $a_{p_1} = 0$  in  $A$ . Equation (5) then implies that for  $j = 1, 2, \dots, q - 1$

$$\omega b_{p_1-1} = 0, \quad b_{p_1-j} + \omega b_{p_1-(j+1)} = 0,$$

with all equations in  $B$ . Combining these equations we get  $\omega^q b_{p_1-q} = 0$  in  $B$ . Using the assumption that  $B$  satisfies property  $M_{q,p_1}$ , it follows that  $b_{p_1-q} = 0$  in  $B$ , which, using the above equations, implies that  $b_{p_1-j} = 0$  for all  $1 \leq j \leq q$ , hence  $c = 0$ , a contradiction.

If  $d$  is odd, then since  $p_1 = p_2$  and  $\dim C_{p_1} = \dim C_{p_1+1}$  the multiplication by  $\omega + T$  map  $C_{p_2} \rightarrow C_{p_2+1}$  is also surjective, hence  $C$  has the WLP. If  $d$  is even, then we use the following argument:

Assume  $c' \in C_{p_2+1}$  with

$$c' = a_{p_2+1} + \sum_{j=1}^q b_{p_2+1-j} T^j$$

where  $a_{p_2+1} \in A_{p_2+1}$  and  $b_{p_2+1-j} \in B_{p_2+1-j}$  for all  $1 \leq j \leq q$ . We will find

$$c = a_{p_2} + \sum_{j=1}^q e_{p_2-j} T^j \in C_{p_2}$$

where  $a_{p_2} \in A_{p_2}$  and  $e_{p_2-j} \in B_{p_2-j}$  for all  $1 \leq j \leq q$ , such that  $(\omega + T)c = c'$ . Hence we need to have (with the first equation in  $A$  and the remaining  $q$  equations in  $B$ )

$$\begin{aligned} a_{p_2+1} &= \omega a_{p_2} \\ b_{p_2+1-1} &= a_{p_2} + \omega e_{p_2-1} \\ b_{p_2+1-2} &= e_{p_2-1} + \omega e_{p_2-2} \\ &\vdots \\ b_{p_2+1-q} &= e_{p_2-q+1} + \omega e_{p_2-q} \end{aligned}$$

Since  $A$  is assumed to have the WLP, the multiplication by  $\omega$  map  $A_{p_2} \rightarrow A_{p_2+1}$  is surjective, hence there exists  $a_{p_2} \in A_{p_2}$  such that  $a_{p_2+1} = \omega a_{p_2}$  in  $A$ . We fix such  $a_{p_2}$ . Given  $e_{p_2-q} \in B_{p_2-q}$ , the last  $q - 1$  equations in  $B$  inductively determine (unique)  $e_{p_2-q+1}, \dots, e_{p_2-1}$  that satisfy them. What we need is to choose  $e_{p_2-q}$  in such a way that we have compatibility with the second equation

$$b_{p_2} = a_{p_2} + \omega e_{p_2-1}.$$

If we express  $e_{p_2-q+1}, \dots, e_{p_2-1}$  in terms of  $e_{p_2-q}$ , the compatibility equation becomes

$$\omega^q e_{p_2-q} = (-1)^q a_{p_2} + \sum_{i=0}^{q-1} (-1)^{q+1-i} \omega^i b_{p_2-i}$$

(equation in  $B$ ). Using the assumption that  $B$  satisfies property  $M_{q,p_1}$ , Remark 3.2 implies that there exists  $e_{p_2-q} \in B_{p_2-q}$  that satisfies the compatibility equation.

We now prove the converse. We assume that  $C$  has the WLP and we prove that  $A$  has the WLP and  $B$  satisfies the  $M_{q,p_1}$  property.

Let  $a \in A_{p_2+1} \subset C_{p_2+1}$ . By Lemma 3.14, there exists  $c \in C_{p_2}$  such that  $a = (\omega + T)c$ . Write

$$c = a_{p_2} + \sum_{j=1}^q e_{p_2-j} T^j \in C_{p_2}$$

with equality in  $C$ , where  $a_{p_2} \in A_{p_2}$  and  $e_{p_2-j} \in B_{p_2-j}$  for all  $1 \leq j \leq q$ . It follows that  $a = \omega a_{p_2}$ , hence the multiplication by  $\omega$  map  $A_{p_2} \rightarrow A_{p_2+1}$  is surjective. Using Lemma 3.4 and Lemma A.3, it follows that  $A$  has the WLP.

We will now prove that  $B$  has the property  $M_{q,p_1}$ . Suppose on the contrary that there exists  $0 \neq b \in B_{p_1-q}$  such that  $\omega^q b = 0$  in  $B$ . We set  $c = \sum_{i=1}^q (-1)^{q-i} \omega^{q-i} b T^i \in C_{p_1}$ . Since the summand of  $c$  corresponding to  $i = q$  is  $b T^q$ , we get from Equation (4) that  $c \neq 0$ . Using that  $T^{q+1} = 0$  in  $C$ , we have

$$\begin{aligned} (\omega + T)c &= \sum_{i=1}^q (-1)^{q-i} \omega^{q-i+1} b T^i + \sum_{i=1}^q (-1)^{q-i} \omega^{q-i} b T^{i+1} \\ &= (-1)^{q-1} \omega^q b T + \sum_{i=2}^q (-1)^{q-i} \omega^{q-i+1} b T^i + \sum_{i=1}^{q-1} (-1)^{q-i} \omega^{q-i} b T^{i+1} \\ &= 0 \end{aligned}$$

which is a contradiction, since, by Lemma 3.14, the multiplication by  $\omega + T$  map  $C_{p_1} \rightarrow C_{p_1+1}$  is injective. This contradiction finishes the proof of Lemma 3.15.  $\square$

## 4 The main results

The present section contains our main results on the Weak Lefschetz Property under stellar subdivisions and their inverses. We continue to use the notation introduced in Section 3.

### 4.1 Weak Lefschetz Property under a stellar subdivision

**Lemma 4.1** *If  $(q > p_1)$  or  $(q = p_1)$  and the field  $k$  has characteristic 0 or a prime number  $> d - q - 1$ , then property  $M_{q,p_1}$  holds for  $k[L]$ .*

**Proof:** If  $q > p_1$ , then  $B_{p_1-q} = 0$  and the result is obvious.

Assume  $q = p_1$ . Since  $B_0 = k$ , property  $M_{q,p_1}$  for  $k[L]$  is equivalent to  $\omega^{p_1} \neq 0$  in  $B$  for general  $\omega \in B_1$ . Suppose on the contrary that  $b^{p_1} = 0$  for all  $b \in B_1$ . This, together with the assumptions on the characteristic of the field  $k$  imply, by Lemma A.5, that  $B_{p_1} = 0$ . Since  $d - q - 1 = d - p_1 - 1 \geq p_1$ , and  $B$  is standard graded, we get  $B_{d-q-1} = 0$  which contradicts Lemma 3.4.  $\square$

**Lemma 4.2** *Assume  $R[T]/I_C$  has the WLP. Then  $k[D_\sigma]$  has the WLP.*

**Proof:** We argue in a very similar way to [18, Proposition 2.2]. Consider the  $k$ -algebra automorphism  $\phi$  of  $R[T]$  defined by  $T \mapsto T, x_i \mapsto x_i + T$  for  $1 \leq i \leq q + 1$  and  $x_i \mapsto x_i$  for  $q + 2 \leq i \leq n$ . We claim that  $I_C$  is the initial ideal of  $\phi(J_{st})$  with respect to any monomial order in  $R[T]$  such that  $T > x_i$  for all  $1 \leq i \leq n$ .

Indeed, it is clear that  $I_C$  is a subset of the initial ideal of  $\phi(J_{st})$  and, by Lemma 3.9,  $\text{HF}(R[T]/I_C) = \text{HF}(k[D_\sigma])$ . It follows from Lemma A.1 that  $R[T]/\phi(J_{st})$  has the WLP, hence also  $R[T]/J_{st} = k[D_\sigma]$  has the WLP. □

We now prove Theorem 1:

**Proof:** We first prove Part iv). By Lemma 3.12,  $A$  has the WLP for general  $f_1, \dots, f_d \in R_1$ . Hence, by Lemma 3.15,  $C$  has the WLP for general  $f_1, \dots, f_d \in R_1$ . As a consequence, by Lemma 3.12,  $R[T]/I_C$  has the WLP. Using Lemma 4.2, Part iv) of Theorem 1 follows.

Part iii) follows from Part iv), since, by Remark 3.3,  $k[L]$  SLP implies that the property  $M_{q,p_1}$  holds for  $k[L]$ .

We finally prove Parts i) and Parts ii) of Theorem 1. Using Lemma 4.1, property  $M_{q,p_1}$  holds for  $k[L]$ . Hence  $k[D_\sigma]$  has the WLP by Part iii). □

### 4.2 Weak Lefschetz Property under the inverse of a stellar subdivision

In the following we prove Theorem 2:

**Proof:**

We assume  $q > p_2$  and  $k[D_\sigma]$  has the WLP.

The assumption that  $k[D_\sigma]$  has the WLP implies, by Lemma 3.8, that for general  $f_1, \dots, f_d, G$  is Artinian Gorenstein with the WLP, and that  $\text{HF}(G) = \text{HF}(C)$ . As a consequence, Lemma 3.5 implies that  $G_d$  is 1-dimensional and,  $G_j = 0$  for  $j > d$ . Hence, for general  $\omega \in R_1$  the multiplication by  $\omega + T$  map  $G_{p_2} \rightarrow G_{p_2+1}$  is surjective.

Using that the map  $\psi$  in the proof of Lemma 3.8 is bijective, it follows that the natural map  $A \rightarrow G$ , with  $[a] \mapsto [a]$  for  $a \in R$ , is injective. Assume  $a \in A_{p_2+1} \subset G_{p_2+1}$ . Then there exists  $g \in G_{p_2}$  such that

$$(\omega + T)g = a \tag{6}$$

Using again that the map  $\psi$  in the proof of Lemma 3.8 is bijective, there exists  $e \in A_{p_2}$  and, for  $1 \leq j \leq q, b_{p_2-j} \in B_{p_2-j}$  such that

$$g = e + \sum_{i=1}^q b_{p_2-i} T^i,$$

with equality in  $G$ . The assumption  $q > p_2$  implies that  $p_2 - q < 0$ , hence  $b_{p_2-q} = 0$  in  $B$ . As a consequence

$$a = (\omega + T)g = \omega e + eT + \sum_{i=1}^{q-1} b_{p_2-i}T^i + \sum_{i=1}^{q-1} b_{p_2-i}T^{i+1}$$

(equality in  $G$ ), which imply that  $a = \omega e$  (equality in  $A \subset G$ ). It follows that the multiplication by  $\omega$  map  $A_{p_2} \rightarrow A_{p_2+1}$  is surjective. Using Lemma 3.4 and Lemma A.3, it follows that  $A$  has the WLP. Hence, Lemma 3.12 implies that  $k[D]$  has the WLP. □

### 4.3 Proof of Corollary 3

**Proof:** We first prove that the statement  $2(\dim \sigma) > \dim D + 1$  is equivalent to  $q > p_2$ . Indeed, by the definitions,  $q = \dim \sigma$ ,  $d = \dim D + 1$ . Assume  $d$  is even. Then  $p_2 = d/2$ . Hence  $q > p_2$  is equivalent to  $\dim \sigma > d/2$  which is equivalent to  $2(\dim \sigma) > d = \dim D + 1$ . Assume now  $d$  is odd. Then  $p_2 = (d - 1)/2$ , hence  $q > p_2$  is equivalent to  $\dim \sigma > (d - 1)/2$  which is equivalent to  $2(\dim \sigma) > d - 1$ . But  $d$  odd implies  $d - 1$  even, hence since  $2(\dim \sigma)$  is always even  $2(\dim \sigma) > d - 1$  is equivalent to  $2(\dim \sigma) > d = \dim D + 1$ .

Assume  $2(\dim \sigma) > \dim D + 1$  and  $k[D]$  has the WLP. As we said above  $q > p_2$ . Since  $p_2 \geq p_1$ , we have  $q > p_1$ , hence Part i) of Theorem 1 implies that  $k[D_\sigma]$  has the WLP.

Assume now  $2(\dim \sigma) > \dim D + 1$  and  $k[D_\sigma]$  has the WLP. As we said above  $q > p_2$ . By Theorem 2,  $k[D]$  has the WLP. □

### 4.4 Final remarks

In the following remarks we keep assuming that  $D$  is a Gorenstein\* simplicial complex and  $k$  is an infinite field.

**Remark 4.3** Suppose

$$D_0 = D, D_1, \dots, D_m$$

is a finite sequence of simplicial complexes such that, for all  $0 \leq i \leq m - 1$ , the complex  $D_{i+1}$  is obtained from  $D_i$  by a stellar subdivision with respect to a face  $\sigma_i$  of  $D_i$  with  $2(\dim \sigma_i) > \dim D + 1$ . Then, by Corollary 3, the Stanley–Reisner ring  $k[D]$  has the WLP if and only if  $k[D_m]$  has the WLP. Is it possible to prove that starting from  $D$  there exists a sequence of stellar subdivisions as above with  $k[D_m]$  WLP? Then it would follow that  $k[D]$  has the WLP. Compare also [12, Conjecture 4.12].

**Remark 4.4** Recall  $I_C = (I, T^{q+1}, TI_L)$ . Assume that  $k[D_\sigma]$  has the WLP. Is it possible to prove that  $R[T]/I_C$  has the WLP? If so, combining Lemmas 3.12 and 3.15 it would then follow that  $k[D]$  has the WLP.

## A Some general lemmas

In the present section we put together a number of general lemmas we use. The following lemma is the analogue for the WLP of [18, Lemma 3.3] which is stated for the SLP and can be proven by the same arguments.

**Lemma A.1** (Wiebe) *Assume  $R$  is a polynomial ring over an infinite field with all variables of degree 1,  $\tau$  is a monomial order on  $R$  and  $J \subset R$  is a homogeneous ideal with  $R/J$  Cohen–Macaulay. Denote by  $in_\tau(J)$  the initial ideal of  $J$  with respect to  $\tau$ . We assume that  $R/in_\tau(J)$  is Cohen–Macaulay and has the WLP. Then  $R/J$  has the WLP.*

We remark that  $R/in_\tau(J)$  is automatically Cohen–Macaulay if  $R/J$  is Cohen–Macaulay and  $\tau$  is the reverse lexicographic order (induced by any linear order of the variables); see [6, Theorem 15.13]. For a proof of the following lemma see [3, Proposition 1.5.12].

**Lemma A.2** *Assume  $k$  is an infinite field,  $R = k[x_1, \dots, x_n]$  with all variables of degree 1,  $J \subset R$  is a homogeneous ideal and  $t$  is a positive integer with  $t \leq \text{depth } R/J$ . Then, there exists a non-empty Zariski open subset  $U \subset (R_1)^t$  such that  $f_1, \dots, f_t$  is an  $R/J$ -regular sequence for all  $(f_1, \dots, f_t) \in U$ .*

**Lemma A.3** *Assume  $k$  is an infinite field and  $F = \bigoplus_{i=0}^d F_i$  is an Artinian standard graded Gorenstein  $k$ -algebra with  $F_d \neq 0$ . If  $d$  is even, we set  $p_1 = d/2 - 1, p_2 = d/2$ , if  $d$  is odd, we set  $p_1 = p_2 = (d - 1)/2$ . Denote by  $\omega \in F_1$  a general linear form. Then the following are equivalent.*

- i)  $F$  has the WLP.*
- ii) The multiplication by  $\omega$  map  $F_{p_1} \rightarrow F_{p_1+1}$  is injective*
- iii) The multiplication by  $\omega$  map  $F_{p_2} \rightarrow F_{p_2+1}$  is surjective.*

**Proof:** It follows from [17, Remark 2.4] □

**Lemma A.4** *Assume  $k$  is an infinite field,  $R = k[x_1, \dots, x_n]$  with all variables of degree 1. Assume  $J \subset R$  is a homogeneous ideal such that  $R/J$  is Cohen–Macaulay and  $g_1, g_2 \in R$  are two nonzero linear forms. We define*

$$\mathcal{S}_1 = \{c \in k : g_1 - cg_2 \notin J\} \subset \mathbb{A}^1,$$

$$\mathcal{S}_2 = \{c \in \mathcal{S}_1 : g_1 - cg_2 \text{ is } R/J\text{-regular}\} \subset \mathbb{A}^1$$

and

$$\mathcal{S}_3 = \{c \in \mathcal{S}_2 : R/(J + (g_1 - cg_2)) \text{ has the WLP}\} \subset \mathbb{A}^1.$$

Then, for all  $1 \leq i \leq 3$ , the subset  $\mathcal{S}_i \subset \mathbb{A}^1$  is Zariski open (but perhaps empty).

**Proof:** Denote by  $\mathcal{B}$  the finite dimensional vector space  $R_1$  considered as an affine variety. Consider the morphism  $\phi : \mathbb{A}^1 \rightarrow \mathcal{B}, c \mapsto g_1 - cg_2$ . It is clear that the image of  $\phi$  is an affine subspace of  $\mathcal{B}$ , and hence Zariski closed. As a consequence, it is enough to prove that, for  $1 \leq i \leq 3$ , the three subsets

$$\mathcal{W}_1 = \{f \in \mathcal{B} : f \notin J\} \subset \mathcal{B},$$

$$\mathcal{W}_2 = \{f \in \mathcal{B} : f \text{ is } R/J\text{-regular}\}$$

and

$$\mathcal{W}_3 = \{f \in \mathcal{B} : f \text{ is } R/J\text{-regular and } R/(J + (f)) \text{ has the WLP}\} \subset \mathcal{B}$$

are Zariski open. For  $\mathcal{W}_1$  it is obvious. For  $\mathcal{W}_2$  it follows from [6, Theorem 3.1]. The case of  $\mathcal{W}_3$  is also well-known, see for example [4, Lemma 4.1]. □

**Lemma A.5** *Assume  $e \geq 1$  is an integer and  $k$  is a field of characteristic 0 or of prime characteristic  $> e$ . Consider the polynomial ring  $R = k[x_1, \dots, x_n]$  with all variables of degree 1 and assume  $V \subset R$  is a  $k$ -vector subspace. If  $a^e \in V$  for all  $a \in R_1$ , then it follows that  $R_e \subset V$ .*

**Proof:** According to [11, Section 3.2, Exercise 2], the linear span of the set  $\{a^e : a \in R_1\}$  is equal to  $R_e$ . The result follows. □

**Lemma A.6** *Assume  $D$  is a Gorenstein\* simplicial complex. Denote by  $k[D]$  the Stanley–Reisner ring of  $D$  over an infinite field  $k$  and by  $F$  an Artinian reduction of  $k[D]$ . We have*

$$F = \bigoplus_{i=0}^{\dim k[D]} F_i$$

and  $F_{\dim k[D]}$  is 1-dimensional.

**Proof:** This follows from [25, Theorems I.12.4-I.12.6]. □

## B A lemma on the WPL property of an Artinian reduction

It is well-known that it can happen that an Artinian reduction of a  $k$ -algebra satisfying the WLP does not satisfy the WLP; see, for example, [5]. The following Lemma B.1 states a condition that guarantees that certain Artinian reductions of a WLP  $k$ -algebra inherit the WLP property.

**Lemma B.1** *Assume  $k$  is an infinite field,  $R = k[x_1, \dots, x_n]$  is a polynomial ring with all variables of degree 1 and  $S = R[T]$ , where  $T$  is a new variable of degree 1. Assume  $J \subset S$  is a homogeneous ideal with  $\dim S/J = d \geq 1$ . Assume  $S/J$  has the WLP, and that there exists  $\tilde{f} \in (R_1)^d$  with  $\dim S/(J + (\tilde{f})) = 0$ . Then there exists a non-empty Zariski open subset  $U \subset (R_1)^d$  such that  $\dim S/(J + (f)) = 0$  and  $S/(J + (f))$  has the WLP for all  $f \in U$ .*

**Proof:**

STEP 1. Since  $S/J$  has the WLP, it is Cohen–Macaulay. It is well-known (see [18, Lemma 3.1]) that there exists a non-empty Zariski open subset  $U_1 \subset (S_1)^{d+1}$  such that  $\dim S/(J + (h)) = 0$  and

$$\text{HF}(S/(J + (h))) = \Delta^+(\Delta^d(\text{HF}(S/J)))$$

for all  $h \in U_1$ .

STEP 2. Denote by  $\phi : (S_1)^{d+1} \rightarrow (S_1)^d$  the morphism of varieties given by

$$\phi(h_1, \dots, h_d, h_{d+1}) = (h_1, \dots, h_d).$$

It is well-known (see, for example, [22, Tag 037G]) that  $\phi$  is an open morphism. We claim that

$$\phi(U_1) \cap (R_1)^d \neq \emptyset.$$

Indeed,  $U_1$  is non-empty, so there exists  $(h_1, \dots, h_{d+1}) \in U_1$ . By Gauss elimination there exist  $f_1, \dots, f_d \in R_1$  and  $f_{d+1} \in S_1$  such that the ideals  $(f_1, \dots, f_{d+1})$  and  $(h_1, \dots, h_{d+1})$  of  $S$  are equal. Hence  $(f_1, \dots, f_{d+1}) \in U_1$ . As a consequence  $(f_1, \dots, f_d) \in \phi(U_1) \cap (R_1)^d$ .

STEP 3. We claim that there exists a non-empty Zariski open subset  $U_2 \subset (R_1)^d$  such that  $\dim S/(J + (f)) = 0$  for all  $f \in U_2$ . Indeed, this follows from the openness of the condition and the hypothesis that there exists  $\tilde{f} \in (R_1)^d$  with the property.

STEP 4. We set

$$U = \phi(U_1) \cap (R_1)^d \cap U_2.$$

The set  $U$  is the intersection of the non-empty Zariski open subsets  $U_2$  and  $\phi(U_1) \cap (R_1)^d$  of  $(R_1)^d$ . Since the field  $k$  is infinite, the affine variety  $(R_1)^d$  is irreducible. Hence  $U$  is a non-empty Zariski open subset of  $(R_1)^d$ .

Assume  $f = (f_1, \dots, f_d) \in U$ . Since  $U \subset U_2$ , we have that  $\dim S/(J + (f)) = 0$ . We claim that  $S/(J + (f))$  has the WLP. Indeed, since  $f \in \phi(U_1)$  there exists  $f_{d+1} \in S_1$  such that  $(f_1, \dots, f_{d+1}) \in U_1$ . Hence

$$\text{HF}(S/(J + (f_1, \dots, f_{d+1}))) = \Delta^+(\Delta^d(\text{HF}(S/J))).$$

Using that  $S/J$  is Cohen–Macaulay of dimension  $d$ , we get that  $\dim S/(J + (f)) = 0$  implies that  $f_1, \dots, f_d$  is a regular sequence for  $S/J$ . Hence

$$\text{HF}(S/(J + (f_1, \dots, f_d))) = \Delta^d(\text{HF}(S/J)).$$

As a consequence,  $S/(J + (f))$  has the WLP. □

**Remark B.2** The existence of  $\tilde{f} \in (R_1)^d$  with  $\dim S/(J + (\tilde{f})) = 0$  in the statement of Lemma B.1 does not follow from the other assumptions since the  $f_i$  do not involve the variable  $T$ . For example, if  $R = k[x_1]$  and  $J = (Tx_1) \subset S = R[T]$ , then such  $\tilde{f}$  does not exist.

**Question B.3** Is there a statement similar to Lemma B.1 for the SLP?

## Acknowledgements

We thank Christos Athanasiadis for suggesting the problem and David Eisenbud, Satoshi Murai and Eran Nevo for useful discussions. We also thank an anonymous referee for minor corrections and comments that improved the presentation of the material. We benefited from experiments with the computer algebra program Macaulay2 [8].

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