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Intersecting families, signed sets, and injection

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Abstract

Let $k, r, n \geq 1$ be integers, and let $\mathcal{S}_{n,k,r}$ be the family of r-signed k-sets on $[n] = \{1, \ldots, n\}$ given by $\mathcal{S}_{n,k,r} = \big\{\{(x_1, a_1), \ldots, (x_k, a_k)\} : \{x_1, \ldots, x_k\} \in \binom{[n]}{k}, a_1, \ldots, a_k \in [r]\big\}$. A family $\mathcal{A} \subseteq \mathcal{S}_{n,k,r}$ is *intersecting* if $A, B \in \mathcal{A}$ implies $A \cap B \neq \emptyset$. A well-known result (first stated by Meyer and proved using different methods by Deza and Frankl, and Bollobás and Leader) states that if $\mathcal{A} \subseteq \mathcal{S}_{n,k,r}$ is intersecting, $r \geq 2$ and $1 \leq k \leq n$, then $|\mathcal{A}| \leq r^{k-1}\binom{n-1}{k-1}$. We provide a proof of this result by injection (in the same spirit as Frankl and Füredi's and Hurlbert and Kamat's injective proofs of the Erdős–Ko–Rado Theorem, and Frankl's and Hurlbert and Kamat's injective proofs of the Hilton–Milner Theorem) whenever $r \geq 2$ and $1 \leq k \leq n/2$, leaving open only some cases when $k \leq n$.

1 Introduction

Let $[n] = \{1, \ldots, n\}$ and let $\binom{[n]}{k}$ denote the collection of all k-subsets of $[n]$. Sets of sets are called *families.* A family $\mathcal{F} \subseteq 2^{[n]}$ is *intersecting* if $F, F' \in \mathcal{F}$ implies $F \cap F' \neq \emptyset$. How large can an intersecting family $\mathcal{F} \subseteq \binom{[n]}{k}$ be? If $2k > n$ then $|\mathcal{F}| = \binom{n}{k}$ is obvious, while if $2k \leq n$ then the answer is given by the classical Erdős–Ko–Rado Theorem [10].

Definition 1.1. Let

$$\mathcal{S} = \Big\{F \in \binom{[n]}{k} : 1 \in F\Big\}.$$

Erdős–Ko–Rado Theorem (Erdős, Ko and Rado [10]). *Let $n, k \geq 0$ be integers, $n \geq 2k$. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be intersecting. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} = |\mathcal{S}|. \tag{1.1}$$

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When $n = 2k$, the proof of the Erdős–Ko–Rado Theorem is easy. Simply partition $\binom{[2k]}{k}$ into complementary pairs. Then, since $\mathcal{F}$ can contain at most one set from each pair, $|\mathcal{F}| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$. To deal with the case $n > 2k$ Erdős, Ko and Rado [10] introduced an important operation on families called *shifting*.

A family is called *non-trivial* if there is no element common to all its members. Hilton and Milner [15] showed that for $n > 2k$, $\mathcal{S}$ is the unique maximal intersecting family.

Definition 1.2. Let $G \in \binom{[n]}{k}$, $1 \notin G$ and

$$\mathcal{N} = \{G\} \cup \Big\{F \in \binom{[n]}{k} : 1 \in F, F \cap G \neq \emptyset\}.$$

Hilton–Milner Theorem (Hilton and Milner [15]). *Suppose that $n \geq 2k \geq 4$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is non-trivial. Then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1 = |\mathcal{N}|. \tag{1.2}$$

There are various proofs of the Erdős–Ko–Rado Theorem (cf. [7, 13, 16, 18]) and the Hilton–Milner Theorem (cf. [11,12,16]). To keep this paper short, let us highlight those which are particularly relevant to us: Frankl and Füredi's [13] and Hurlbert and Kamat's [16] injective proofs of (1.1), and Frankl's [11] and Hurlbert and Kamat's [16] injective proofs of (1.2).

We should mention that by "injective proof" we mean an explicit or implicit injection from $\mathcal{F}$ into a given intersecting family (usually a family whose members contain a prescribed element). We believe that such proofs are of interest, particularly in yielding further insight for the cases when the size of intersecting families cannot be determined *a priori*; as an example of such a case see [4, Conjecture 1.4]. For further results in extremal set theory, we refer the reader to the excellent monograph by Gerbner and Patkos [14].

We now define *signed sets*. Let $k, r, n \geq 1$ be integers, and let $\mathcal{S}_{n,k,r}$ be the family of r-signed k-sets on $[n]$ given by

$$\mathcal{S}_{n,k,r} = \Big\{\{(x_1, a_1), \ldots, (x_k, a_k)\} : \{x_1, \ldots, x_k\} \in \binom{[n]}{k}, a_1, \ldots, a_k \in [r]\Big\}.$$

A well-known analogue of the Erdős–Ko–Rado Theorem for signed sets was first stated by Meyer [20], and later proved by Deza and Frankl [8] using the shifting technique, and by Bollobás and Leader [3] using Katona's elegant cycle method [18].

Definition 1.3. Let

$$\mathcal{W} = \Big\{W \in \mathcal{S}_{n,k,r} : (1, 1) \in W\Big\}.$$

Theorem 1.1 (Deza and Frankl [8]; Bollobás and Leader [3])**.** *Let $n, k, r \geq 1$ be integers, $n \geq k$. Let $\mathcal{F} \subseteq \mathcal{S}_{n,k,r}$ be intersecting. Then*

$$|\mathcal{F}| \leq r^{k-1}\binom{n-1}{k-1} = |\mathcal{W}|. \tag{1.3}$$

We should mention that there are several generalisations, extensions and variations of Theorem 1.1; see for example [1, 2, 4–6, 9, 19].

Motivated by the afore-mentioned results we consider the following problem.

Problem 1. *Find an injective proof of (1.3).*

The object of this paper is to present the following theorem that provides extensive solutions to Problem 1 leaving open only some cases when $k \leq n$.

Theorem 1.2. *There is an injective proof of (1.3) whenever $r \geq 2$ and $k \leq n/2$.*

2 The proof

One of the main tools in our proof is Katona's *Intersection Shadow Theorem.* For integers $k > s \geq 0$ and a family $\mathcal{F} \subseteq \binom{[n]}{k}$, define its s-shadow $\partial_s(\mathcal{F})$ by

$$\partial_s(\mathcal{F}) := \left\{G \in \binom{[n]}{s} : \exists F \in \mathcal{F}, G \subset F\right\}.$$

Suppose that $\mathcal{F} \subseteq \binom{[n]}{s}$ such that $|F \cap F'| \geq t \geq 0$ for all $F, F' \in \mathcal{F}$. Katona [17] then showed that

$$|\partial_{s-t}(\mathcal{F})| \geq |\mathcal{F}|. \tag{2.1}$$

Let mod* be the usual modulo operation except that for integers x and y, (xy) mod$^*y$ is y instead of 0. Following Borg [4], for a signed sets A and integers q and r, let $\theta_r^q(A)$ be the shifting operation given by

$$\theta_r^q(A) = \{(x, (a+q) \text{ mod}^* r) : (x, a) \in A\},$$

and, for a family $\mathcal{A}$ of signed sets,

$$\theta_r^q(\mathcal{A}) = \{\theta_r^q(A) : A \in \mathcal{A}\}.$$

Proof of Theorem 1.2. The proof is an adaptation of the proof in [13], with more or less simple changes. Let $\mathcal{A} \subseteq \mathcal{S}_{n,k,r}$ be intersecting, let $\mathcal{A}_0 = \{A \in \mathcal{A} : A \cap \{(1,1), \dots, (1,r)\} = \emptyset\}$ and $\mathcal{A}_i = \{A \in \mathcal{A} : (1,i) \in A\}$ for $1 \leq i \leq r$. Note that $\mathcal{A}_0, \dots, \mathcal{A}_r$ partition $\mathcal{A}$. Let $\mathcal{A}'_0 = \mathcal{A}_0$ and $\mathcal{A}'_i = \{A \setminus \{(1,i)\} : A \in \mathcal{A}_i\}$ for $1 \leq i \leq r$.

Let $\mathcal{A}' = \bigcup_{i=0}^r \mathcal{A}'_r$. For $A \in \mathcal{A}'$, let $M_A = \{x : (x,a) \in A\}$. We say that M_A *represents* A. Let $\mathcal{M}_0 = \{M_A : A \in \mathcal{A}'_0\}$, $\mathcal{M}_1 = \{M_A : A \in \mathcal{A}' \setminus \mathcal{A}'_0\}$, $\mathcal{N} = \{[2,n] \setminus M : M \in \mathcal{M}_0\}$ and

$$\mathcal{B} = \Big\{(x_1, a_1), \dots, (x_{k-1}, a_{k-1}) : \{x_1, \dots, x_{k-1}\} \in \partial_{k-1}(\mathcal{N}), a_1, \dots, a_{k-1} \in [r]\Big\}.$$

Claim 1. $|\mathcal{A}'_0| \leq |\mathcal{B}|$.

Proof. Since $\mathcal{A}'_0$ is intersecting,

$$\text{each set in } \mathcal{M}_0 \text{ can represent at most } r^{k-1} \text{ sets in } \mathcal{A}'_0. \tag{2.2}$$

Let $N, N' \in \mathcal{N}$. Since $1 \leq k \leq n/2$, we infer

$$|N \cap N'| = |([2,n] \setminus M) \cap ([2,n] \setminus M')| = n - 1 - 2k + |M \cap M'| \geq n - 2k \geq 0,$$

so that applying (2.1) with $s = n - 1 - k$ and $t = n - 2k$ gives us

$$|\mathcal{M}_0| = |\mathcal{N}| \leq |\partial_{k-1}(\mathcal{N})|. \tag{2.3}$$

Then (2.2) and (2.3) yield

$$|\mathcal{A}'_0| \leq r^{k-1}|\mathcal{M}_0| \leq r^{k-1}|\partial_{k-1}(\mathcal{N})| = |\mathcal{B}|.$$

□

Claim 2. *The families $\mathcal{A}'_1, \theta^1_r(\mathcal{A}'_2), \ldots, \theta^{r-1}_r(\mathcal{A}'_r), \mathcal{B}$ are pairwise disjoint.*

Proof. Since $\mathcal{A}$ is intersecting,

$$\text{for } i,j \in \{0\} \cup [r] \text{ with } i \neq j \text{ each set in } \mathcal{A}'_i \text{ intersects each set in } \mathcal{A}'_j. \tag{2.4}$$

Suppose there exists $B \in \theta^{i-1}_r(\mathcal{A}'_i) \cap \theta^{j-1}_r(\mathcal{A}'_j)$ for some distinct $i, j \in [2, r]$. Let $A_i = \theta_r^{-(i-1)}(B) \in \mathcal{A}'_i$ and $A_j = \theta_r^{-(j-1)}(B) \in \mathcal{A}'_j$. Then $A_i \cap A_j = \emptyset$, which contradicts (2.4). Similarly, if we suppose $B \in \mathcal{A}'_1 \cap \theta^{i-1}_r(\mathcal{A}'_i)$ for some $i \in [2, r]$, then we get a contradiction to (2.4). Therefore, families $\mathcal{A}'_1, \theta^1_r(\mathcal{A}'_2), \ldots, \theta^{r-1}_r(\mathcal{A}'_r)$ are pairwise disjoint. By (2.4), each set in $\mathcal{M}_0$ intersects each set in $\mathcal{M}_1$. Therefore $\mathcal{M}_1 \cap \partial_{k-1}(\mathcal{N}) = \emptyset$, which is to say

$$\mathcal{B} \cap \left(\mathcal{A}'_1 \cup \bigcup_{i=2}^{r} \theta^{i-1}_r(\mathcal{A}'_i)\right) = \emptyset$$

and the claim is proved. □

Let $\mathcal{A}^*_0 = \{B \cup \{(1,1)\} : B \in \mathcal{B}\}$, $\mathcal{A}^*_1 = \mathcal{A}_1$ and $\mathcal{A}^*_i = \{A \cup \{(1,1)\} : A \in \theta^{i-1}_r(\mathcal{A}'_i)\}$ for $2 \leq i \leq r$. For $0 \leq i \leq r$, $\mathcal{A}^*_i \subseteq \mathcal{W}$. By Claim 2, $\sum_{i=0}^{p} |\mathcal{A}^*_i| \leq |\mathcal{W}|$. By Claim 1, $|\mathcal{A}_0| \leq |\mathcal{A}^*_0|$. We have

$$|\mathcal{A}| = \sum_{i=0}^{r} |\mathcal{A}_i| = |\mathcal{A}_0| + \sum_{i=1}^{r} |\mathcal{A}^*_i| \leq \sum_{i=0}^{r} |\mathcal{A}^*_i| \leq |\mathcal{W}|,$$

and the theorem is proved. □

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