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Maximum packing and minimum covering of the line graph of the complete graph with kites

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Abstract

Packings and coverings of complete graphs and complete multipartite graphs have been extensively studied. In this paper, the study of packings and coverings of line graphs of complete graphs is initiated. The graph with vertex set $\{a, b, c, d\}$ and edge set $\{ab, bc, ca, cd\}$ is called a *kite*. In this paper, maximum kite-packings and minimum kite-coverings of $L(K_n)(\lambda)$, that is, the λ-fold line graph of the complete graph K_n, with every possible leave and padding are obtained. In particular, it is shown that for $n \geq 4$, the graph $L(K_n)$ has a kite-decomposition if and only if $n \equiv 0 \pmod 2$ or $n \equiv 1 \pmod 8$.

1 Introduction

Throughout the paper, we consider only finite graphs without loops. Let $V(G)$ and $E(G)$ denote the vertex set and edge set of a graph G, respectively. The *path* (respectively, *cycle*) on k vertices is denoted by P_k (respectively, C_k). A cycle of length 3 is called a *triangle*. Let nG denote n vertex-disjoint copies of G. The complete graph on n vertices is denoted by K_n and the complete bipartite graph with bipartition (X, Y), where $|X| = m$ and $|Y| = n$, is denoted by $K_{m,n}$. The graph $H_1 \cup H_2$ denotes the disjoint union of the graphs H_1 and H_2. The graph $G(\lambda)$ is obtained by replacing each edge of G by λ parallel edges. For disjoint subsets A and B of the vertex set $V(G)$ of G, let $E(A, B) = \{e = ab \in E(G) | a \in A$ and $b \in B\}$. The graph with vertex set $\{a, b, c, d\}$ and edge set $\{ab, bc, ca, cd\}$ is called a *kite* and it is denoted by $[(a, b, c); cd]$; see Figure 1. We denote a kite by K. A graph G is said to be $H_1, H_2, \ldots, H_k$*-decomposable* if the edge set of G can be partitioned into $E_1, E_2, \ldots, E_k$ such that, for each i, $\langle E_i \rangle \simeq H_i$, where $\langle E_i \rangle$ denotes the subgraph induced by E_i; we denote this by $\{H_1, H_2, \ldots, H_k\} \,|\, G$. If each $\langle E_i \rangle \simeq H$, then we say that G has an H*-decomposition*. In this case, we write

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$H|G$. If $H = K$ then we say that G has a *kite-decomposition*. By an $\{H_1^\alpha, H_2^\beta, H_3^\gamma\}$-*decomposition* of a graph G, we mean a decomposition of G into α copies of H_1, β copies of H_2 and γ copies of H_3, where α, β, γ are non-negative integers and $\alpha|E(H_1)| + \beta|E(H_2)| + \gamma|E(H_3)| = |E(G)|$.

The *line graph* of a graph G, denoted by $L(G)$, is the graph with $V(L(G)) = E(G)$ and $e_1e_2 \in E(L(G))$ if and only if the edges e_1 and e_2 are incident at a common vertex of G. Let $\mathcal{P}_k(t)$ be the set of all k-element subsets of the t element set $\{1, 2, \ldots, t\}$. For a set S with $|S| \geq 2$, $\mathcal{P}_2(S)$ denotes all 2-element subsets of S. Let $V(K_n) = \{1, 2, \ldots, n\}$; then $V(L(K_n)(\lambda)) = \mathcal{P}_2(\{1, 2, \ldots, n\})$ and $|E(L(K_n)(\lambda))| = \frac{\lambda n(n-1)(n-2)}{2}$.

A *packing* of the graph G with kite K is a triple (V, E, L), where V is the vertex set of G, E is a set of edge-disjoint kites of G, and L is the set of edges of G not belonging to any of the kites of E. The collection of edges L is the *leave*. If $|E|$ is as large as possible, or equivalently if $|L|$ is as small as possible, then (V, E, L) is called a *maximum packing* of G with kites. A *covering* of the graph G with kite K is a triple (V, E, P), where V is the vertex set of G, P is a subset of the edge set of $G(\lambda)$, and E is a set of edge-disjoint kites which partitions the union of P and the edge set of G. The collection of edges P is called the *padding*. If $|P|$ is as small as possible, then (V, E, P) is called a *minimum covering* of G with kites. See [15] for definitions.

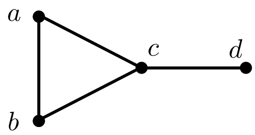


Figure 1: The graph kite, $K = [(a, b, c); cd]$.

For brevity, we focus only on the literature related to decompositions of graphs into kites. Bermond and Schönheim [2] proved the existence of a kite-decomposition of K_n. Roditty [18] obtained a maximum packing of K_n with kites. Küçükçifçi and Milici [14] obtained a complete solution for the decomposition of $K_n(\lambda)$ into kites and 4-cycles. Hu et al. [11] proved the existence of a maximum kite-packing of the complete m-partite graph in which each partite set has n vertices, with every possible leave. Tamil Elakkiya and Muthusamy [21] obtained a gregarious kite-decomposition of $K_m \times K_n$, where $\times$ denotes the tensor product of graphs. The kite decompositions of certain graphs are considered in [4, 16, 17]. Maximum packings of K_n with graphs C_4, C_5, C_6, K_4, the graphs having four or fewer vertices, certain graphs on five vertices and the 3-cube are studied in [1, 3, 13, 18, 19, 20, 23]. Hamilton cycle decompositions, 4-cycle decompositions, 2^ℓ-cycle decompositions, $\ell \geq 2$, and 6-cycle decompositions of $L(K_n)$, the line graph of the complete graph K_n, have been studied in [5, 6, 7, 9, 10, 22]. Very recently, Ganesamurthy et al. [8] obtained a characterization for the existence of a $\{C_3^\alpha, P_4^\beta, B^\gamma\}$-decomposition of $L(K_n)$, where B is the bowtie, that is, the graph with two triangles having exactly one common vertex. In the same paper, they also proved the existence of a $\{C_3^{\alpha},\ K_{1,3}^{\beta}\}$-decomposition of $L(K_n)$, $n \geq 4$. In this paper, complete solutions to the maximum kite-packing and minimum kite-covering of $L(K_n)$ are given.

To state the main Theorems 1.1 and 1.3 we define a graph P'. Let P' be the multigraph on three vertices a, b, c and three edges ab, ab, bc.

We prove the following results:

Theorem 1.1. *A maximum K-packing of $L(K_n)(\lambda)$, the λ-fold line graph of K_n, with all possible leaves exist. The leaves are given in the following table:*

$\lambda \equiv a \pmod 4$	$n \geq 4$ *and* $n \equiv b \pmod 8$	*Possible leaves in* $L(K_n)(\lambda)$
$a = 0$	$n \geq 4$	$\emptyset$
$a \in \{1, 2, 3\}$	n *even or* $b = 1$	$\emptyset$
$a = 1$	$b = 3$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
	$b = 5$	$P_3, 2K_2, K_2(2)$
	$b = 7$	K_2
$a = 2$	$b \in \{3, 7\}$	$P_3, 2K_2, K_2(2)$
	$b = 5$	$\emptyset$
$a = 3$	$b = 3$	K_2
	$b = 5$	$P_3, 2K_2, K_2(2)$
	$b = 7$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$

Corollary 1.2. *For $n \geq 4$, the graph $L(K_n)$, the line graph of K_n, has a kite-decomposition if and only if $n \equiv 0 \pmod 2$ or $n \equiv 1 \pmod 8$.*

Theorem 1.3. *A maximum K-covering of $L(K_n)(\lambda)$ with all possible paddings exist. The paddings are given in the following table:*

$\lambda \equiv a \pmod 4$	$n \geq 4$ *and* $n \equiv b \pmod 8$	*Possible paddings in* $L(K_n)(\lambda)$
$a = 0$	$n \geq 4$	$\emptyset$
$a \in \{1, 2, 3\}$	n *even or* $b = 1$	$\emptyset$
$a = 1$	$b = 3$	K_2
	$b = 5$	$P_3, 2K_2, K_2(2)$
	$b = 7$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
$a = 2$	$b \in \{3, 7\}$	$P_3, 2K_2, K_2(2)$
	$b = 5$	$\emptyset$
$a = 3$	$b = 3$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
	$b = 5$	$P_3, 2K_2, K_2(2)$
	$b = 7$	K_2

2 Basic lemmas

In this section, we prove some lemmas which are required to prove the main result of this paper. Throughout this paper, we assume that $\{1, 2, \dots, n\}$ is the vertex set of K_n.

Lemma 2.1. *The graph $L(K_4)$ has a K-decomposition.*

Proof. Let $V(L(K_4)) = \mathcal{P}_2(4)$. A K-decomposition of $L(K_4)$ is given here: $[(\{1,2\},\{2,3\},\{1,3\});\{1,3\}\{1,4\}]$, $[(\{2,4\},\{2,3\},\{3,4\});\{3,4\}\{1,3\}]$ and $[(\{2,4\},\{1,2\},\{1,4\});\{1,4\}\{3,4\}]$. □

Observation 2.1. For a graph G, $S_1(G)$ denotes the graph that arises out of the subdivision of each edge of G exactly once; $S_1(G)$ is the *first subdivision graph* of G. Let $G^\star$ be the graph obtained from G by adding to each edge $e = uv$ of G a new vertex $\{u, v\}$ such that the vertex $\{u, v\}$ is adjacent to both the vertices u and v, and $\{u, v\}$ is a vertex of degree two in $G^\star$; see Figure 2. If we delete all the edges of G in $G^\star$, then the resulting graph is isomorphic to $S_1(G)$, the first subdivision graph of G and hence $G^\star = G \oplus S_1(G)$.

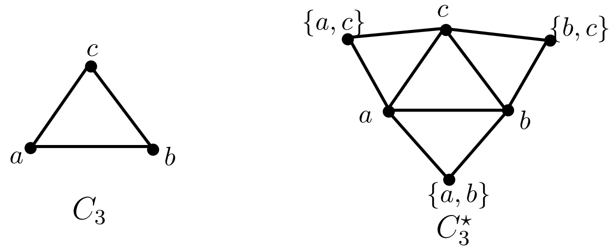


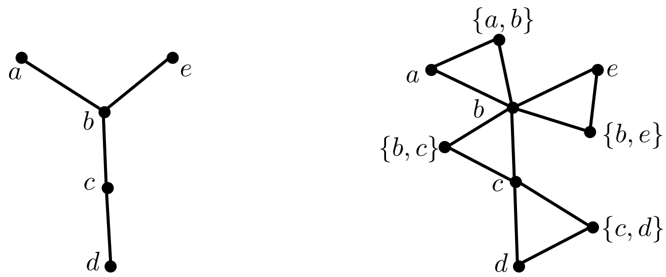
Figure 2: The graphs C_3 and $C_3^\star$.

Observation 2.2. Let $V(L(K_{n+1})) = \mathcal{P}_2(\{1, 2, \dots, n+1\})$. We partition the vertex set of $L(K_{n+1})$ into two sets A_1 and A_2, where $V(A_1) = \mathcal{P}_2(\{1, 2, \dots, n\})$ and $V(A_2) = \cup_{i=1}^{n}\{i, n+1\}$. The subgraph of $L(K_{n+1})$ induced by A_1 (respectively, A_2) is isomorphic to $L(K_n)$ (respectively, K_n). Clearly, $E(A_1, A_2)$, in $L(K_{n+1})$, is $\{\{i,j\}\{i,n+1\}, \{i,j\}\{j,n+1\}\}$, $1 \le i < j \le n$; note that each 2-element subset represents a vertex in the line graph. Then $L(K_{n+1}) = \langle A_1\rangle \oplus \langle A_2\rangle \oplus \langle E(A_1, A_2)\rangle = L(K_n) \oplus K_n^\star$; see Observation 2.1.

Let T denote the tree on five vertices $\{a, b, c, d, e\}$ with edge set $\{ab, bc, cd, be\}$; we denote this T by $[a, b, c, d; be]$; see Figure 3. We use this T at many places in the later part of this paper.

Lemma 2.2. *If G admits a T-decomposition, then $G^\star$ admits a K-decomposition.*

Proof. Let the vertices of T be $\{a, b, c, d, e\}$. Since G has a T-decomposition, $G^\star = T^\star \oplus T^\star \oplus \cdots \oplus T^\star$. A K-decomposition of $T^\star$ is given here:



The tree $T = [a, b, c, d; be]$. $T^{\star}$

Figure 3: The graphs T and $T^{\star}$ are shown above.

$[(a, \{a, b\}, b); bc], [(e, \{b, e\}, b); b\{b, c\}]$ and $[(d, \{c, d\}, c); c\{b, c\}]$.

□

Lemma 2.3. *The graph $K^{\star}_{4,6}$ admits a K-decomposition.*

Proof. By Lemma 2.2, it suffices to prove that $K_{4,6}$ admits a T-decomposition. Let $V(K_{4,6}) = X \cup Y$, where $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, \dots, y_6\}$. A T-decomposition of $K_{4,6}$ is $[y_1, x_1, y_3, x_3; x_1y_2]$, $[y_4, x_1, y_5, x_3; x_1y_6]$, $[y_4, x_3, y_1, x_2; x_3y_2]$, $[y_4, x_2, y_6, x_3; x_2y_5]$, $[y_1, x_4, y_2, x_2; x_4y_6]$ and $[y_5, x_4, y_3, x_2; x_4y_4]$. □

Lemma 2.4. *For all $k \geq 1$, each of the graphs $K^{\star}_{4,8k}, K^{\star}_{6,8k}$ and $K^{\star}_{10,8k}$ admits a K-decomposition.*

Proof. By Lemma 2.2, it suffices to show that the graphs $K_{4,8k}$, $K_{6,8k}$ and $K_{10,8k}$, $k \geq 1$, admit T-decompositions.

(i) Clearly, $K_{4,8k} = \underbrace{K_{4,4} \oplus K_{4,4} \oplus \cdots \oplus K_{4,4}}_{2k-times}$. Let $V(K_{4,4}) = X \cup Y$, where $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4\}$. Let $\rho = (x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4)$ be a permutation on $V(K_{4,4})$. A T-decomposition of $K_{4,4}$ is $\rho^i[y_1, x_1, y_2, x_4; x_1y_4]$, where $\rho^i[y_1, x_1, y_2, x_4; x_1y_4] = [\rho^i(y_1), \rho^i(x_1), \rho^i(y_2), \rho^i(x_4); \rho^i(x_1)\rho^i(y_4)]$, $0 \leq i \leq 3$.

(ii) Clearly, $K_{6,8k} = \underbrace{K_{6,8} \oplus K_{6,8} \oplus \cdots \oplus K_{6,8}}_{k-times}$, where $X = \{x_1, x_2, \dots, x_6\}$ and $Y = \{y_1, y_2, \dots, y_8\}$ is the bipartition of $K_{6,8}$. Let $\rho = (x_1, x_2 \dots, x_6)(y_1, y_2, \dots, y_6)(y_7)(y_8)$ be a permutation on $V(K_{6,8})$. A T-decomposition of $K_{6,8}$ is $\rho^i[y_7, x_1, y_2, x_6; x_1y_1]$ and $\rho^i[y_8, x_6, y_4, x_1; x_6y_5]$, where $0 \leq i \leq 5$.

(iii) Clearly, the graph $K_{10,8k} = \underbrace{K_{10,8} \oplus K_{10,8} \oplus \cdots \oplus K_{10,8}}_{k-times}$. We now produce a T-decomposition of $K_{10,8}$. The graph $K_{8,10}$ is the union of four edge-disjoint copies of $K_{4,5}$. A T-decomposition of $K_{4,5}$ is described here, where we assume that the bipartition (X, Y) of $K_{4,5}$ is $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$:
$[y_1, x_1, y_3, x_4; x_1y_2]$, $[y_3, x_2, y_1, x_4; x_2y_2]$, $[y_1, x_3, y_2, x_4; x_3y_3]$, $[y_5, x_4, y_4, x_3; y_4x_2]$ and $[y_4, x_1, y_5, x_2; y_5x_3]$. □

Lemma 2.5. *For each $L \in \{P_3, 2K_2\}$, there exists a maximum K-packing of $K_4^\star$ with leave L.*

Proof. Let $V(K_4) = \{1, 2, 3, 4\}$. A K-packing of $K_4^\star$ with leave P_3 is $[(1, \{1, 3\}, 3); 34]$, $[(1, \{1, 4\}, 4); 4\{3, 4\}]$, $[(2, \{2, 3\}, 3); 3\{3, 4\}]$, $[(4, \{2, 4\}, 2); 21]$ and the leave is $\{1\{1, 2\}, \{1, 2\}2\}$.

A K-packing of $K_4^\star$ with leave $2K_2$ is $[(1, \{1, 4\}, 4); 4\{3, 4\}]$, $[(3, \{1, 3\}, 1); 12]$, $[(2, \{2, 4\}, 4); 43]$, $[(3, \{2, 3\}, 2); 2\{1, 2\}]$ and the leave is $\{3\{3, 4\}, 1\{1, 2\}\}$. □

Lemma 2.6. *The graph $P_4^\star$ has a maximum K-packing with leave K_2.*

Proof. Let $V(P_4) = \{a, b, c, d\}$. A K-packing of $P_4^\star$ with leave K_2 is the following: $[(a, \{a, b\}, b); b\{b, c\}]$, $[(d, \{c, d\}, c); bc]$ and the leave is $c\{b, c\}$.

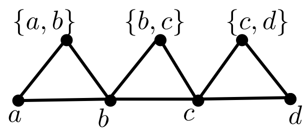


Figure 4: The graph $P_4^\star$.

□

Lemma 2.7. *The graph $K_6^\star$ has a maximum K-packing with leave K_2.*

Proof. Let $V(K_6) = \{1, 2, \dots, 6\}$. A T-packing of K_6 with leave P_4 is $[1, 2, 3, 4; 25]$, $[4, 5, 3, 1; 56]$, $[4, 6, 1, 5; 63]$ and the leave is $\{14, 42, 26\}$. Hence the graph $K_6^\star$ has a $T^\star$-packing with leave $P_4^\star$. By Lemma 2.2, the graph $T^\star$ has a K-decomposition. Thus the graph $K_6^\star$ has a K-packing with leave K_2, because $P_4^\star$ can be decomposed into two copies of the kite K and one copy of K_2, by Lemma 2.6. □

Let M be the graph with vertex set $\{a, b, c, d, e, f\}$ and edge set $\{ab, bc, cd, de, cf\}$. We denote M by $[a, b, c, d, e; cf]$; see Figure 5.

Lemma 2.8. *For each $L \in \{P_4, K_{1,3}\}$, there exists a maximum K-packing of $M^\star$ with leave L.*

Proof. A K-packing of $M^\star$ with leave P_4 is $[(e, \{d, e\}, d); dc]$, $[(a, \{a, b\}, b); b\{b, c\}]$, $[(f, \{c, f\}, c); c\{b, c\}]$, and the leave is $\{bc, c\{c, d\}, \{c, d\}d\}$. A K-packing of $M^\star$ with leave $K_{1,3}$ is $[(a, \{a, b\}, b); b\{b, c\}]$, $[(f, \{c, f\}, c); c\{c, d\}]$, $[(e, \{d, e\}, d); d\{c, d\}]$, and the leave is $\{cb, c\{b, c\}, cd\}$.

□

Lemma 2.9. *For each $L \in \{K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2\}$, there exists a maximum K-packing of $K_{10}^\star$ with leave L.*

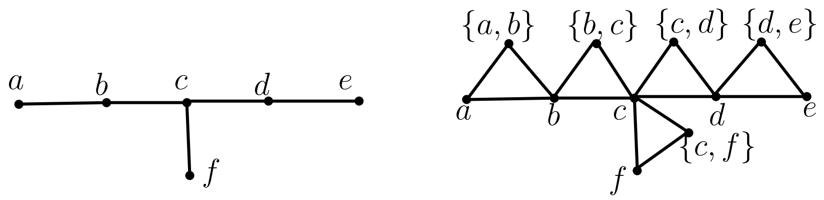


Figure 5: The graphs M and $M^\star$.

Proof. In this lemma, for convenience we assume that $V(K_{10}) = \{a_1, a_2, \dots, a_{10}\}$.

(i) A K-packing of $K_{10}^\star$ with leave K_3 is obtained through a T-packing of K_{10}. A T-packing of K_{10} with leave K_2 is

$$\begin{array}{lll} [a_1, a_2, a_3, a_4; a_2a_5], & [a_1, a_3, a_5, a_9; a_3a_6], & [a_1, a_5, a_7, a_3; a_7a_4], \\ [a_1, a_6, a_5, a_4; a_6a_7], & [a_7, a_9, a_1, a_4; a_1a_{10}], & [a_9, a_2, a_8, a_4; a_8a_5], \\ [a_2, a_6, a_{10}, a_5; a_{10}a_7], & [a_3, a_8, a_9, a_{10}; a_8a_7], & [a_7, a_1, a_8, a_{10}; a_8a_6], \\ [a_{10}, a_2, a_4, a_6; a_2a_7], & [a_{10}, a_3, a_9, a_4; a_9a_6], & \end{array}$$

and the leave is the edge a_4a_{10}. Since K_{10} has a T-packing with leave K_2, $K_{10}^\star$ admits a $T^\star$-packing with leave $K_2^\star$. By Lemma 2.2, the graph $K_{10}^\star$ has a K-packing with leave K_3; observe that $K_2^\star \simeq K_3$.

(ii) A T-packing of K_{10} with leave M is

$$\begin{array}{lll} [a_1, a_2, a_3, a_4; a_2a_5], & [a_1, a_3, a_5, a_9; a_3a_6], & [a_1, a_5, a_7, a_3; a_7a_4], \\ [a_1, a_6, a_5, a_4; a_6a_7], & [a_9, a_2, a_8, a_4; a_8a_5], & [a_2, a_6, a_{10}, a_5; a_{10}a_7], \\ [a_3, a_8, a_9, a_{10}; a_8a_7], & [a_7, a_1, a_{10}, a_2; a_1a_9], & [a_7, a_9, a_6, a_8; a_6a_4], \\ [a_7, a_2, a_4, a_9; a_4a_1], & & \end{array}$$

with leave $\{a_1a_8, a_8a_{10}, a_{10}a_3, a_3a_9, a_{10}a_4\}$. Hence the graph $K_{10}^\star$ admits a $T^\star$-packing with leave $M^\star$. By Lemma 2.2, $T^\star$ has a K-decomposition. Thus the graph $K_{10}^\star$ has a K-packing with leave P_4, because $M^\star$ has three copies of the kite K and one copy of P_4, by Lemma 2.8.

(iii) Next we describe a K-packing of $K_{10}^\star$ with leave $K_{1,3}$. From (ii) above, $K_{10}^\star$ has a $T^\star$-packing with leave $M^\star$. By Lemmas 2.2 and 2.8, the result follows.

(iv) A K-packing of $K_{10}^\star$ with leave $3K_2$ is obtained by taking a decomposition of $K_{10}^\star$ into $K_4^\star$, $K_6^\star$ and $K_{4,6}^\star$. Now apply Lemmas 2.5, 2.7 and 2.3 to the appropriate graphs to get a desired K-packing with leave $3K_2$.

(v) A K-packing of $K_{10}^\star$ with leave $P_3 \cup K_2$ is obtained by taking a decomposition of $K_{10}^\star$ into $K_4^\star$, $K_6^\star$ and $K_{4,6}^\star$. Applying Lemmas 2.5, 2.7 and 2.3 to the appropriate graphs, $K_{10}^\star$ has a K-packing with leave $P_3 \cup K_2$. □

We use the following theorem in the proof of Lemma 2.11.

Theorem 2.10. [12] *The graph K_n admits a T-decomposition if and only if $n \equiv 0$ or 1 (mod 8).*

By Lemma 2.2 and Theorem 2.10, both the graphs $K^\star_{8k}$ and $K^\star_{8k+1}$, $k \geq 1$, admit K-decompositions.

Lemma 2.11. *For all $k \geq 1$ and for each $i \in \{0, 2, 4, 6\}$, the graph $K^\star_{8k+i}$ admits a maximum K-packing with every possible leave. The possible leaves are given in the table below:*

i	*Possible leaves*
0	$\emptyset$
2	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2$
4	$P_3, 2K_2$
6	K_2

Proof. For $i = 0$, the graph K_{8k} has a T-decomposition by Theorem 2.10, and the graph $K^\star_{8k}$ admits a K-decomposition by Lemma 2.2. For $i = 2$, the graph $K^\star_{8k+2} = K^\star_{10} \oplus K^\star_{8(k-1)} \oplus K^\star_{10,8(k-1)}$. By Theorem 2.10 and Lemma 2.4, the graphs $K^\star_{8(k-1)}$ and $K^\star_{10,8(k-1)}$ have K-decompositions, and the rest follows by Lemma 2.9. For $i = 4$, the graph $K^\star_{8k+4} = K^\star_4 \oplus K^\star_{8k} \oplus K^\star_{4,8k}$. Now by Theorem 2.10, Lemmas 2.4 and 2.5, we get a K-packing with leave P_3 or $2K_2$. For $i = 6$, the graph $K^\star_{8k+6} = K^\star_6 \oplus K^\star_{8k} \oplus K^\star_{6,8k}$. Now applying Theorem 2.10, Lemmas 2.4 and 2.7 to the respective graphs, we get a K-packing with leave K_2. □

3 Maximum packing of $L(K_n)$ with kites

Let G be the graph given in Figure 6. From the proof of Theorem 1.4 of [8], we have the following lemma.

Lemma 3.1. [8] *For $n = 2k$, $k \geq 3$, the edge set of $L(K_n)$ can be partitioned into $\binom{k}{2}$ copies of $L(K_4)$ and $\binom{k}{3}$ copies of G, where G is isomorphic to the graph given in Figure 6.*

Lemma 3.2. *The graph G, (given in Figure 6) has a K-decomposition.*

Proof. A K-decomposition of G consists of the kites

$$[(a_2, b_2, c_4); c_4b_4], \quad [(a_3, c_3, b_4); b_4a_1], \quad [(a_3, b_3, c_1); c_1a_4],$$
$$[(c_2, b_3, a_1); a_1c_4], \quad [(c_2, a_2, b_1); b_1c_1], \quad [(c_3, b_2, a_4); a_4b_1].$$

□

Lemma 3.3. *If $n \geq 4$ is even, then the graph $L(K_n)$ has a K-decomposition.*

Proof. Let $n = 2k$, $k \geq 2$. The case $k = 2$ follows by Lemma 2.1. If $k \geq 3$, then the graph $L(K_n) = \underbrace{L(K_4) \oplus L(K_4) \oplus \cdots \oplus L(K_4)}_{\binom{k}{2}-times} \oplus \underbrace{G \oplus G \oplus \cdots \oplus G}_{\binom{k}{3}-times}$, by Lemma 3.1. We obtain a K-decomposition of $L(K_{2k})$ by applying Lemmas 2.1 and 3.2 to the graphs $L(K_4)$ and G, respectively. □

Theorem 3.4. *A maximum K-packing of $L(K_n)$ with every possible leave is given in the following table:*

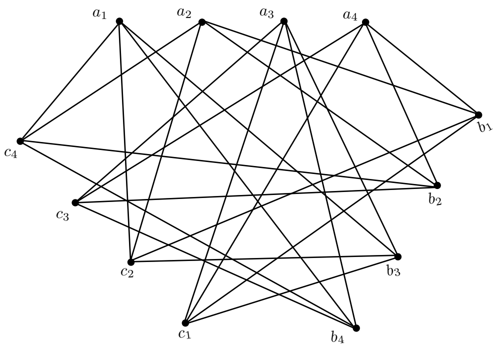


Figure 6: The graph G.

n	*Possible leaves*
$n \equiv 0\,(mod\ 2)$ *or* $1\,(mod\ 8)$	$\emptyset$
$n \equiv 3\,(mod\ 8)$	$K_3, P_4, K_{1,3}, 3K_2, P_2 \cup K_2$
$n \equiv 5\,(mod\ 8)$	$P_3, 2K_2$
$n \equiv 7\,(mod\ 8)$	K_2

Proof. *Case 1:* $n = 2k$, $k \geq 2$.

By Lemma 3.3, $L(K_n)$ has a K-decomposition.

Case 2: $n = 2k + 1$, $k \geq 2$.

Then $n \equiv 1, 3, 5$ or $7 \pmod 8$. Let $n = 8k + i$, $i \in \{1, 3, 5, 7\}$. The graph $L(K_{8k+i}) = L(K_{8k+i-1}) \oplus K^{\star}_{8k+i-1}$, by Observation 2.2. For each $i \in \{1, 3, 5, 7\}$, applying Lemmas 3.3 and 2.11 to the graphs $L(K_{8k+i-1})$ and $K^{\star}_{8k+i-1}$, respectively, we obtain K-packings with every possible leave. □

Proof of Corollary 1.2. The proof immediately follows from the above theorem.

4 Maximum packing of $L(K_n)(\lambda)$ with kites

In this section, we prove the existence of a maximum kite-packing of $L(K_n)(\lambda)$, $\lambda \geq 2$, with every possible leave.

Lemma 4.1. *The graph $K^{\star}_4(2)$ has a K-decomposition.*

Proof. A K-decomposition of $K^{\star}_4(2)$ is

$$[(2, \{1, 2\}, 1); 13], \quad [(2, \{2, 3\}, 3); 3\{1, 3\}], \quad [(4, \{2, 4\}, 2); 2\{2, 3\}],$$
$$[(3, \{1, 3\}, 1); 12], \quad [(4, \{2, 4\}, 2); 2\{1, 2\}], \quad [(4, \{1, 4\}, 1); 1\{1, 3\}],$$
$$[(4, \{3, 4\}, 3); 32], \quad [(4, \{3, 4\}, 3); 3\{2, 3\}], \quad [(4, \{1, 4\}, 1); 1\{1, 2\}].$$
□

Lemma 4.2. *For each $L \in \{P_3, 2K_2, K_2(2)\}$, there exists a maximum K-packing of each of the graphs $K^{\star}_4(3)$ and $K^{\star}_4(5)$ with leave L.*

Proof. (i) Clearly, the graph $K^{\star}_4(3) = K^{\star}_4 \oplus K^{\star}_4(2)$. By Lemmas 2.5 and 4.1, a required K-packing of $K^{\star}_4(3)$ with leave P_3 or $2K_2$ follows.

A K-packing of $K_4^\star(3)$ with leave $K_2(2)$ is

$[(2,\{1,2\},1);13]$, $[(1,\{1,4\},4);4\{2,4\}]$, $[(2,\{2,3\},3);3\{1,3\}]$,
$[(3,\{1,3\},1);12]$, $[(3,\{2,3\},2);2\{2,4\}]$, $[(3,\{1,3\},1);1\{1,2\}]$,
$[(3,\{3,4\},4);42]$, $[(4,\{2,4\},2);2\{2,3\}]$, $[(4,\{3,4\},3);3\{2,3\}]$,
$[(4,\{3,4\},3);32]$, $[(4,\{1,4\},1);1\{1,3\}]$, $[(4,\{1,4\},1);1\{1,2\}]$,
$[(4,\{2,4\},2);21]$.

and the leave is $\{2\{1,2\},2\{1,2\}\}$.

(ii) A K-packing of $K_4^\star(5)$ with leave $L \in \{P_3, 2K_2, K_2(2)\}$ follows by (i) above and Lemma 4.1, since $K_4^\star(5) = K_4^\star(2) \oplus K_4^\star(3)$. □

The following lemma is an easy observation.

Lemma 4.3. *If $H|G$ then $H|G(\lambda)$, for any $\lambda \geq 2$.* □

Lemma 4.4. *The graphs $K_6^\star(2), K_6^\star(3)$ and $K_6^\star(5)$ have maximum K-packings with leave $2K_2, K_{1,3}$ and K_2, respectively.*

Proof. From the proof of Lemma 2.7, the graph $K_6 = T_1 \oplus T_2 \oplus T_3 \oplus P_4$, where $T_1 = [1,2,3,4;25]$, $T_2 = [4,5,3,1;56]$, $T_3 = [4,6,1,5;63]$ and $P_4 = [1,4,2,6]$. Here T_1, T_2 and T_3 are isomorphic to T, and P_4 is the path of length 3. Then $K_6^\star = T_1^\star \oplus T_2^\star \oplus T_3^\star \oplus P_4^\star$. Now consider the graph $K_6^\star = T_1^\star \oplus T_2^\star \oplus H^\star$, where $H^\star = T_3^\star \oplus P_4^\star$; see Figure 7.

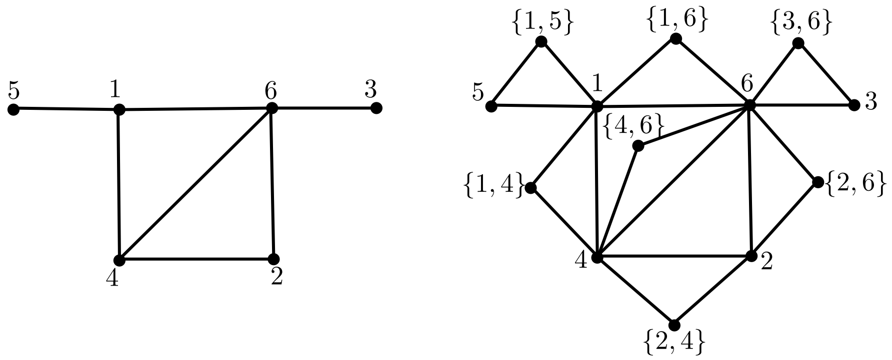


Figure 7: The graphs H and $H^\star$.

(i) The graph $K_6^\star(2) = T_1^\star(2) \oplus T_2^\star(2) \oplus H^\star(2)$. By Lemmas 2.2 and 4.3, the graphs $T_1^\star(2)$ and $T_2^\star(2)$ have K-decompositions. A K-packing of $H^\star(2)$ with leave $2K_2$ is

$[(1,\{1,4\},4);16]$, $[(2,\{2,4\},4);4\{1,4\}]$, $[(3,\{3,6\},6);6\{1,6\}]$,
$[(2,\{2,6\},6);46]$, $[(2,\{2,6\},6);6\{4,6\}]$, $[(3,\{3,6\},6);6\{1,6\}]$,
$[(2,\{2,4\},4);14]$, $[(5,\{1,5\},1);1\{1,6\}]$, $[(5,\{1,5\},1);1\{1,6\}]$,
$[(4,\{4,6\},6);16]$.

and the leave is $\{1\{1,4\},4\{4,6\}\}$.

(ii) The graph $K_6^\star(3) = T_1^\star(3) \oplus T_2^\star(3) \oplus H^\star(3)$. The graphs $T_1^\star(3)$ and $T_2^\star(3)$ have K-decompositions, by Lemmas 2.2 and 4.3.

A K-packing of $H^\star(3)$ with leave $K_{1,3}$ is

$$\begin{array}{lll}
[(1,\{1,4\},4);4\{2,4\}], & [(1,\{1,4\},4);4\{4,6\}], & [(1,\{1,4\},4);42], \\
[(3,\{3,6\},6);6\{1,6\}], & [(3,\{3,6\},6);6\{1,6\}], & [(4,\{4,6\},6);61], \\
[(3,\{3,6\},6);6\{1,6\}], & [(5,\{1,5\},1);1\{1,6\}], & [(4,\{2,4\},2);26], \\
[(4,\{2,4\},2);2\{2,6\}], & [(5,\{1,5\},1);1\{1,6\}], & [(4,\{4,6\},6);61], \\
[(5,\{1,5\},1);1\{1,6\}], & [(6,\{2,6\},2);2\{2,4\}], & [(6,\{2,6\},2);16].
\end{array}$$

and the leave is $\{6\{4,6\},6\{2,6\},64\}$.

(iii) Clearly, the graph $K_6^\star(5) = K_6^\star \oplus K_6^\star(4) = K_6^\star \oplus T_1^\star(4) \oplus T_2^\star(4) \oplus H^\star(4)$. By Lemmas 2.2 and 4.3, the graphs $T_1^\star(4)$ and $T_2^\star(4)$ have K-decompositions, and by Lemma 2.7 the graph $K_6^\star$ has a K-packing with leave K_2. A K-decomposition of $H^\star(4)$ is

$$\begin{array}{lll}
[(1,\{1,4\},4);4\{2,4\}], & [(2,\{2,6\},6);2\{2,4\}], & [(1,\{1,4\},4);16], \\
[(1,\{1,4\},4);4\{2,4\}], & [(2,\{2,4\},4);4\{4,6\}], & [(1,\{1,4\},4);24], \\
[(2,\{2,6\},6);2\{2,4\}], & [(3,\{3,6\},6);6\{1,6\}], & [(2,\{2,6\},6);24], \\
[(2,\{2,6\},6);6\{4,6\}], & [(3,\{3,6\},6);6\{1,6\}], & [(2,\{2,4\},4);46], \\
[(3,\{3,6\},6);6\{1,6\}], & [(5,\{1,5\},1);1\{1,6\}], & [(4,\{4,6\},6);16], \\
[(3,\{3,6\},6);6\{1,6\}], & [(5,\{1,5\},1);1\{1,6\}], & [(4,\{4,6\},6);16], \\
[(5,\{1,5\},1);1\{1,6\}], & [(5,\{1,5\},1);1\{1,6\}], & [(4,\{4,6\},6);16].
\end{array}$$

□

Lemma 4.5. *For each $L \in \{P_3, 2K_2, K_2(2)\}$, there exists a maximum K-packing of $K_6^\star(2)$ with leave L.*

Proof. *(i)* From the proof of Lemma 2.7, the graph $K_6 = T \oplus T \oplus T \oplus P_4$. The graph $K_6^\star(2) = T^\star(2) \oplus T^\star(2) \oplus T^\star(2) \oplus P_4^\star(2)$. By Lemmas 2.2 and 4.3, the graph $T^\star(2)$ has a K-decomposition. A K-packing of $P_4^\star(2)$ with leave P_3 when $P_4 = [1,4,2,6]$ is $[(1,\{1,4\},4);24]$, $[(1,\{1,4\},4);4\{2,4\}]$, $[(6,\{2,6\},2);24]$, $[(6,\{2,6\},2);2\{2,4\}]$, and the leave is $\{2\{2,4\},\{2,4\}4\}$.

(ii) A K-packing of $K_6^\star(2)$ with leave $2K_2$ follows by Lemma 4.4.

(iii) Since $K_6^\star$ has a K-packing with leave K_2, $K_6^\star(2)$ has a K-packing with leave $K_2(2)$, by Lemma 2.7. □

Recall that P' is the multigraph on three vertices a, b, c and three edges ab, ab, bc. We use it often in the rest of the paper.

Lemma 4.6. *For each $L \in \{K_3, P_4, P_3 \cup K_2, P'\}$, there exists a maximum K-packing of $P_4^\star(3)$ with leave L.*

Proof. Let $V(P_4) = \{a, b, c, d\}$.

(i) A K-packing of $P_4^\star(3)$ with leave K_3 is

$$\begin{array}{lll}
[(a,\{a,b\},b);bc], & [(a,\{a,b\},b);b\{b,c\}], & [(a,\{a,b\},b);b\{b,c\}], \\
[(d,\{c,d\},c);bc], & [(d,\{c,d\},c);c\{b,c\}], & [(d,\{c,d\},c);c\{b,c\}]
\end{array}$$

and the leave is $\{\{b,c\}c, cb, b\{b,c\}\}$.

(ii) A K-packing of $P_4^\star(3)$ with leave P_4 is

$$[(a,\{a,b\},b);bc],\quad [(d,\{c,d\},c);bc],\quad [(a,\{a,b\},b);b\{b,c\}],$$
$$[(b,\{b,c\},c);cd],\quad [(a,\{a,b\},b);b\{b,c\}],\quad [(d,\{c,d\},c);c\{b,c\}]$$

and the leave is $\{\{b,c\}c, c\{c,d\}, \{c,d\}d\}$.

(iii) A K-packing of $P_4^\star(3)$ with leave $P_3 \cup K_2$ is

$$[(a,\{a,b\},b);bc],\quad [(a,\{a,b\},b);bc],\quad [(a,\{a,b\},b);b\{b,c\}],$$
$$[(b,\{b,c\},c);c\{c,d\}],\quad [(d,\{c,d\},c);c\{b,c\}],\quad [(d,\{c,d\},c);c\{b,c\}]$$

and the leave is $\{b\{b,c\}, cd, d\{c,d\}\}$.

(iv) A K-packing of $P_4^\star(3)$ with leave P' is

$$[(a,\{a,b\},b);bc],\quad [(a,\{a,b\},b);bc],\quad [(a,\{a,b\},b);b\{b,c\}],$$
$$[(d,\{c,d\},c);bc],\quad [(d,\{c,d\},c);c\{b,c\}],\quad [(d,\{c,d\},c);c\{b,c\}]$$

and the leave is $\{b\{b,c\}, b\{b,c\}, \{b,c\}c\}$. □

Lemma 4.7. *For each $L \in \{K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)\}$, there exists a maximum K-packing of $K_6^\star(3)$ with leave L.*

Proof. (i) A K-packing of $K_6^\star(3)$ with leave $K_3, P_4, P_3 \cup K_2$ or P' is described below. Since the graph $K_6 = T \oplus T \oplus T \oplus P_4$ (see the proof of Lemma 2.7), the graph $K_6^\star(3) = T^\star(3) \oplus T^\star(3) \oplus T^\star(3) \oplus P_4^\star(3)$. By Lemmas 2.2, 4.3 and 4.6, the graph $K_6^\star(3)$ has a K-packing with leave $L \in \{K_3, P_4, P_3 \cup K_2, P'\}$.

(ii) The graph $K_6^\star(3) = K_6^\star \oplus K_6^\star(2)$. For each $L \in \{K_2 \cup K_2(2), 3K_2\}$, the graph $K_6^\star(3)$ has a K-packing with leave L, by Lemmas 2.7 and 4.5.

(iii) Since $K_6^\star$ has a K-packing with leave K_2, $K_6^\star(3)$ has a K-packing with leave $K_2(3)$, by Lemma 2.7.

(iv) A K-packing of $K_6^\star(3)$ with leave $K_{1,3}$ follows by Lemma 4.4. □

Lemma 4.8. *For each $L \in \{P_3, 2K_2, K_2(2)\}$, there exists a maximum K-packing of $K_{10}^\star(2)$ with leave L.*

Proof. Clearly, $K_{10}^\star(2) = K_4^\star(2) \oplus K_6^\star(2) \oplus K_{4,6}^\star(2)$. By Lemmas 2.3, 4.1 and 4.3, the graphs $K_4^\star(2)$ and $K_{4,6}^\star(2)$ have K-decompositions. Thus a required K-packing follows by Lemma 4.5. □

Recall that the graph $M = [a,b,c,d,e;cf]$; see Figure 5.

Lemma 4.9. *The graph $M^\star(3)$ has a maximum K-packing with leave K_2.*

Proof. A K-packing of $M^\star(3)$ with leave K_2 is

$$[(a,\{a,b\},b);b\{b,c\}],\quad [(a,\{a,b\},b);bc],\quad [(a,\{a,b\},b);b\{b,c\}],$$
$$[(b,\{b,c\},c);c\{c,d\}],\quad [(c,\{c,d\},d);de],\quad [(e,\{d,e\},d);d\{c,d\}],$$
$$[(f,\{c,f\},c);c\{b,c\}],\quad [(f,\{c,f\},c);bc],\quad [(c,\{c,d\},d);d\{d,e\}],$$
$$[(f,\{c,f\},c);c\{b,c\}],\quad [(e,\{d,e\},d);cd]$$

and the leave is $e\{d, e\}$. □

Lemma 4.10. *For each $L \in \{P', K_2 \cup K_2(2), K_2(3)\}$, there exists a maximum K-packing of $M^\star(5)$ with leave L.*

Proof. (i) A K-packing of $M^\star(5)$ with leave P' is

$$\begin{array}{lll} [(a,\{a,b\},b);bc], & [(a,\{a,b\},b);b\{b,c\}], & [(e,\{d,e\},d);cd],\\ [(a,\{a,b\},b);bc], & [(a,\{a,b\},b);b\{b,c\}], & [(b,\{b,c\},c);c\{c,d\}],\\ [(b,\{b,c\},c);cd], & [(d,\{c,d\},c);c\{b,c\}], & [(e,\{d,e\},d);d\{c,d\}],\\ [(f,\{c,f\},c);bc], & [(e,\{d,e\},d);d\{c,d\}], & [(a,\{a,b\},b);b\{b,c\}],\\ [(f,\{c,f\},c);cd], & [(e,\{d,e\},d);d\{c,d\}], & [(e,\{d,e\},d);d\{c,d\}],\\ [(f,\{c,f\},c);cd], & [(f,\{c,f\},c);c\{c,d\}], & [(f,\{c,f\},c);c\{c,d\}] \end{array}$$

and the leave is $\{\{b, c\}c, \{b, c\}c, c\{c, d\}\}$.

(ii) A K-packing of $M^\star(5)$ with leave $K_2 \cup K_2(2)$ is

$$\begin{array}{lll} [(a,\{a,b\},b);bc], & [(a,\{a,b\},b);b\{b,c\}], & [(f,\{c,f\},c);bc],\\ [(d,\{c,d\},c);bc], & [(a,\{a,b\},b);b\{b,c\}], & [(b,\{b,c\},c);c\{c,d\}],\\ [(c,\{c,d\},d);de], & [(c,\{c,d\},d);d\{d,e\}], & [(a,\{a,b\},b);b\{b,c\}],\\ [(e,\{d,e\},d);cd], & [(e,\{d,e\},d);d\{c,d\}], & [(e,\{d,e\},d);d\{c,d\}],\\ [(e,\{d,e\},d);cd], & [(f,\{c,f\},c);c\{c,d\}], & [(a,\{a,b\},b);b\{b,c\}],\\ [(f,\{c,f\},c);bc], & [(f,\{c,f\},c);c\{b,c\}], & [(f,\{c,f\},c);c\{b,c\}] \end{array}$$

and the leave is $\{\{b, c\}c, \{b, c\}c, \{d, e\}e\}$.

(iii) A K-packing of $M^\star(5)$ with leave $K_2(3)$ is

$$\begin{array}{lll} [(a,\{a,b\},b);bc], & [(a,\{a,b\},b);b\{b,c\}], & [(f,\{c,f\},c);cd],\\ [(d,\{c,d\},c);bc], & [(a,\{a,b\},b);b\{b,c\}], & [(b,\{b,c\},c);c\{c,d\}],\\ [(e,\{d,e\},d);cd], & [(a,\{a,b\},b);b\{b,c\}], & [(e,\{d,e\},d);d\{c,d\}],\\ [(e,\{d,e\},d);cd], & [(e,\{d,e\},d);d\{c,d\}], & [(d,\{c,d\},c);c\{b,c\}],\\ [(f,\{c,f\},c);bc], & [(f,\{c,f\},c);c\{c,d\}], & [(f,\{c,f\},c);c\{c,d\}],\\ [(f,\{c,f\},c);bc], & [(e,\{d,e\},d);d\{c,d\}], & [(a,\{a,b\},b);b\{b,c\}] \end{array}$$

and the leave is $\{\{b, c\}c, \{b, c\}c, \{b, c\}c\}$. □

Lemma 4.11. *The graph $K^\star_{10}(3)$ has a maximum K-packing with leave K_2.*

Proof. Since the graph $K^\star_{10}$ has a $T^\star$-packing with leave $M^\star$ (see the proof of Lemma 2.9), the graph $K^\star_{10}(3) = T^\star(3) \oplus T^\star(3) \oplus \cdots \oplus T^\star(3) \oplus M^\star(3)$. The graph $K^\star_{10}(3)$ has a K-packing with leave K_2, by applying Lemmas 2.2 and 4.3 to the graph $T^\star(3)$ and Lemma 4.9 to the graph $M^\star(3)$. □

Lemma 4.12. *For each $L \in \{K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)\}$ there exists a maximum K-packing of $K^\star_{10}(5)$ with leave L.*

Proof. (i) The graph
$$\begin{aligned} K^\star_{10}(5) &= K^\star_{10} \oplus K^\star_{10}(4)\\ &= K^\star_{10} \oplus K^\star_4(4) \oplus K^\star_6(4) \oplus K^\star_{4,6}(4). \end{aligned}$$

From the proof of Lemma 4.4 (*iii*) and Lemmas 4.1, 2.3 and 4.3, the graphs $K_4^\star(4)$, $K_6^\star(4)$ and $K_{4,6}^\star(4)$ have K-decompositions. Now a K-packing of $K_{10}^\star(5)$ with leave $L \in \{K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2\}$ follows by Lemma 2.9.

(*ii*) From the proof of Lemma 2.9, the graph $K_{10}^\star$ has a $T^\star$-packing with leave $M^\star$ and hence the graph $K_{10}^\star(5)$ has a $T^\star(5)$-packing with leave $M^\star(5)$. The required K-packing with leave $L \in \{P', K_2 \cup K_2(2), K_2(3)\}$ follows by applying Lemmas 2.2 and 4.3 to the graph $T^\star(5)$ and Lemma 4.10 to the graph $M^\star(5)$. □

Lemma 4.13. *For each* $i \in \{2, 6\}$ *and each* $L \in \{P_3, 2K_2, K_2(2)\}$*, there exists a maximum* K*-packing of* $K_{8k+i}^\star(2)$*,* $k \geq 1$*, with leave* L*.*

Proof. For $i = 2$, the graph $K_{8k+2}^\star(2) = K_{10}^\star(2) \oplus K_{8(k-1)}^\star(2) \oplus K_{10,8(k-1)}^\star(2)$. Now apply Lemmas 2.11, 2.4 and 4.3 to the graphs $K_{8(k-1)}^\star(2)$ and $K_{10,8(k-1)}^\star(2)$ and Lemma 4.8 to the graph $K_{10}^\star(2)$. For $i = 6$, the graph $K_{8k+6}^\star(2) = K_6^\star(2) \oplus K_{8k}^\star(2) \oplus K_{6,8k}^\star(2)$. Now apply Lemmas 2.11, 2.4 and 4.3 to the graphs $K_{8k}^\star(2)$ and $K_{6,8k}^\star(2)$ and Lemma 4.5 to the graph $K_6^\star(2)$. □

Lemma 4.14. *For all* $k \geq 1$ *and each* $i \in \{2, 4, 6\}$ *the graph* $K_{8k+i}^\star(3)$ *admits a maximum* K*-packing with every possible leave as described in the table below:*

i	*Possible leaves*
2	K_2
4	$P_3, 2K_2, K_2(2)$
6	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$

Proof. For $i = 2$, the graph $K_{8k+2}^\star(3) = K_{10}^\star(3) \oplus K_{8(k-1)}^\star(3) \oplus K_{10,8(k-1)}^\star(3)$. By Lemmas 2.11, 2.4 and 4.3, the graphs $K_{8(k-1)}^\star(3)$ and $K_{10,8(k-1)}^\star(3)$ have K-decompositions. The graph $K_{8k+2}^\star(3)$ has a K-packing with leave K_2, by Lemma 4.11. For $i = 4$, the graph $K_{8k+4}^\star(3) = K_4^\star(3) \oplus K_{8k}^\star(3) \oplus K_{4,8k}^\star(3)$. Now by Lemmas 2.11, 2.4 and 4.3, the graphs $K_{8k}^\star(3)$ and $K_{4,8k}^\star(3)$ have K-decompositions. The result now follows by Lemma 4.2. For $i = 6$, the graph $K_{8k+6}^\star(3) = K_6^\star(3) \oplus K_{8k}^\star(3) \oplus K_{6,8k}^\star(3)$. The result follows by Lemmas 2.11, 2.4, 4.3 and 4.7. □

Lemma 4.15. *For all* $k \geq 1$ *and each* $i \in \{2, 4, 6\}$ *the graph* $K_{8k+i}^\star(5)$ *admits a maximum* K*-packings with every possible leave as described in the table below:*

i	*Possible leaves*
2	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
4	$P_3, 2K_2, K_2(2)$
6	K_2

Proof. For $i = 2$, the graph $K_{8k+2}^\star(5) = K_{10}^\star(5) \oplus K_{8(k-1)}^\star(5) \oplus K_{10,8(k-1)}^\star(5)$. By Lemmas 2.11, 2.4 and 4.3, the graphs $K_{8(k-1)}^\star(5)$ and $K_{10,8(k-1)}^\star(5)$ have K-decompositions and, by Lemma 4.12 applied to $K_{10}^\star(5)$, give the required K-packings with every possible leave.

For $i = 4$, the graph $K_{8k+4}^\star(5) = K_4^\star(5) \oplus K_{8k}^\star(5) \oplus K_{4,8k}^\star(5)$. As above, by Lemmas 2.11, 2.4 and 4.3, the graphs $K_{8k}^\star(5)$ and $K_{4,8k}^\star(5)$ have K-decompositions.

Now apply Lemma 4.2 to $K_4^\star(5)$ to get a required K-packing with every possible leave.

For $i = 6$, the graph $K_{8k+6}^\star(5) = K_6^\star(5) \oplus K_{8k}^\star(5) \oplus K_{6,8k}^\star(5)$. By Lemmas 2.11, 2.4 and 4.3, the graphs $K_{8k}^\star(5)$ and $K_{6,8k}^\star(5)$ have K-decompositions. The result now follows by applying Lemma 4.4 to the graph $K_6^\star(5)$. □

We use the following theorem in the proof of the next lemma.

Theorem 4.16. [14] *Let α and β be non-negative integers. For any integer $n \geq 4$ and $\lambda \geq 1$, the graph $K_n(\lambda)$ has a $\{K^\alpha, C_4^\beta\}$-decomposition if and only if $4(\alpha+\beta) = \lambda\binom{n}{2}$, where K denotes the kite.*

Lemma 4.17. *For $\lambda \geq 2$ and $n \geq 4$, the graph $L(K_n)(\lambda)$ admits a K-decomposition if and only if n is even or $n \equiv 1 \pmod 8$ or $\lambda \equiv 0 \pmod 4$.*

Proof. The proof of the necessity is obvious. We prove the sufficiency. If $n \equiv 0 \pmod 2$ or $n \equiv 1 \pmod 8$, the proof follows by Corollary 1.2 and Lemma 4.3. If $\lambda \equiv 0 \pmod 4$, then $\lambda = 4k'$, for some $k' \geq 1$. The graph $L(K_n)(4k') = K_{n-1}(4k') \oplus K_{n-1}(4k') \oplus \cdots \oplus K_{n-1}(4k')$, as the star at each vertex of K_n yields a K_{n-1} in $L(K_n)$. By Theorem 4.16, a K-decomposition of $L(K_n)(\lambda)$ exists. □

Lemma 4.18. *For each $\lambda \in \{2, 3, 5\}$, the graph $L(K_n)(\lambda)$ admits a maximum K-packing with every possible leave as described in the table below:*

λ	$n \equiv a \pmod 8$	*Possible leaves*
2	$a = 5$	$\emptyset$
	$a \in \{3, 7\}$	$P_3, 2K_2, K_2(2)$
3	$a = 3$	K_2
	$a = 5$	$P_3, 2K_2, K_2(2)$
	$a = 7$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
5	$a = 3$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
	$a = 5$	$P_3, 2K_2, K_2(2)$
	$a = 7$	K_2

Proof. Case 1. $\lambda = 2$.
If $n \equiv 3$ or $7 \pmod 8$, then for each $i \in \{3, 7\}$, the graph $L(K_{8k+i})(2) = L(K_{8k+i-1})(2) \oplus K_{8k+i-1}^\star(2)$, by Observation 2.2. We apply Lemmas 4.17 and 4.13 to the graphs $L(K_{8k+i-1})(2)$ and $K_{8k+i-1}^\star(2)$ to get a required maximum K-packing. If $n \equiv 5 \pmod 8$, then the graph $L(K_{8k+5})(2) = K_{8k+4}(2) \oplus K_{8k+4}(2) \oplus \cdots \oplus K_{8k+4}(2)$, as the star at each vertex of K_{8k+5} yields a K_{8k+4} in $L(K_{8k+5})$. Now apply Theorem 4.16 to the graph $K_{8k+4}(2)$.

Case 2. $\lambda = 3$.
If $n \equiv 3$, 5 or $7 \pmod 8$, then for each $i \in \{3, 5, 7\}$, the graph $L(K_{8k+i})(3) = L(K_{8k+i-1})(3) \oplus K_{8k+i-1}^\star(3)$, by Observation 2.2. We apply Lemmas 4.17 and 4.14 to the graphs $L(K_{8k+i-1})(3)$ and $K_{8k+i-1}^\star(3)$ to get a maximum K-packing.

Case 3. $\lambda = 5$.

If $n \equiv 3, 5$ or $7 \pmod 8$, then for each $i \in \{3,5,7\}$, the graph $L(K_{8k+i})(5) = L(K_{8k+i-1})(5) \oplus K^{\star}_{8k+i-1}(5)$, by Observation 2.2. We apply Lemmas 4.17 and 4.15 to the graphs $L(K_{8k+i-1})(5)$ and $K^{\star}_{8k+i-1}(5)$, respectively, to obtain a maximum K-packing with every possible leave. □

Proof of Theorem 1.1. Because of Theorem 3.4 and Lemmas 4.17 and 4.18, the proof follows for $\lambda \in \{1,2,3,4,5\}$. Now we consider the proof for $\lambda \geq 6$. Let $\lambda = 4k' + i$, where $i \in \{0,1,2,3\}$ and $k' \geq 1$. For $i = 0$, the graph $L(K_n)(\lambda)$ has a K-decomposition, by Lemma 4.17. For $i = 1$, the graph $L(K_n)(4k'+1) = L(K_n)(5) \oplus L(K_n)(4k'-4)$. The result now follows by Lemmas 4.17 and 4.18. For each $i \in \{2,3\}$, the graph $L(K_n)(4k'+i) = L(K_n)(i) \oplus L(K_n)(4k')$. The result now follows by Lemmas 4.17 and 4.18. □

5 Minimum covering of $L(K_n)(\lambda)$ with kites

In this section, we prove the existence of a minimum kite-covering of $L(K_n)(\lambda)$, $\lambda \geq 1$, with every possible padding.

Lemma 5.1. *For all $k \geq 1$ and for each $i \in \{2,4,6\}$, the graph $K^{\star}_{8k+i}$ admits a minimum K-covering with every possible padding. The possible paddings are given in the following table:*

i	*Possible paddings*
2	K_2
4	$P_3, 2K_2, K_2(2)$
6	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2,$ $P', K_2 \cup K_2(2), K_2(3)$

Proof. The graph $K^{\star}_{8k+2} = K^{\star}_{10} \oplus K^{\star}_{8(k-1)} \oplus K^{\star}_{10,8(k-1)}$, $K^{\star}_{8k+4} = K^{\star}_{4} \oplus K^{\star}_{8k} \oplus K^{\star}_{4,8k}$ and $K^{\star}_{8k+6} = K^{\star}_{6} \oplus K^{\star}_{8k} \oplus K^{\star}_{6,8k}$. By Theorem 2.10 and Lemmas 2.2 and 2.4, the graphs $K^{\star}_{8(k-1)}$, $K^{\star}_{8k}$, $K^{\star}_{10,8(k-1)}$, $K^{\star}_{4,8k}$ and $K^{\star}_{6,8k}$ have K-decompositions. It suffices to show that each of the graphs $K^{\star}_{4}$, $K^{\star}_{6}$ and $K^{\star}_{10}$ has a minimum K-covering with every possible padding.

(i) A minimum K-covering of $K^{\star}_{4}$ with padding $P \in \{P_3, 2K_2, K_2(2)\}$ is given below: A minimum K-covering of $K^{\star}_{4}$ with padding P_3 is

$[(1,\{1,4\},4);4\{3,4\}]$, $[(1,\{1,3\},3);34]$, $[(1,\{1,2\},2);2\{2,3\}]$,
$[(2,\{2,3\},3);3\{3,4\}]$, $[(4,\{2,4\},2);12]$

and the padding is $\{12, 2\{2,3\}\}$.

A minimum K-covering of $K^{\star}_{4}$ with padding $2K_2$ is

$[(1,\{1,4\},4);4\{3,4\}]$, $[(1,\{1,3\},3);23]$, $[(2,\{2,4\},4);34]$,
$[(2,\{2,3\},3);3\{3,4\}]$, $[(2,\{1,2\},1);14]$

and the padding is $\{14, 23\}$. A minimum K-covering of $K^{\star}_{4}$ with padding $K_2(2)$ is

$[(1,\{1,2\},2);24]$, $[(3,\{2,3\},2);2\{2,4\}]$, $[(3,\{3,4\},4);4\{2,4\}]$,
$[(3,\{1,3\},1);1\{1,2\}]$, $[(4,\{1,4\},1);1\{1,2\}]$
and the padding is $\{1\{1,2\},1\{1,2\}\}$.

(ii) A minimum K-covering of $K_6^\star$ with padding $P \in \{K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)\}$ is described below:
The graph $K_6^\star$ can be decomposed in two different ways, namely, $K_6^\star = T^\star \oplus T^\star \oplus T^\star \oplus P_4^\star$ and $K_6^\star = T^\star \oplus T^\star \oplus H^\star$. If $K_6^\star = T^\star \oplus T^\star \oplus T^\star \oplus P_4^\star$, then by Lemma 2.2 and Appendix A, the paddings are $P \in \{K_3, P_4, K_{1,3}, P'\}$. If $K_6^\star = T^\star \oplus T^\star \oplus H^\star$, then again by Lemma 2.2 and Appendix A, the paddings are $P \in \{3K_2, P_3 \cup K_2, K_2 \cup K_2(2), K_2(3)\}$.

(iii) A minimum K-covering of $K_{10}^\star$ with padding K_2 is given below.
From the proof of Lemma 2.9(ii), the graph $K_{10}^\star = \underbrace{T^\star \oplus T^\star \oplus \cdots \oplus T^\star}_{10-times} \oplus M^\star$. Thus a required K-covering with padding K_2 follows by Lemma 2.2 and Appendix A. □

Lemma 5.2. *For all $k \geq 1$ and for each $i \in \{2, 6\}$, the graph $K_{8k+i}^\star(2)$ admits a minimum K-covering with padding P, where $P \in \{P_3, 2K_2, K_2(2)\}$.*

Proof. The graph $K_{8k+2}^\star(2) = K_{10}^\star(2) \oplus K_{8(k-1)}^\star(2) \oplus K_{10,8(k-1)}^\star(2)$ and $K_{8k+6}^\star(2) = K_6^\star(2) \oplus K_{8k}^\star(2) \oplus K_{6,8k}^\star(2)$. By Theorem 2.10 and Lemmas 2.2, 2.4 and 4.3, the graphs $K_{8(k-1)}^\star(2)$, $K_{8k}^\star(2)$, $K_{10,8(k-1)}^\star(2)$ and $K_{6,8k}^\star(2)$ have K-decompositions. It is enough to show that the graphs $K_6^\star(2)$ and $K_{10}^\star(2)$ have minimum K-coverings with every possible padding.

(i) A minimum K-covering of $K_6^\star(2)$ with padding P_3, $2K_2$ or $K_2(2)$ is described below:
The graph $K_6^\star(2)$ can be decomposed in two different ways, namely, $K_6^\star(2) = T^\star(2) \oplus T^\star(2) \oplus T^\star(2) \oplus P_4^\star(2)$ and $K_6^\star(2) = T^\star(2) \oplus T^\star(2) \oplus H^\star(2)$. If $K_6^\star(2) = T^\star(2) \oplus T^\star(2) \oplus T^\star(2) \oplus P_4^\star(2)$, then by Lemmas 2.2, 4.3 and Appendix A, the paddings are P_3 or $2K_2$. If $K_6^\star(2) = T^\star(2) \oplus T^\star(2) \oplus H^\star(2)$, then again by Lemmas 2.2, 4.3 and Appendix A, the padding is $K_2(2)$.

(ii) A minimum K-covering of $K_{10}^\star(2)$ with padding P, where $P \in \{P_3, 2K_2, K_2(2)\}$ is described here. The result follows by (i) above and Lemmas 4.1, 2.3 and 4.3, since the graph $K_{10}^\star(2) = K_6^\star(2) \oplus K_4^\star(2) \oplus K_{4,6}^\star(2)$. □

Lemma 5.3. *For all $k \geq 1$ and for each $i \in \{2, 4, 6\}$, the graph $K_{8k+i}^\star(3)$ admits a minimum K-covering with every possible padding. The possible paddings are given in the table below:*

i	*Possible paddings*
2	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2,$ $P', K_2 \cup K_2(2), K_2(3)$
4	$P_3, 2K_2, K_2(2)$
6	K_2

Proof. The graph $K^\star_{8k+2}(3) = K^\star_{10}(3) \oplus K^\star_{8(k-1)}(3) \oplus K^\star_{10,8(k-1)}(3)$, $K^\star_{8k+4}(3) = K^\star_4(3) \oplus K^\star_{8k}(3) \oplus K^\star_{4,8k}(3)$ and $K^\star_{8k+6}(3) = K^\star_6(3) \oplus K^\star_{8k}(3) \oplus K^\star_{6,8k}(3)$. By Theorem 2.10 and Lemmas 2.2, 2.4 and 4.3, the graphs $K^\star_{8(k-1)}(3)$, $K^\star_{8k}(3)$, $K^\star_{10,8(k-1)}(3)$, $K^\star_{4,8k}(3)$ and $K^\star_{6,8k}(3)$ have K-decompositions. It is enough to show that each of the graphs $K^\star_4(3)$, $K^\star_6(3)$ and $K^\star_{10}(3)$ has a minimum K-covering with every possible padding.

(i) By the proof of Lemmas 5.1 and 4.1, a K-covering of $K^\star_4(3)$ with padding P, where $P \in \{P_3, 2K_2, K_2(2)\}$, follows as the graph $K^\star_4(3) = K^\star_4 \oplus K^\star_4(2)$.

(ii) A K-covering of $K^\star_6(3)$ with padding K_2 follows by Lemmas 2.2, 4.3 and Appendix A, as the graph $K^\star_6(3) = T^\star(3) \oplus T^\star(3) \oplus T^\star(3) \oplus P^\star_4(3)$.

(iii) Clearly, $K^\star_{10}(3) = \underbrace{T^\star(3) \oplus T^\star(3) \oplus \cdots \oplus T^\star(3)}_{10-times} \oplus M^\star(3)$, by the proof of Lemma 2.9. Now a K-covering of $K^\star_{10}(3)$ with padding P, where $P \in \{K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)\}$, follows by Lemmas 2.2, 4.3 and Appendix A. ☐

Lemma 5.4. *For each $\lambda \in \{1, 2, 3\}$, the graph $L(K_n)(\lambda)$ admits a minimum K-covering with every possible padding as given in the table below:*

λ	$n \equiv a \pmod 8$	*Possible paddings*
1 *or* 2 *or* 3	n *even or* $a = 1$	$\emptyset$
1	$a = 3$	K_2
	$a = 5$	$P_3, 2K_2, K_2(2)$
	$a = 7$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
2	$a \in \{3, 7\}$	$P_3, 2K_2, K_2(2)$
	$a = 5$	$\emptyset$
3	$a = 3$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
	$a = 5$	$P_3, 2K_2, K_2(2)$
	$a = 7$	K_2

Proof. Let $i \in \{3, 5, 7\}$. The graph $L(K_{8k+i})(\lambda) = L(K_{8k+i-1})(\lambda) \oplus K^\star_{8k+i-1}(\lambda)$, by Observation 2.2. By Lemma 4.17, the graph $L(K_{8k+i-1})(\lambda)$ has a K-decomposition. For $\lambda = 1$ and for each $i \in \{3, 5, 7\}$, apply Lemma 5.1 to the graph $K^\star_{8k+i-1}$ to obtain a minimum K-covering of $L(K_{8k+i})$. For $\lambda = 2$ and for each $i \in \{3, 7\}$, apply Lemma 5.2 to the graph $K^\star_{8k+i-1}(2)$ to get a K-covering of $L(K_{8k+i})(2)$. Now the result follows by applying Lemma 5.3 to the graph $K^\star_{8k+i-1}(3)$ for $\lambda = 3$ and for each $i \in \{3, 5, 7\}$. ☐

Proof of Theorem 1.3. Because of Lemmas 4.17 and 5.4, we only consider the proof for $\lambda \geq 5$. Let $\lambda = 4k' + i$, where $i \in \{0, 1, 2, 3\}$ and $k' \geq 1$. For $i = 0$, the graph $L(K_n)(\lambda)$ has a K-decomposition, by Lemma 4.17. For $i \in \{1, 2, 3\}$, the graph $L(K_n)(\lambda) = L(K_n)(4k' + i) = L(K_n)(i) \oplus L(K_n)(4k')$. Now the result follows by Lemmas 5.4 and 4.17.

We summarise our main theorems in the following:

Theorem 5.5. *Maximum kite-packings and minimum kite-coverings of $L(K_n)(\lambda)$ with every possible leave and padding exist. The leaves and paddings are described in the following table:*

$\lambda \equiv a \pmod 4$	$n \geq 4$ *and* $n \equiv b \pmod 8$	*Possible leaves in* $L(K_n)(\lambda)$	*Possible paddings in* $L(K_n)(\lambda)$
$a = 0$	$n \geq 4$	$\emptyset$	$\emptyset$
$a \in \{1, 2, 3\}$	n *even or* $b = 1$	$\emptyset$	$\emptyset$
$a = 1$	$b = 3$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$	K_2
	$b = 5$	$P_3, 2K_2, K_2(2)$	$P_3, 2K_2, K_2(2)$
	$b = 7$	K_2	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
$a = 2$	$b \in \{3, 7\}$	$P_3, 2K_2, K_2(2)$	$P_3, 2K_2, K_2(2)$
	$b = 5$	$\emptyset$	$\emptyset$
$a = 3$	$b = 3$	K_2	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$
	$b = 5$	$P_3, 2K_2, K_2(2)$	$P_3, 2K_2, K_2(2)$
	$b = 7$	$K_3, P_4, K_{1,3}, 3K_2, P_3 \cup K_2, P', K_2 \cup K_2(2), K_2(3)$	K_2

Proof. The proof follows by Theorems 1.1 and 1.3. □

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Appendix A: Minimum coverings of some graphs with kites

Graph	Padding	Covering
$P_4^\star$	$K_3 : \{cd, d\{c,d\}, \{c,d\}c\}$	$[(a,\{a,b\},b); b\{b,c\}], [(d,\{c,d\},c); bc]$, $[(d,\{c,d\},c); c\{b,c\}]$
$P_4^\star$	$P_4 : \{b\{b,c\}, \{b,c\}c, c\{c,d\}\}$	$[(a,\{a,b\},b); b\{b,c\}], [(b,\{b,c\},c); c\{c,d\}]$, $[(d,\{c,d\},c); c\{b,c\}]$
$P_4^\star$	$P' : \{bc, bc, b\{a,b\}\}$	$[(a,\{a,b\},b); bc], [(c,\{b,c\},b); b\{a,b\}]$, $[(d,\{c,d\},c); bc]$
$P_4^\star$	$K_{1,3} : \{ab, bc, b\{b,c\}\}$	$[(a,\{a,b\},b); b\{b,c\}], [(c,\{b,c\},b); ab]$, $[(d,\{c,d\},c); bc]$
$M^\star$	$K_2 : \{cd\}$	$[(a,\{a,b\},b); b\{b,c\}], [(d,\{c,d\},c); bc]$, $[(e,\{d,e\},d); cd], [(f,\{c,f\},c); c\{b,c\}]$
$H^\star$	$P_3 \cup K_2 : \{14, 4\{1,4\}, 6\{4,6\}\}$	$[(2,\{2,4\},4); 4\{1,4\}], [(2,\{2,6\},6); 6\{4,6\}]$, $[(3,\{3,6\},6); 6\{1,6\}], [(4,\{1,4\},1); 16]$, $[(5,\{1,5\},1); 1\{1,6\}], [(6,\{4,6\},4); 14]$
$H^\star$	$K_2 \cup K_2(2) : \{14, 14, 6\{4,6\}\}$	$[(2,\{2,4\},4); 14], [(2,\{2,6\},6); 6\{4,6\}]$, $[(3,\{3,6\},6); 6\{1,6\}], [(4,\{1,4\},1); 16]$, $[(5,\{1,5\},1); 1\{1,6\}], [(6,\{4,6\},4); 14]$
$H^\star$	$K_2(3) : \{46, 46, 46\}$	$[(1,\{1,4\},4); 46], [(2,\{2,4\},4); 46]$, $[(2,\{2,6\},6); 46], [(3,\{3,6\},6); 6\{1,6\}]$, $[(4,\{4,6\},6); 16], [(5,\{1,5\},1); 1\{1,6\}]$
$H^\star$	$3K_2 : \{1\{1,6\}, 36, 4\{1,4\}\}$	$[(2,\{2,4\},4); 4\{1,4\}], [(2,\{2,6\},6); 36]$, $[(3,\{3,6\},6); 6\{1,6\}], [(4,\{1,4\},1); 1\{1,6\}]$, $[(4,\{4,6\},6); 16], [(5,\{1,5\},1); 1\{1,6\}]$
$P_4^\star(2)$	$P_3 : \{bc, c\{c,d\}\}$	$[(a,\{a,b\},b); bc], [(a,\{a,b\},b); b\{b,c\}]$, $[(b,\{b,c\},c); c\{c,d\}], [(d,\{c,d\},c); c\{b,c\}]$, $[(d,\{c,d\},c); bc]$
$P_4^\star(2)$	$2K_2 : \{b\{b,c\}, c\{c,d\}\}$	$[(a,\{a,b\},b); b\{b,c\}], [(a,\{a,b\},b); b\{b,c\}]$, $[(b,\{b,c\},c); c\{c,d\}], [(d,\{c,d\},c); c\{b,c\}]$, $[(d,\{c,d\},c); bc]$
$H^\star(2)$	$K_2(2) : \{4\{4,6\}, 4\{4,6\}\}$	$[(1,\{1,4\},4); 4\{4,6\}], [(2,\{2,4\},4); 4\{4,6\}]$, $[(2,\{2,4\},4); 4\{4,6\}], [(2,\{2,6\},6); 6\{4,6\}]$, $[(2,\{2,6\},6); 46], [(3,\{3,6\},6); 6\{1,6\}]$, $[(3,\{3,6\},6); 6\{1,6\}], [(4,\{1,4\},1); 16]$, $[(4,\{4,6\},6); 16], [(5,\{1,5\},1); 1\{1,6\}]$, $[(5,\{1,5\},1); 1\{1,6\}]$

$P_4^{\star}(3)$	$K_2 : \{cd\}$	$[(a, \{a, b\}, b); bc], [(a, \{a, b\}, b); b\{b, c\}],$ $[(a, \{a, b\}, b); b\{b, c\}], [(b, \{b, c\}, c); cd],$ $[(d, \{c, d\}, c); c\{b, c\}], [(d, \{c, d\}, c); c\{b, c\}],$ $[(d, \{c, d\}, c); bc]$
$M^{\star}(3)$	$K_3 : \{d\{d, e\}, de, e\{d, e\}\}$	$[(a, \{a, b\}, b); b\{b, c\}], [(a, \{a, b\}, b); b\{b, c\}],$ $[(a, \{a, b\}, b); bc], [(b, \{b, c\}, c); c\{c, d\}],$ $[(d, \{c, d\}, c); c\{b, c\}], [(e, \{d, e\}, d); d\{c, d\}],$ $[(e, \{d, e\}, d); d\{c, d\}], [(e, \{d, e\}, d); cd]$ $[(e, \{d, e\}, d); cd], [(f, \{c, f\}, c); c\{b, c\}],$ $[(f, \{c, f\}, c); c\{c, d\}], [(f, \{c, f\}, c); bc]$
$M^{\star}(3)$	$P_4 : \{ab, bc, c\{b, c\}\}$	$[(a, \{a, b\}, b); b\{b, c\}], [(a, \{a, b\}, b); bc],$ $[(a, \{a, b\}, b); bc], [(b, \{b, c\}, c); c\{c, d\}],$ $[(c, \{b, c\}, b); ab], [(d, \{c, d\}, c); c\{b, c\}],$ $[(e, \{d, e\}, d); d\{c, d\}], [(e, \{d, e\}, d); d\{c, d\}],$ $[(e, \{d, e\}, d); cd], [(f, \{c, f\}, c); c\{b, c\}],$ $[(f, \{c, f\}, c); c\{c, d\}], [(f, \{c, f\}, c); cd]$
$M^{\star}(3)$	$P_3 \cup K_2 : \{b\{b, c\}, cd, c\{c, d\}\}$	$[(a, \{a, b\}, b); b\{b, c\}], [(a, \{a, b\}, b); b\{b, c\}],$ $[(a, \{a, b\}, b); b\{b, c\}], [(b, \{b, c\}, c); c\{c, d\}],$ $[(d, \{c, d\}, c); c\{b, c\}], [(d, \{c, d\}, c); c\{b, c\}],$ $[(e, \{d, e\}, d); cd], [(e, \{d, e\}, d); cd],$ $[(e, \{d, e\}, d); d\{c, d\}], [(f, \{c, f\}, c); bc],$ $[(f, \{c, f\}, c); bc], [(f, \{c, f\}, c); c\{c, d\}]$
$M^{\star}(3)$	$P' : \{bc, bc, c\{b, c\}\}$	$[(a, \{a, b\}, b); b\{b, c\}], [(a, \{a, b\}, b); bc],$ $[(a, \{a, b\}, b); bc], [(b, \{b, c\}, c); c\{c, d\}],$ $[(b, \{b, c\}, c); c\{c, d\}], [(d, \{c, d\}, c); c\{b, c\}],$ $[(e, \{d, e\}, d); d\{c, d\}], [(e, \{d, e\}, d); d\{c, d\}],$ $[(e, \{d, e\}, d); cd], [(f, \{c, f\}, c); bc],$ $[(f, \{c, f\}, c); cd], [(f, \{c, f\}, c); c\{b, c\}]$

$M^{\star}(3)$	$K_2 \cup K_2(2) : \{b\{b,c\}, cd, cd\}$	$[(a,\{a,b\},b); b\{b,c\}], [(a,\{a,b\},b); b\{b,c\}],$ $[(a,\{a,b\},b); b\{b,c\}], [(b,\{b,c\},c); cd],$ $[(d,\{c,d\},c); bc], [(d,\{c,d\},c); bc],$ $[(e,\{d,e\},d); cd], [(e,\{d,e\},d); cd],$ $[(e,\{d,e\},d); d\{c,d\}], [(f,\{c,f\},c); c\{b,c\}],$ $[(f,\{c,f\},c); c\{b,c\}], [(f,\{c,f\},c); c\{c,d\}]$
$M^{\star}(3)$	$3K_2 : \{b\{a,b\}, cf, d\{c,d\}\}$	$[(a,\{a,b\},b); b\{b,c\}], [(a,\{a,b\},b); b\{b,c\}],$ $[(a,\{a,b\},b); bc], [(c,\{b,c\},b); b\{a,b\}],$ $[(d,\{c,d\},c); c\{b,c\}], [(d,\{c,d\},c); cf],$ $[(e,\{d,e\},d); d\{c,d\}], [(e,\{d,e\},d); cd],$ $[(e,\{d,e\},d); d\{c,d\}], [(f,\{c,f\},c); bc],$ $[(f,\{c,f\},c); c\{c,d\}], [(f,\{c,f\},c); c\{b,c\}]$
$M^{\star}(3)$	$K_2(3) : \{bc, bc, bc\}$	$[(a,\{a,b\},b); b\{b,c\}], [(a,\{a,b\},b); bc],$ $[(a,\{a,b\},b); bc], [(b,\{b,c\},c); c\{c,d\}],$ $[(b,\{b,c\},c); c\{c,d\}], [(d,\{c,d\},c); bc],$ $[(e,\{d,e\},d); d\{c,d\}], [(e,\{d,e\},d); d\{c,d\}],$ $[(e,\{d,e\},d); cd], [(f,\{c,f\},c); bc],$ $[(f,\{c,f\},c); cd], [(f,\{c,f\},c); c\{b,c\}]$
$M^{\star}(3)$	$K_{1,3} : \{d\{d,e\}, de, cd\}$	$[(a,\{a,b\},b); b\{b,c\}], [(a,\{a,b\},b); b\{b,c\}],$ $[(a,\{a,b\},b); bc], [(b,\{b,c\},c); c\{c,d\}],$ $[(c,\{c,d\},d); de], [(c,\{c,d\},d); d\{d,e\}],$ $[(e,\{d,e\},d); cd], [(e,\{d,e\},d); d\{c,d\}]$ $[(e,\{d,e\},d); cd], [(f,\{c,f\},c); c\{b,c\}],$ $[(f,\{c,f\},c); c\{b,c\}], [(f,\{c,f\},c); bc]$

The graphs M and H are as in Figures 5 and 7.

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