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The eternal domination number for $3 \times n$ grid graphs

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Abstract

In the eternal dominating set problem, guards form a dominating set on a graph, and at each step, a vertex is attacked. To defend against the attack, each guard either remains in place or moves to a neighboring vertex in order to form a new dominating set that contains the attacked vertex. We wish to determine the minimum number of guards required to successfully defend against any possible sequence of attacks, the eternal domination number. This number is known for $3 \times n$ grid graphs when $n < 26$. This paper determines exact values of eternal domination numbers for $3 \times n$ grid graphs when $n \geq 26$.

1 Introduction

A dominating set for a graph is a positioning of guards on vertices so that every vertex is monitored from a distance of at most one. A graph's domination number is the smallest size of such a set. An eternal dominating family is a collection of dominating sets resulting from having to respond to an arbitrary infinitely long sequence of attacks at individual vertices. Each response permits each guard to either remain stationary or move a distance of one, but must have one guard move to the attacked vertex. The smallest size of the sets for such a family is the eternal domination number. This has been referred to as the "all guards move model" or "eternal m-security" [7] and as "m-eternal domination" [4], and is one of a number of variations of problems involving mobile guards [8]. This paper builds on previous results on the eternal domination number for members of the family of $3 \times n$ grid graphs [2, 3, 4, 9], determining the outstanding values.

The domination number, denoted $\gamma(G)$, has been determined for all grid graphs [5, 6], with the value for $3 \times n$ grid graphs being $\gamma(P_3 \,\square\, P_n) = \lceil \frac{3n+1}{4} \rceil$, making this

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a lower bound for the eternal domination number of such graphs, which we denote $\gamma_{all}^{\infty}(P_3 \square P_n)$. Goldwasser, Klostermeyer, and Mynhardt [4] found that

$$\gamma_{all}^{\infty}(P_3 \square P_n) \leq \left\lceil \frac{8n}{9} \right\rceil \text{ for } n \geq 9$$

and conjectured that $\gamma_{all}^{\infty}(P_3 \square P_n) = \lceil \frac{4n+5}{5} \rceil$ for $n > 9$. Finbow, Messinger, and van Bommel [2] disproved the conjecture and provided a tightening of the bounds to

$$\left\lceil \frac{4n+6}{5} \right\rceil \leq \gamma_{all}^{\infty}(P_3 \square P_n) \leq \left\lceil \frac{6n+2}{7} \right\rceil \text{ for } n \geq 11.$$

Messinger and Delaney [9] developed a set of configurations for eternal dominating families which helped reduce the upper bound to

$$\gamma_{all}^{\infty}(P_3 \square P_n) \leq \left\lceil \frac{4n+10}{5} \right\rceil + \begin{cases} 1 & \text{if } n \equiv 0, 1, 3 \pmod 5 \\ 0 & \text{otherwise.} \end{cases}$$

This paper employs a variation of their configurations and proves the lower bound is not tight, specifically when $n \equiv 1 \pmod 5$, $n \geq 26$, in order to remove the gap completely, construct eternal dominating families, and establish the value $\gamma_{all}^{\infty}(P_3 \square P_n) = \lceil \frac{4n+7}{5} \rceil$ for $n \geq 26$.

2 Definitions

Let $G = (V, E)$ be a graph. A *dominating set* of G is a subset of V whose closed neighbourhood is V. The smallest cardinality of a dominating set is denoted $\gamma(G)$ and is called the *domination number* of G. Let $\mathbb{D}_q(G)$ be the set of all dominating sets of G which have cardinality q. Let $D, D' \in \mathbb{D}_q(G)$. We will say D can be *transformed* to D' (or D *transforms* to D') if $D = \{v_1, v_2, \ldots, v_q\}$, $D' = \{u_1, u_2, \ldots, u_q\}$ and $u_i \in N[v_i]$ for $i = 1, 2, \ldots, q$.

In the "eternal dominating set problem," a defender is given q guards to protect the graph from a series of attacks on vertices made by an attacker. An *eternal dominating family of* G is a subset $\mathcal{E} \subseteq \mathbb{D}_q(G)$ for some q so that for every $D \in \mathcal{E}$ and every possible attack $v \in V(G)$, there is a dominating set $D' \in \mathcal{E}$ such that $v \in D'$ and D transforms to D'. When the value of q in the above definition is known, we refer to this family as an eternal dominating family with q guards. For a graph G, the minimum value of q such that there exists an eternal dominating family with q guards is denoted $\gamma_{all}^{\infty}(G)$. A set $D \in \mathbb{D}_q(G)$ is an *eternal dominating set* or a *q-eternal dominating set* if it is a member of some eternal dominating family. Note that the set of all eternal dominating sets of a particular cardinality is an eternal dominating family, provided the family is non-empty.

The *Cartesian product* of graphs G and H is denoted by $G \square H$. The vertex set of $G \square H$ is $V(G \square H) = \{(u, v) | u \in V(G), v \in V(H)\}$, and two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

When $G = P_m$ and $H = P_n$, these graphs are also known as a *grids* or *grid graphs* of dimensions $m \times n$. The vertices of P_m (respectively P_n) are labeled in their usual ordering $u_1, u_2, \dots u_m$ (resp. $v_1, v_2, \dots v_n$). In this paper, we focus on the eternal domination numbers of grid graphs with $m = 3$. Each copy of P_3, corresponding to a vertex of P_n, is referred to as a column. We refer to each of the columns as the first column, second column, etc. and as column 1, column 2, etc. starting from one the columns corresponding to a leaf of P_n and proceeding consecutively.

In constructing eternal dominating families we make use of the symmetries of the $3 \times n$ grid graph. Given a dominating set $D \in \mathbb{D}_q(P_3 \Box P_n)$, a vertical reflection of D (about the horizontal line of symmetry) is denoted D_v, while a horizontal reflection (about the vertical line of symmetry) is denoted D_h. A rotation of a dominating set D by 180° (which is the same as both the vertical reflection of D_h and the horizontal reflection of D_v) is denoted D_r. When we wish to discuss an arbitrary symmetry of a dominating set D, we denote it D_s.

3 Previous Results and Extensions

We begin with several observations of Beaton et. al [1] and Finbow et. al [2], and extend two of these results. We note that, by symmetry, statements and arguments referring to the first i columns also apply to the last i columns, for any i.

Theorem 3.1 *[1] Given dominating sets $D, E \in \mathbb{D}_q(P_m \Box P_n)$ and any arbitrary symmetry s resulting from a reflection or rotation, D transforms to E if and only if D_s transforms to E_s.*

Corollary 3.2 *Let $\mathcal{E}$ be an eternal dominating family of $P_3 \Box P_n$. Then the family*

$$\mathcal{F} = \mathcal{E} \cup \{D_h | D \in \mathcal{E}\} \cup \{D_v | D \in \mathcal{E}\} \cup \{D_r | D \in \mathcal{E}\}$$

is an eternal dominating family of $P_3 \Box P_n$.

Proof: Let $F \in \mathcal{F}$ be some dominating set in $\mathcal{F}$. If $F \in \mathcal{E}$, then since $\mathcal{E}$ is an eternal dominating family, for every possible attack $v \in V(P_3 \Box P_n)$, there exists a dominating set $D' \in \mathcal{E}$ so that $v \in D'$ and F transforms to D'.

Otherwise, if $F \notin \mathcal{E}$, there must exist a dominating set $D \in \mathcal{E}$ and some symmetry s of D such that $F = D_s$. Consider an attack on some $v \in V(P_3 \Box P_n)$. Let $v_s \in V(P_3 \Box P_n)$ be the image of v under the symmetry s. Since $\mathcal{E}$ is an eternal dominating family, there exists a dominating set $D' \in \mathcal{E}$ so that $v_s \in D'$ and D transforms to D'. It follows that $v \in D'_s$, the symmetry of D', and $F = D_s$ transforms to D'_s. As $D'_s \in \mathcal{F}$, and F was an arbitrary member of $\mathcal{F}$, it follows that $\mathcal{F}$ is an eternal dominating family of $P_3 \Box P_n$. ■

Lemma 3.3 *Let $\mathcal{E}$ be an eternal dominating family of $P_3 \Box P_n$. If there are at least k guards in the first i columns for each dominating set $D \in \mathcal{E}$, then for any set $D' \in \mathcal{E}$ all of the following hold.*

1. *If there are at most k guards in the first $i+1$ columns, then there are k guards in the first i columns, no guards in column $i+1$, and three guards in column $i+2$.*

2. *If there are at most $k+1$ guards in the first $i+2$ columns, then there are $k+1$ guards in the first $i+1$ columns, no guards in column $i+2$, and at least two guards in column $i+3$.*

3. *If there are at most $k+2$ guards in the first $i+3$ columns, then there are $k+2$ guards in the first $i+2$ columns, no guards in column $i+3$, and at least two guards in column $i+4$.*

4. *If there are at most $k+3$ guards in the first $i+4$ columns, then there are $k+3$ guards in the first $i+3$ columns, no guards in column $i+4$, and at least one guard in column $i+5$.*

5. *There are at least $k+4$ guards in the first $i+5$ columns.*

6. *If there are at most $k+4$ guards in the first $i+6$ columns, then there are k guards in the first i columns, one guard in the middle of column $i+1$, no guards in column $i+2$, two guards (in the top row and bottom row) of column $i+3$, no guards in column $i+4$, one guard in the middle of column $i+5$, no guards in column $i+6$, and three guards in column $i+7$, as shown in Figure 1.*

1 ··· i	i+1	i+2	i+3	i+4	i+5	i+6	i+7	
			X				X	
k guards	X				X		X	
			X				X	

Figure 1: Only possible configuration for Lemma 3.3 (6.).

Proof: Items (1.) through (5.) were proven in [2]. We proceed to proving (6.).

Given the assumption of at most $k+4$ guards in the first $i+6$ columns, (5.) and (1.) show there are no guards in column $i+6$ and three guards in column $i+7$.

By assumption, there are at least k guards in the first i columns. Since there are no guards in column $i+6$, we require at least two guards in columns $i+3$ through $i+5$ to dominate the vertices of columns $i+4$ and $i+5$. This shows there are at most $k+2$ guards in the first $i+2$ columns.

Suppose there are exactly $k+2$ guards in the first $i+2$ columns. Then there are at most two guards in column $i+3$, column $i+4$ and column $i+5$. As D is a dominating set, it can be seen that there must be a guard in column $i+5$, a guard in column $i+4$ and two guards in column $i+2$. Hence there are k guards in the first $i+1$ columns. It follows from (1.) there are $k+3$ guards in the first $i+2$ columns, a contradiction showing there at most $k+1$ guards in the first $i+2$ columns.

By (2.), there are $k+1$ guards in the first $i+1$ columns, no guards in column $i+2$ and at least two guards in column $i+3$. As D is dominating, the remaining guard must be in the middle of column $i+5$, the two guards in column $i+3$ must be in the top and bottom row, and there is a guard in column $i+1$ which must be in the middle row. ■

Corollary 3.4 ([2]) *In any eternal dominating set of $P_3 \square P_n$, for any $\ell \geq 2$, the first ℓ columns contain at least $\lceil \frac{4\ell-3}{5} \rceil$ guards.*

Lemma 3.5 ([2]) *Let $\mathcal{E}$ be a family of eternal dominating sets of $P_3 \square P_n$ and let $i \in \{0,1,2,3,4\}$. For every $D \in \mathcal{E}$, there are at least i guards in the first $i+1$ columns.*

4 Improving the Lower Bound

In this section, we establish an improved lower bound in the case $n \equiv 1 \pmod 5$. We note when $n \equiv 1 \pmod 5$, the lower bound is $\lceil \frac{4n+6}{5} \rceil = \frac{4n+6}{5}$.

Lemma 4.1 *Let $\mathcal{E}$ be an eternal dominating family of $P_3 \square P_n$. If there are at least 10 guards in the first 12 columns for each dominating set $D \in \mathcal{E}$, then for any $\ell \geq 3$, $\ell \neq 7$, the first ℓ columns contain at least $\lceil \frac{4\ell-2}{5} \rceil$ guards.*

Proof: For $3 \leq l \leq 11$, $\lceil \frac{4\ell-3}{5} \rceil = \lceil \frac{4\ell-2}{5} \rceil$ unless $l = 7$. For $l = 12$, $\lceil \frac{4\ell-2}{5} \rceil = 10$. So by assumption and Corollary 3.4, the first l columns contain at least $\lceil \frac{4\ell-2}{5} \rceil$ guards for $3 \leq \ell \leq 12$, unless $\ell = 7$. Further, by Lemma 3.3 (5.), if the result holds for $\ell = k$, then the result holds for $\ell = k+5$. By induction, for any $\ell \geq 8$, the first ℓ columns contain at least $\lceil \frac{4\ell-2}{5} \rceil$ guards. ■

Lemma 4.2 *Let $n \equiv 1 \pmod 5$, $n \geq 26$. Let $\mathcal{E}$ be an eternal dominating family of $P_3 \square P_n$ with at most $\frac{4n+6}{5}$ guards and with the property that if $D \in \mathcal{E}$, then $D_r \in \mathcal{E}$. There is an eternal dominating set in $\mathcal{E}$ with 9 guards in the first 12 columns.*

Proof: By Corollary 3.4, there must be at least 9 guards in the first 12 columns. Suppose then, by way of contradiction, each dominating set in $\mathcal{E}$ has at least 10 guards in the first 12 columns. Hence, since $D \in \mathcal{E}$ implies $D_r \in \mathcal{E}$, each dominating set in $\mathcal{E}$ has at least 10 guards in the last 12 columns.

Let l be the largest integer such that for any $k \leq l$, every eternal dominating set in $\mathcal{E}$ has at least k guards in the first $k+2$ columns, but there is a set $D \in \mathcal{E}$ with at most l guards in the first $l+3$ columns. Such an l must exist since there are at most $\frac{4n+6}{5}$ guards and $n \geq 26$, thus there are at most $n-4$ guards in the n columns.

By assumption and Corollary 3.4, $l \geq 10$. We claim $l \leq n-9$. Suppose an eternal dominating set has $n-8$ guards in the first $n-6$ columns. By Corollary 3.4, there are at least five guards in the last 6 columns, hence the number of guards is at least

$n - 8 + 5 \leq \frac{4n+6}{5}$. This implies $n \leq 21$, a contradiction, and therefore $l \leq n - 9$. Recall there is a dominating set D which has at most l guards in the first $l+3$ columns. By Lemma 3.3 (1.), D has l guards in the first $l+2$ columns, no guards in column $l+3$ and three guards in column $l+4$. It must be the case that either $n-(l+4) = 7$ or $n-(l+4) \neq 7$.

Case 1: $n-(l+4) = 7$. By Corollary 3.4, D has at least 5 guards in the remaining 7 columns. Therefore $\frac{4n+6}{5} \geq |D| \geq l+0+3+5 = (l+4)+4 = (n-7)+4 = n-3$. This implies $n \leq 21$, a contradiction.

Case 2: $n-(l+4) \neq 7$. Since $l \leq n-9$, $n-(l+4) \geq 5$. By Lemma 4.1, D has at least $\left\lceil \frac{4(n-(l+4))-2}{5} \right\rceil$ guards in the remaining $n-(l+4)$ columns. Hence,

$$\frac{4n+6}{5} \geq |D| \geq l+0+3+\left\lceil \frac{4(n-(l+4))-2}{5} \right\rceil .$$

Rearranging we obtain

$$\frac{4n+6}{5} \geq \frac{4n+6}{5} + \left\lceil \frac{l-9}{5} \right\rceil$$

which is false since $l \geq 10$.

As both cases lead to a contradiction, our original assumption that each dominating set in $\mathcal{E}$ has at least 10 guards in the first 12 columns is false. Thus, from this result and Corollary 3.4, we can conclude there is a dominating set in $\mathcal{E}$ with exactly 9 guards in the first 12 columns. ■

Observation 4.3 ([2]) *The unique minimum dominating set (up to reflection) on $P_3 \square P_6$ is the following.*

1	2	3	4	5	6
		X			
X				X	X
		X			

Lemma 4.4 *For any $n \equiv 1 \pmod 5$, $n \geq 26$, $\gamma_{all}^{\infty}(P_3 \square P_n) \geq \frac{4n+11}{5}$.*

Proof: Let $n \equiv 1 \pmod 5$, $n \geq 26$ be given and let $\mathcal{E}$ be an eternal dominating family of $P_3 \square P_n$ which uses $\frac{4n+6}{5}$ guards. Set $d = \frac{4n+6}{5}$ and let

$$\mathcal{E}' = \mathcal{E} \cup \{D_h | D \in \mathcal{E}\} \cup \{D_v | D \in \mathcal{E}\} \cup \{D_r | D \in \mathcal{E}\}.$$

By Corollary 3.2, $\mathcal{E}'$ is an eternal dominating family of $P_3 \square P_n$. It follows from Lemma 4.2 that there exists an eternal dominating set $D \in \mathcal{E}'$ with 9 guards in the first 12 columns. By Corollary 3.4, any eternal dominating set of $P_3 \square P_n$ must contain at least 9 guards in the first 11 columns, and hence by Lemma 3.3 (6.), D has 5 guards in the first 6 columns, one guard in the middle of column 7, no guards in column 8, two guards (in the top row and bottom row) of column 9, no guards in column 10, one guard in the middle of column 11, no guards in column 12, and three guards in column 13.

We now wish to determine the possible positions of the 5 guards in the first 6 columns in D. Let u be the vertex in the middle row of column 6, v be the vertex in the middle row of column 7, and w be the vertex in the middle row of column 8. As $\mathcal{E}'$ is an eternal dominating family, D must transform to a set $E \in \mathcal{E}'$ with $w \in E$, corresponding to the response to an attack on w. A guard in v is the only possible guard in D that could respond to the attack, implying E has 5 guards in the first 7 columns. By Corollary 3.4, there are at least 5 guards in the first 6 columns of every set in $\mathcal{E}'$, and hence E has no guards in column 7 and 5 guards in the first 6 columns. Further these 5 guards must dominate each vertex in the first 6 columns and hence, by Observation 4.3, $u \in E$. It follows that of the 5 guards in the first 6 columns in D, one is adjacent to u, and thus the 5 guards in the first 6 columns in D dominate each vertex in the first six columns of D. Therefore, from Observation 4.3, there are two possibilities, call them **A** and **B**, for the guards locations in the first 13 columns, as illustrated in Figure 2.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
			X						X				X	
A	X				X	X	X				X		X	
			X						X				X	
				X					X				X	
B	X	X				X	X				X		X	
				X					X				X	

Figure 2: The two possible configurations for Lemma 4.4.

We first wish to show that **A** is not part of a d-eternal dominating set. To establish this, we consider a sequence of two attacks and the corresponding transformations, as depicted in Figure 3. Specifically, the first attack is at the middle vertex of column 2. To successfully defend against the attack, the defender must move the guards to transform an eternal dominating set containing **A** to a d-eternal dominating set D' containing the attacked vertex (and thus also containing **A**$'$).

In particular, the defender must:

- move the guard in column 1 to the attacked vertex.
- move the guards in column 3 to column 2 so that D' has a vertex in the neighbourhood of each of the vertices of the first column.
- move the guards in the middle vertices of columns 5 through 7 one column to the left so that D' has guards in the neighbourhood of all the vertices in columns 4 through 6.
- move the guards in the top and bottom vertices in column 9 to column 8 so that D' has guards in the neighbourhood of all the vertices in columns 7 and 8.
- move the guard in the middle vertex of column 11 to the middle vertex of column 10 so that D' has guards in the neighbourhood of all the vertices in columns 9 and 10.
- move the guards in the top and bottom vertices in column 13 to column 12 so that D' has guards in the neighbourhood of all the vertices in column 11.

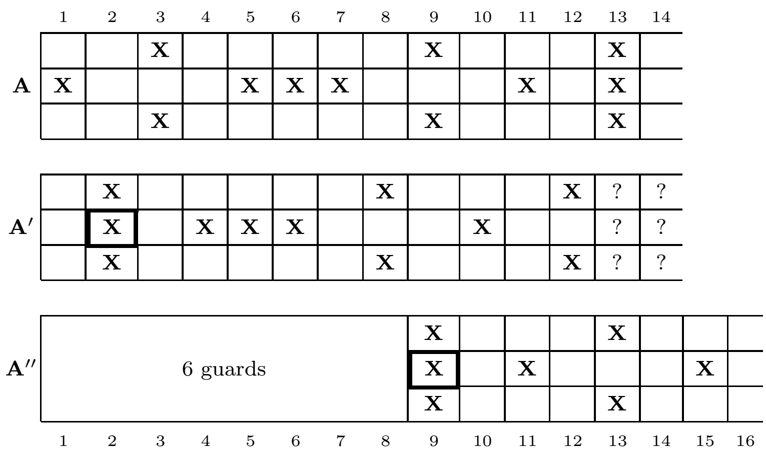

Figure 3: Two attacks and defender responses, starting from **A**.

According to Corollary 3.4, the last $n-12$ columns of D' contain at least $\left\lceil \frac{4(n-12)-3}{5} \right\rceil$ guards, or $d-11$ guards as $n \equiv 1 \pmod 5$. This implies there are no other guards in the first 12 columns of D'.

With guards located on the vertices of D' (containing $\mathbf{A}'$), consider an attack on the middle vertex of column 9. To successfully defend against the attack, the defender must transform D' to a d-eternal dominating set D'' containing the attacked vertex (and thus $\mathbf{A}''$). The defender must move the guard in column 10 to the attacked vertex, and it is clear that in D'' there are 9 guards in the first 9 columns. Hence, there are $d-9$ guards in the remaining $n-9$ columns.

By Corollary 3.4, every dominating set in $\mathcal{E}'$ has at least $\left\lceil\frac{4(n-15)-3}{5}\right\rceil$ guards in the last $n-15$ columns, or $d-13$ as $n \equiv 1 \pmod 5$.

With these two values ($i = n-15$ and $k = d-13$), Lemma 3.3 (6.) implies D'' has $d-13$ guards in the last $n-15$ columns, one guard in the middle of column 15, no guards in column 14, two guards (in the top and bottom row) of column 13, no guards in column 12, one guard in the middle of column 11, no guards in column 10, and three guards in column 9. However, after moving the guard in column 10 to column 9 in response to the attack, no guard can be moved to the middle of column 11. Thus D' cannot transform to a d-eternal dominating set containing $\mathbf{A}''$, so $\mathbf{A}$ is not part of an eternal dominating set.

An almost identical argument can be used to establish $\mathbf{B}$ is not part of an eternal dominating set by considering an attack in the middle of the third column, followed by an attack in the middle of the ninth column. Therefore D does not exist. ∎

This result, combined with the previous value $\gamma^{\infty}_{all}(P_3 \square P_n) \geq \left\lceil\frac{4n+6}{5}\right\rceil$ for $n \geq 11$, improves the lower bound to the following.

Corollary 4.5 *For all* $n \geq 26$, $\gamma^{\infty}_{all}(P_3 \square P_n) \geq \left\lceil\frac{4n+7}{5}\right\rceil$.

5 Arrangement for $n \equiv 2 \pmod 5$

In this section we provide an eternal dominating family with sets of cardinality $\frac{4n+7}{5}$ when $n \equiv 2 \pmod 5$. Two of the building blocks for guard arrangements are illustrated in Figure 4. We note these building blocks were also the basis of the configurations provided by Messinger and Delaney [9], however, modifications to their work, including careful attention to the first four and last three columns, allow for fewer guards than in their results.

Consider the two guard arrangements as illustrated in Figure 4. Shifting the repeating patterns of these building blocks within the $3 \times n$ grid graph where $n \equiv 2 \pmod 5$ is fixed, and providing an adjustment for the start and end of each dominating set, lead to several possible configurations. Some specific ones are illustrated in Figure 5, where a starting pattern of four columns adjusts for the start, a pattern of three columns adjusts for the end, and the ellipses in each configuration represent a continuation of the pattern from the central part in five-column block increments. Note that in the configurations for G, O, and P (see Figure 4), the last three columns shown complete a dominating set only in the case when $n \equiv 2 \pmod{10}$. The given ending is used in this case, and while we do not include it in Figure 5, the vertical reflection of each ending is necessarily used to complete the dominating set in G, O, and P when $n \equiv 7 \pmod{10}$.

The central part of Configuration B (Blue) follows Pattern 1. The central part of Configuration Y (Yellow) also follows Pattern 1 shifted but is shifted one column to the right, and the central part of Configuration R (Red) follows Pattern 1 shifted two columns to the right. The central parts of Configurations G (Green), O (Orange) and P (Purple) similarly follow Pattern 2. The central parts of the three configurations which follow Pattern 2 alternate between an original block of five columns and its vertical reflection, but must start with the original block as shown to form a dominating set.

We claim the sets depicted in Figure 5 are part of an eternal dominating family consisting of all possible configurations illustrated in Figure 5, Figure 6, and Figure 7, as well as their symmetries. Each configuration illustrated represents either a particular configuration or a set of configurations in the eternal dominating family. We provide an explanation of the configurations themselves before discussing how a defence to every possible attack is encoded in the figures. Those in Figure 5 are described above.

Configurations listed in Figure 6 all contain one column with three guards. We call the block containing three guards in one of its columns an x block or, when appropriate, an x start. For example, the configuration xO illustrated in Figure 6 is equivalent to configuration O except we do not use an orange start. A set of possible configurations is illustrated by the configuration labeled RxO in Figure 6. These configurations begin identically to the pattern of R, followed by an x block somewhere in the central part, and with the remaining blocks following the O pattern. The ellipses in this configuration represent zero or more repetitions of the pattern of the closest five columns from the central block. Using the notation of Messinger and Delaney [9], we use the C to represent any sequence of combinations of Y and B blocks. In Figure 6, the configuration labeled xC represents the set of configurations with an x start and any combination of Y and B blocks in the central part of the configuration, and either choice of ending from Y or B. Similarly RxC represents a set of configurations which begin with the pattern of R, followed by an x block somewhere in the central part, and finishes with a sequence of Y and B blocks of the appropriate length.

The six additional sets of configurations illustrated in Figure 7, represent some additional combinations of patterns from Figure 5. The sets of configurations labeled PO and PG begin with the pattern of P and finish with the pattern of O and G, respectively; that is, they contain the purple start followed by zero or more five-column blocks of the pattern of P, and the remaining blocks follow the orange or green pattern. In each case shown in Figure 7, the last column of the last purple block has a guard in the top column and the guard in the first column of the first

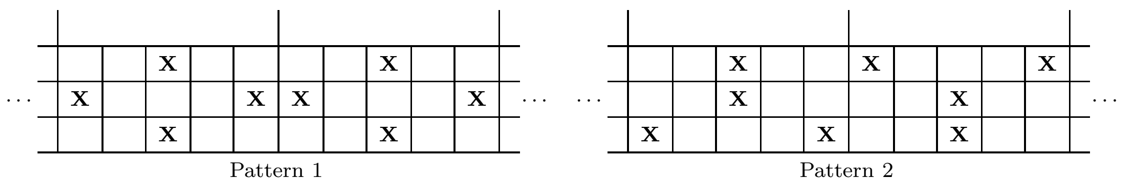

Figure 4: Two basic guard patterns.

orange or green pattern is in the bottom row. For configurations in these sets where the last column of the last purple block has a guard in the bottom column, the guard in the first column of the first orange or green pattern is in the top row. The set of configurations labeled $\mathrm{R}\overline{\mathrm{O}}$ are the configurations which start with the pattern of R, continue with the horizontal reflection of the middle blocks of the O pattern, and end with a distinctive ending pattern. Note that in these sets the second column of the first flipped orange pattern always has a guard in the top row. The final three sets of configurations can be expressed in terms of patterns we have seen, except they have a single column which we call m, since it contains one guard in the middle row, separating two distinct patterns. Configurations labeled RmO start with blocks following the pattern of R, and finish with blocks following the pattern of O so that there is a guard in the top row of the first column of the first orange pattern (with a unique ending containing only two columns); however, these two are separated by a single m column. The configurations in the sets $\overline{\mathrm{Y}}\mathrm{mO}$ and $\overline{\mathrm{O}}\mathrm{mO}$ start with the horizontal reflection of the Y and O patterns, respectively, starting with horizontal reflections of their specific endings, followed by zero or more blocks which follow the pattern of the horizontal reflections of their central blocks. In either case, this is followed by the m column and the appropriate number of blocks which follow the O pattern. For configurations in $\overline{\mathrm{Y}}\mathrm{mO}$, the first column of the first block has a guard in the bottom row. For configurations in $\overline{\mathrm{O}}\mathrm{mO}$, if there is a guard in the top row of the last column of the last flipped orange pattern, there is a guard in the bottom row of

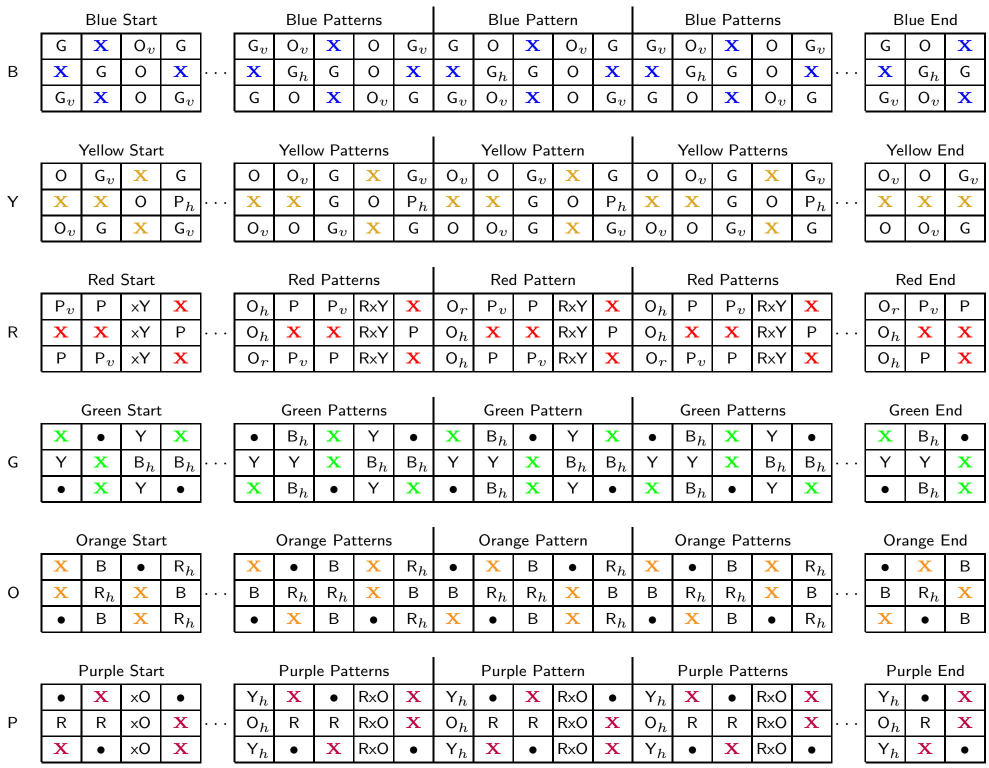

Figure 5: Six monochromatic guard configurations when $n \equiv 2 \pmod 5$.

the first column of the first orange pattern, otherwise, there is a guard in the bottom row of the last column of the last flipped orange pattern and a guard in the top row of the first column of the first orange pattern. As in the previous configurations, when appropriate, the vertical reflection of the ending pattern is used instead of the one shown in Figure 7. In all cases, the ellipses in these configurations represent zero or more repetitions of the pattern of the closest five columns from the central block.

For a set of configurations A, we will denote the set of configurations of the vertical reflection (respectively horizontal reflection and rotation by 180°) of all configurations in A by A_v (respectively A_h and A_r). To justify the claim that all possible configurations illustrated in Figure 5, Figure 6, and Figure 7, as well as their symmetries, form an eternal dominating set, a defence to every possible attack on each configuration is encoded in the figures.

Consider the B configuration, which consists of a blue start and a repetition of one or more blue patterns, followed by a blue end. In order to defend each possible attack, we must ensure that every vertex not guarded can be defended by a move of the guards. An attack on an unguarded vertex that appears in the same column as some guard can be defended by a move of all guards either up or down within each of their columns, leading to the G configuration (or the vertical reflection of G). This is illustrated in Figure 5, where a G (or G_v) in each of these positions of the B configuration indicates an attack on one of these vertices can be defended by transforming B to the G configuration (or the vertical reflection of G). We note

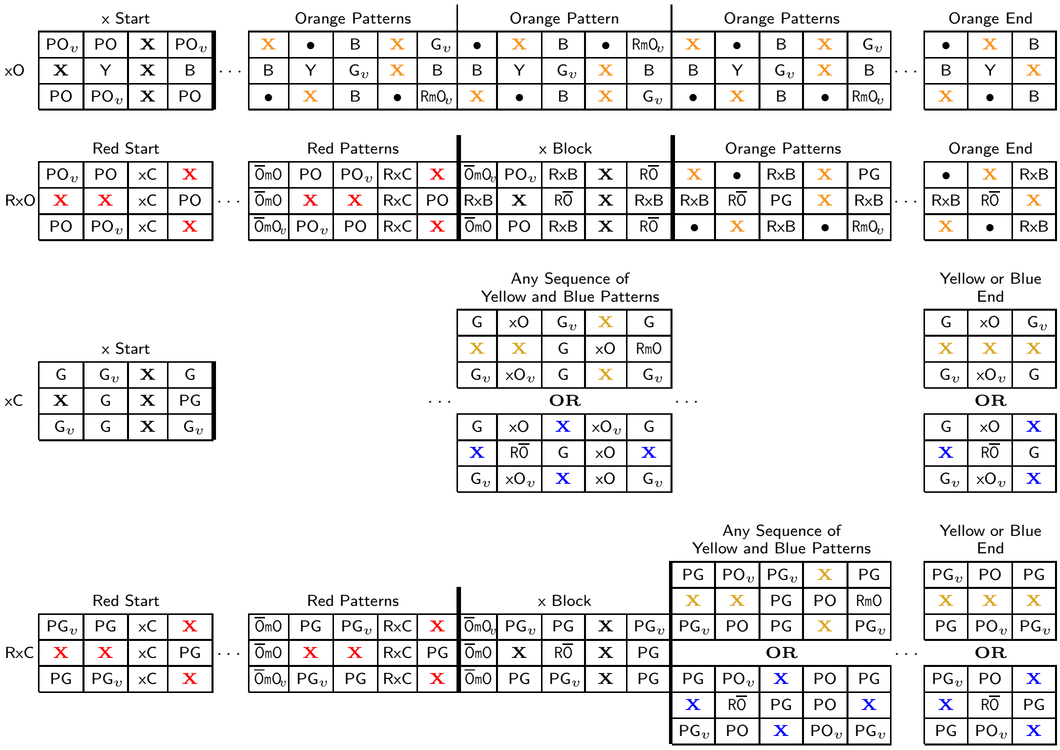

Figure 6: Configurations with x Blocks for $n \equiv 2 \pmod 5$.

that in first blue pattern displayed in the central part, G_v is used to defend the top row of the first column and G used to defend the bottom row of the first column, but this is reversed in the second blue pattern displayed. Which defence is used is dependent on the parity of the number of patterns which precede the pattern of the vertex we are trying to defend. This is due in part to the central parts of the three configurations which follow Pattern 2 alternating between an original block and its vertical reflection. Every second iteration of a pattern has the same defensive scheme and defensive schemes in adjacent blocks have a vertical symmetry. It is important to note that the actual defence of an attack on the vertices in the blue end is also dependent on the parity of the number of patterns which precede it as well. Specifically, when $n \equiv 7 \pmod{10}$ we must defend the listed vertex with the vertical reflection of the configuration shown in Figure 5. This is to reflect the fact that the vertical reflection of the given ending is necessarily used to complete the dominating set in G, O, and P when $n \equiv 7 \pmod{10}$.

Continuing with the B configuration, an attack on an unguarded vertex not in the same column as some guard must be defended by moving guards from the adjacent columns. The vertices labeled O (or O_v) indicate an attack on any one of these vertices can be defended by a transformation to the O configuration (or the vertical reflection of O). The vertices labeled G_h indicates an attack on any one of these vertices can be defended by a transition of the guards to the horizontal reflection of the G configuration. As an example, while the guards are in the B configuration, an

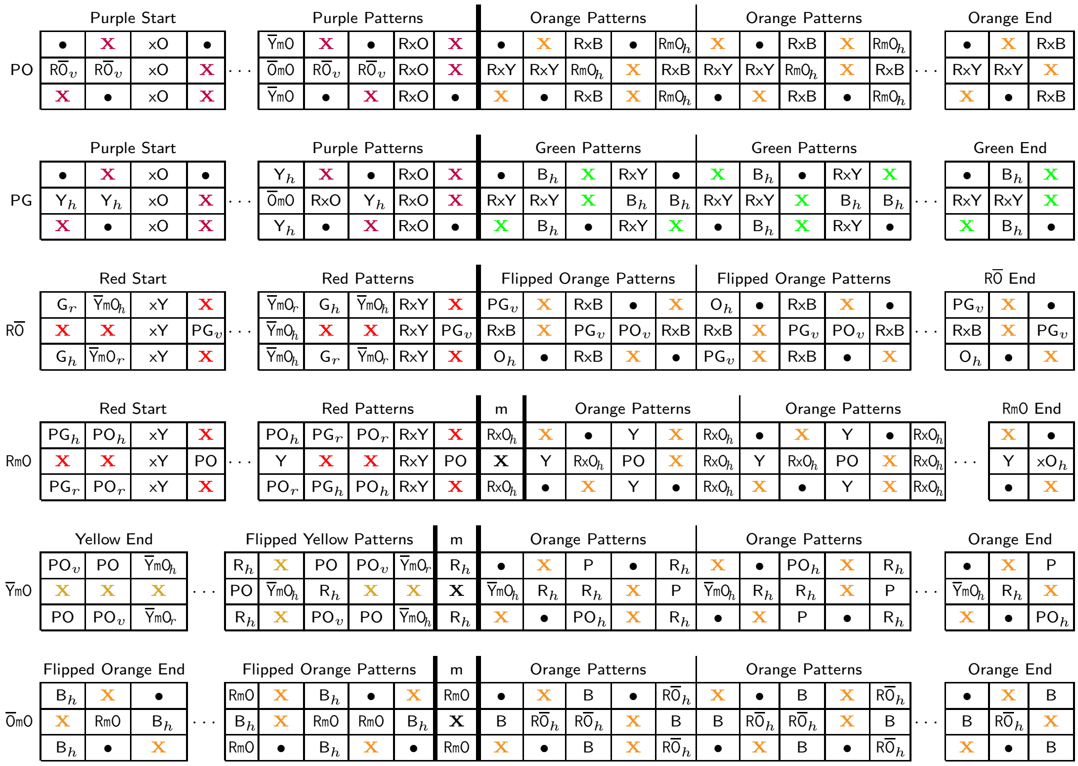

Figure 7: Remaining Configurations for $n \equiv 2 \pmod{5}$.

attack on the vertex in the middle of column 6 (appearing in the second column of the first central block) can be defended by a move of the guard in column 5 to that vertex. This defence is represented by the guard in the middle of the 6^{th} last column in the G configuration (appearing in the middle column of the last central block of the G configuration). Note the difference in the number of columns in the start and end of the configurations. An attack on the middle vertex in the middle column of the blue end can be defended by a move of the guard in the first column of the blue end, which is represented by the guard in the middle of the second column of the green start in G_h.

Now consider the Y configuration. An attack on an unguarded vertex of the Y configuration that appears in the same column as some guard can be defended by a move of most guards either up or down within each of their columns, leading to the O configuration (or the vertical reflection of O) for most columns, and to the G configuration (or the vertical reflection of G) for column 2 and the last column. This is indicated in Figure 5 by the appropriate positions labeled O, O_v, G, or G_v in the Y configuration. All remaining vertices are labeled G, G_v, or P_h, the horizontal reflection of P, which represent an appropriate response to an attack on those vertices. For example, if there is an attack on the top row of column 4, the guards could respond by moving to the G pattern. Similarly, if there is an attack on the middle row of the fourth last column while the guards are in configuration Y, which is the last column of the last central block, the guards could respond by moving to the P_h pattern, so that the new position of the guards in the forth last column is the same as the position of the guards in the forth column of P. As before (and through out), the appropriate defence to use is also dependent on the parity of the number of patterns which precede the pattern of the vertex we are trying to defend.

Many attacks, when the guards are in the G configuration, can be defended by a transformation to G_v, indicated by the bullets (•). This requires the guard at the top of the first column to move right, the guard at the bottom of the second column to move left, all other guards in columns with two guards to move up or down, and all other guards in columns with a single guard to move left or right. As indicated in Figure 5, any remaining attack can be defended by a transformation to either Y or B_h, the horizontal reflection of B. Recall that when $n \equiv 7 \pmod{10}$, the vertical reflection of the illustrated ending is necessarily used to complete the dominating set. In a similar manner, guards in the O configuration can transform to O_v (indicated by •), B, or R_h to defend against any possible attack.

Consider the P configuration. Most of the possible attacks can be defended by a transformation to P_v, R, O_h, or Y_h. However, consider an attack on the top vertex of the third column. The guard in the second column must move defend this attack, leaving only one guard in the first two columns, which must move to the middle of the first column. Every configuration in Figure 5, and every symmetry of a configuration in Figure 5, has at least two guards in the first two columns and hence this attack cannot be defended by a transformation to a configuration in Figure 5. One possible response leads to the pattern of configuration xO illustrated in Figure 6. An attack on the P configuration at one of the vertices labeled RxO in the repeating pattern may also be defended by transforming to a similar configuration where the guard defending the attack is in a column of three guards.

Finally, for the configurations in Figure 5, consider the R configuration. Most of the possible attacks can be defended by a transformation to P, O_h, or their vertical symmetries. However, consider an attack on the middle vertex of the third column. The response can lead to the configuration we have denoted xY, which is an element of the set of configurations labeled xC in Figure 6, consisting of an x start followed by the necessary number of Y blocks, and ending with the end of the Y configuration. An attack on the R configuration at one of the positions labeled RxY in the repeating red pattern may be defended by transforming to a similar configuration where the guard defending the attack is in the column of three guards. A set of possible configurations is illustrated by the configuration RxC in Figure 6 where only Y patterns are chosen, thus it begins with the pattern of R, including zero or more red blocks in the central part, followed by the x pattern in the necessary position, and finishes with the Y pattern, using zero or more yellow blocks in the central part, followed by the yellow end.

Consider an attack on configuration xO in Figure 6. Attacks on some vertices in the first four columns can be defended moving the guards to configurations in PO or PO_v with no P blocks in the central part. Moving the guards to configurations in xO_v, B, Y, or G_v can be used to defend most of the remaining possible attacks. An attack on the top or bottom of the last column of each of the central orange blocks that is not defended by G_v can be defended by moving the guards to the configuration in RmO_v with no central red blocks; that is, the starting four columns of R, the m column as the fifth column, and the remaining columns following the orange pattern and an RmO end (or its vertical reflection as necessary).

For a configuration in the set denoted as RxO, attacks on the vertices labeled with • can be defended by leaving the guards in the R and x patterns stationary, and moving the guards in the O pattern to their vertical reflection. Moving the guards in the R and x patterns to the P pattern and leaving the guards in the O pattern stationary (or as necessary to their vertical reflection), provides a defence for attacks on those vertices labeled PO and PO_v. An attack a vertex labeled xC can be defended by moving the starting block to the x pattern, moving the guards in the R patterns in the central part to the Y pattern, and moving the guards in the x and O patterns to the B pattern. Similarly, an attack on a vertex labeled RxC can be defended by moving the guards in the attacked block to the x pattern, any R blocks to the right of the attacked block to the Y pattern, and the guards in the x and O blocks to the B pattern. To transform guards in a configuration in RxO to a configuration in $\overline{\text{O}}$mO (or a vertical reflection of a configuration in $\overline{\text{O}}$mO), move the guard in the middle of the column with three guards one unit right. This guard becomes the lone guard in the m column, the guards in the O pattern remain stationary (or as necessary to their vertical reflection), and the remaining guards move as needed. The vertices labeled RxB can be defended by the transition of guards in the last R block to an x block and the x and O blocks to B blocks, leading to a configuration in RxC with no Y blocks.

Moving the guards in the x block to an R block pattern, moving those in O blocks to the horizontal reflection of O, and moving those in the ending to the pattern of the ending of R$\overline{\text{O}}$, leads to the R$\overline{\text{O}}$ configuration as a defence for attacks on those vertices labeled R$\overline{\text{O}}$. Attacks on vertices labeled $\mathtt{RmO}_v$ can be defended by moving the guards in the x pattern to the R pattern, using the first guard of the first O block for the m column, and finishing with the orange pattern and $\mathtt{RmO}$ end. The resulting configuration is the vertical reflection of a configuration in $\mathtt{RmO}$. Finally, to defend the vertices labeled PG one may move the guards in the R blocks and the x block to a P pattern (or the vertical reflection of a P pattern) and move those guards in an O block to the G pattern, transforming to a configuration in PG or PG_v. Whether a configuration in RxO transforms to a configuration in PG or PG_v depends on the parity of the number of R blocks.

Consider an attack on a configuration in xC in Figure 6. Most vertices can be defended by moving the guards to G or G_v configurations, since guards in x, Y, and B patterns can all move to the patterns of G and its vertical reflection. An attack on the middle of column 4 can be defended by moving the guards to a PG pattern, where the guards in x starting block transition to the P starting block, and all others move to the G pattern. A transformation to xO (or its vertical reflection) would defend most of the other attacks. To guard attacks on those vertices labeled R$\overline{\text{O}}$, guards in the x block move to the pattern of the R starting block, guards in Y and B patterns move to the pattern of the horizontal reflection of O, and guards in the end move to the pattern of the R$\overline{\text{O}}$ ending. Finally, any configuration in xC transforms to the configuration in $\mathtt{RmO}$ where the m column is the fifth column.

For configurations in RxC, defence of the vertices labeled PG, PG_v, PO, and PO_v can be achieved by the transition of guards in the R blocks and the x block to P and P_v patterns, and the transition of guards in the Y and B blocks to G and G_v or O and O_v patterns. As with RxO, an attack on a vertex labeled xC or RxC in configurations in RxC can be defended by moving the guards in the attacked block to the x block pattern, moving any guards in the R blocks on the right of the attacked block to the Y pattern, and leaving the guards in the Y and B blocks stationary, yielding a configuration in xC or another configuration in RxC itself. Similar parallels to the defence of RxO can be made here for the transformations to the $\mathtt{RmO}$, $\overline{\mathtt{O}}\mathtt{mO}$ and $\overline{\mathtt{O}}\mathtt{mO}_v$ configurations, as well as the R$\overline{\text{O}}$ configuration, where the guards in the x block are moved to an R pattern, the guards in the Y and B blocks are moved to horizontal reflections of O patterns, and the guards in the Y or B ending are moved to the R$\overline{\text{O}}$ ending.

Moving to Figure 7, we note all represented configurations move to their vertical reflections, providing a defence for any attack on vertex labeled •. We will first focus on the response to some attacks on configurations in PO, PG, and R$\overline{\text{O}}$ simultaneously.

Guards in a configuration in PO or PG can defend attacks on vertices labeled xO or RxO by moving the guards in the attacked block to the x pattern, moving guards in any blocks to the left of the attacked block to the R pattern, and moving guards in all blocks to the right, including any guards in P patterns, to the O pattern. In the case the vertex is labeled RxO, the described transformation leads to either a configuration in RxO or a vertical reflection of a configuration in RxO, depending on the number of P blocks in the original configuration. Configurations in $\mathrm{R}\overline{\mathrm{O}}$ can similarly transform to xY and RxY. Guards in a configuration in PO, PG, or $\mathrm{R}\overline{\mathrm{O}}$ can also transform to RxY and RxB, to defend attacks of vertices with these labels, by moving the guards in the last P or R block to an x pattern, and moving the guards in O, G, or flipped O patterns to either Y or B patterns.

Guards in a configuration in PO can defend attacks on vertices labeled $\mathrm{R}\overline{\mathrm{O}}_v$ with a transformation to a configuration in $\mathrm{R}\overline{\mathrm{O}}$ or $\mathrm{R}\overline{\mathrm{O}}_v$ by moving the guards in their P blocks to the R pattern and the guards in their O blocks to horizontal reflections of O patterns. Note that encoding of the defence of a configuration in PO in Figure 7, we use the label $\mathrm{R}\overline{\mathrm{O}}_v$ to indicate an attack on the vertex can be defended by a transformation to a configuration in $\mathrm{R}\overline{\mathrm{O}}$ or a configuration in $\mathrm{R}\overline{\mathrm{O}}_v$, where the actual defence the guards can use depends on the parity of the number of P blocks in the given configuration. Guards in a configuration in PO can defend vertices labeled $\overline{\mathtt{Y}}\mathtt{m}\mathtt{O}$ and $\overline{\mathtt{O}}\mathtt{m}\mathtt{O}$ with transformations to a particular configuration in $\overline{\mathtt{Y}}\mathtt{m}\mathtt{O}$ and a particular configuration in $\overline{\mathtt{O}}\mathtt{m}\mathtt{O}$. In each case, the desired transformation is achieved by having the guard in the middle of the last column of the last P block after the transformation in what becomes the m column of the configuration in $\overline{\mathtt{Y}}\mathtt{m}\mathtt{O}$ or $\overline{\mathtt{O}}\mathtt{m}\mathtt{O}$. The transition of the remaining guards in the P blocks to the appropriate starting blocks completes the transformation. For configuration PO, the remaining vertices whose defence is not outline above are labeled $\mathtt{RmO}_h$. Guards can successfully respond to attacks on these vertices by having the guard in the middle row of the last column of the last P block moving one unit to the left, having the remaining guard in the last column of the last P block and all guards in the O blocks move to the horizontal reflection of R blocks (including the horizontal reflection of the red start at the end), and have the remaining guards remain stationary (or move to the vertical reflection of their current positions as necessary). This results in a configuration in $\mathtt{RmO}_h$ or $\mathtt{RmO}_v$.

We now consider the configuration in PG. It is fairly straightforward to see that any set in PG transforms to both B_h and Y_h, and, as discussed above, the set also transforms to xO, RxO, and RxY. As shown in Figure 7, all vertices in one of these configurations can be defended by a transformation to one of these sets or to $\overline{\mathtt{O}}\mathtt{m}\mathtt{O}$, achieved by having the guard in the middle of the last column of the last P block remain stationary in what becomes the m column of the configuration $\overline{\mathtt{O}}\mathtt{m}\mathtt{O}$, and the remaining guards move as needed.

Above we established that any configuration in $\mathrm{R}\overline{\mathrm{O}}$ can transform to xY, RxY, and RxB. Note that configurations in $\mathrm{R}\overline{\mathrm{O}}$ also transform to both G_r and G_h, providing a defence for an attack on the vertices so labeled. Each configuration in $\mathrm{R}\overline{\mathrm{O}}$ also transforms to a particular configuration in $\overline{\mathtt{Y}}\mathtt{m}\mathtt{O}_h$ and its vertical reflection (attacks on vertices defended by the vertical reflection of this configuration are denoted by $\overline{\mathtt{Y}}\mathtt{m}\mathtt{O}_r$) by moving the guards so that the last column of the last red block becomes the m column of the configuration in $\overline{\mathtt{Y}}\mathtt{m}\mathtt{O}_h$. Guards in a given configuration in $\mathrm{R}\overline{\mathrm{O}}$ can defend attacks on vertices labeled PO_v with a transformation to a configuration in PO or in PO_v by moving the guards in their R blocks to the P pattern and moving the guards in their flipped O blocks to O patterns.

In a similar manner, it can be seen that a configuration in $\mathtt{R\overline{O}}$ transforms to a configuration in PG or in PG_v by moving the guards in their R blocks to the P pattern and moving the guards in flipped O blocks to G patterns. As before, the actual response to an attack (a transformation to a configuration in PG or a transformation to a configuration in PG_v) depends on the parity of the number of R blocks in the original transformation. Finally, an attack on a configuration, say A, in $\mathtt{R\overline{O}}$ could occur on a vertex labeled O_h. If $n \equiv 2 \pmod{10}$ and A has an even number of red blocks or if $n \equiv 7 \pmod{10}$ and A has an odd number of red blocks, then A transforms to O_h. Otherwise, A transforms to O_r, providing a defence for an attack on these vertices.

Consider a configuration in $\mathtt{RmO}$. A transformation to defend an attack on one of the vertices labeled PG_h, PG_r, PO_h, or PO_r can be achieved if the guards in the orange blocks remain stationary (or move so they are in the vertical reflection of their current positions), the guard in the $\mathtt{m}$ column moves one unit left, and the remaining guards move to positions as indicated by the appropriate label. For an attack on one of the vertices labeled PO, the guards in the red blocks move to the P pattern and the remaining guards shift to a repositioned orange pattern. A configuration in $\mathtt{RmO}$ can also defend attacks on the third column of its starting block and the fourth columns of any of its R blocks with transformations to xY or RxY by moving the guards in the attacked block to the x pattern and moving the remaining guards as needed. The x block of the horizontal reflections of (or rotational symmetry of) xO and RxO can defend attacks on corresponding positions in $\mathtt{RmO}$ labeled $\mathsf{x}\mathrm{O}_h$ or $\mathrm{R}\mathsf{x}\mathrm{O}_h$, including the top and bottom of the $\mathtt{m}$ column. A transformation to Y can be used to defend the remaining vertices.

As indicated in Figure 7, every configuration in $\mathtt{\overline{Y}mO}$ transforms to R_h. By moving the guards in yellow end to an orange end (or its vertical image), guards in the flipped yellow blocks to flipped orange blocks (or the vertical reflection), guards in the orange blocks and orange end to yellow blocks and a yellow end, a configuration in $\mathtt{\overline{Y}mO}$ is transformed into the horizontal image or 180 degree rotation of some configuration in $\mathtt{\overline{Y}mO}$, providing a defence to an attack on vertices labeled $\mathtt{\overline{Y}mO}_h$ or $\mathtt{\overline{Y}mO}_r$. Attacks on vertices of a configuration in $\mathtt{\overline{Y}mO}$ labeled PO or PO_v can be defended by a transformation to a set in PO or its vertical symmetry (achieved by moving the guards in the yellow end, flipped yellow patterns, and $\mathtt{m}$ column to P blocks, and having the guards in the orange blocks remain stationary (or move so they are in the vertical reflection of their current positions). Attacks on vertices of a set in $\mathtt{\overline{Y}mO}$ labeled P may be defended similarly, except almost all guards in the orange blocks move one unit right, to transform the configuration in $\mathtt{\overline{Y}mO}$ into either P or P_v. For an attack on vertices labeled PO_h, transition the guards in yellow end to a flipped orange end, the guards in the flipped yellow patterns to flipped orange patterns, and the guards in the $\mathtt{m}$ column and orange patterns to purple patterns (or as needed to the vertical reflection of these blocks) to obtain a configuration in either PO_h or PO_r.

Finally, any attack on a configuration in $\mathtt{\overline{O}mO}$ can be defended by transformations to B, B_h, a configuration in $\mathtt{RmO}$ (where the first guard in the first orange pattern moves to become the m column), or a configuration in $\mathrm{R}\overline{\mathrm{O}}_h$ or $\mathrm{R}\overline{\mathrm{O}}_r$ (where the number of central red blocks matches the number of orange blocks in the attacked $\overline{\mathrm{O}}\mathtt{m}\mathrm{O}$ configuration).

Lemma 5.1 *For $n \equiv 2 \pmod 5$, $n \geq 17$, $\gamma^{\infty}_{all}(P_3 \square P_n) \leq \frac{4n+7}{5}$.*

Proof: The configurations illustrated in Figures 5, 6, and 7 together with all their symmetries demonstrate how to construct one such eternal dominating family for any such possible value of n. As discussed in this section a possible transformation for any attack on any of the dominating sets. ∎

6 Arrangement for $n \equiv 3 \pmod 5$

In this section, we construct an eternal dominating family of $P_3 \square P_n$ when $n \equiv 3 \pmod 5$ by building on the configurations in Figures 5, 6, and 7. The basic idea is to create dominating sets by adding one column with one guard, either at the front or end of these configurations, and show the result is an eternal dominating family.

Lemma 6.1 *For any $n \equiv 3 \pmod 5$ with $n \geq 28$, $\gamma^{\infty}_{all}(P_3 \square P_n) \leq \dfrac{4n+8}{5}$.*

Proof: Let $n \equiv 3 \pmod 5$ be given and let $\mathcal{E}$ be the eternal dominating family presented in Section 5 for a $P_3 \square P_{n-1}$ grid. We form $\mathcal{F}'$ from $\mathcal{E}$ as follows.

1. For each configuration $D \in \mathcal{E}$, add to $\mathcal{F}'$ the configuration with a guard in the middle of the first column and which is identical to D on the remaining $n-1$ columns. The configuration created by this process (and its vertical reflection) will be denoted $\mathtt{m}D$ (and $\mathtt{m}D_v$).

   We note that $\mathtt{m}(D_h)$ is the horizontal reflection of the configuration identical to D in the first $n-1$ columns with a guard m in the middle of the last column. Therefore we will denote $(\mathtt{m}(D_h))_h$ with the notation $D\mathtt{m}$.

2. For each configuration $D \in \mathcal{E}$ with a guard in the bottom row (respectively in the top row) of the first column, add to $\mathcal{F}'$ the configuration of $P_3 \square P_n$ which is identical to D on the last $n-1$ columns and with a guard in the top row (respectively bottom row) of the new first column. A configuration of this form (and its vertical reflection) will be denoted $\mathtt{t}D$, and $(\mathtt{t}(D_h))_h$ will be denoted by $D\mathtt{t}$.

   In the three cases where configurations in $\mathcal{E}$ have a guard in both the top row and the bottom row - B$\mathtt{t}$, RxC$\mathtt{t}$, and xC$\mathtt{t}$ - the two sets that may be formed with this process are vertical reflections of each other.

For example, the sets that have been added to $\mathcal{F}'$ that are associated with the configurations B and B_h are shown in Figure 8.

Let $\mathcal{F} = \mathcal{F}' \cup \{D_h | D \in \mathcal{F}'\} \cup \{D_v | D \in \mathcal{F}'\} \cup \{D_r | D \in \mathcal{F}'\}$. To establish the Lemma, it suffices to show the family $\mathcal{F}$ is an eternal dominating family of $P_3 \square P_n$.

According to the construction defined above, each set in $\mathcal{F}$ is a dominating set with $\frac{4n+8}{5}$ guards. Let $D \in \mathcal{F}'$. We note there exists a set $E \in \mathcal{E}$ such that D is either $\mathtt{m}E$ or $\mathtt{t}E$. Consider an attack on a vertex $v \in V(P_3 \square P_n) - D$.

We have four cases.

Case 1: v is not in the first column.

The defender considers only the position of the guards and the attacker on the last $n-1$ columns. The guards on these vertices are positioned on the set E. As $\mathcal{E}$ is an eternal dominating family for a $P_3 \square P_{n-1}$ grid, there is a set $E' \in \mathcal{E}$ so that $v \in E'$ and E transforms to E'. It follows that D transforms to $D' = \mathtt{m}E'$ and $v \in D' \in \mathcal{F}$.

Case 2: v is the middle row of the first column.

Clearly $v \notin D$, so $D = \mathtt{t}E$. The set D transforms to $D' = \mathtt{m}E$ and $v \in D' \in \mathcal{F}$.

Case 3: v is the top row (or bottom row) of the first column and $D = \mathtt{m}E$.

Let u be the vertex in the bottom row of the second column. If $u \in D$, then D transforms to $\mathtt{t}D$ and $v \in \mathtt{t}D$. If $u \notin D$, the defender momentarily considers only the position of the guards and the attacker on the last $n-1$ columns. The guards on these vertices are positioned on the set E. As $\mathcal{E}$ be the eternal dominating family for a $P_3 \square P_{n-1}$ grid, there is a set $E' \in \mathcal{E}$ so that $u \in E'$ and E transforms to E'. It follows that D transforms to $D' = \mathtt{t}E'$ and $u, v \in D' \in \mathcal{F}$.

Case 4: v is the top row (or bottom row) of the first column and $D = \mathtt{t}E$.

Let u be the vertex in the bottom row of the second column. By definition of $\mathcal{F}'$, $u \in D$. Careful inspection shows D is one of the sets listed in Table 1. The second column represents a set $D' \in \mathcal{F}$ so that $v \in D'$ and D transforms to D'.

We conclude that for any set $D \in \mathcal{F}'$ and for any attack on a vertex $v \in V(P_3 \square P_n) - D$, there exists a set $D' \in \mathcal{F}$ such that $v \in D'$ and D transforms to D'. If $D \in \mathcal{F} - \mathcal{F}'$, for some symmetry s, $D_s \in \mathcal{F}'$. Consider an attack on some $v \in V(P_3 \square P_n) - D$. Let $v_s \in V(P_3 \square P_n)$ be the image of v under the symmetry s. As $D_s \in \mathcal{F}'$, there exists a dominating set $D' \in \mathcal{F}$ so that $v_s \in D'$ and D_s transforms

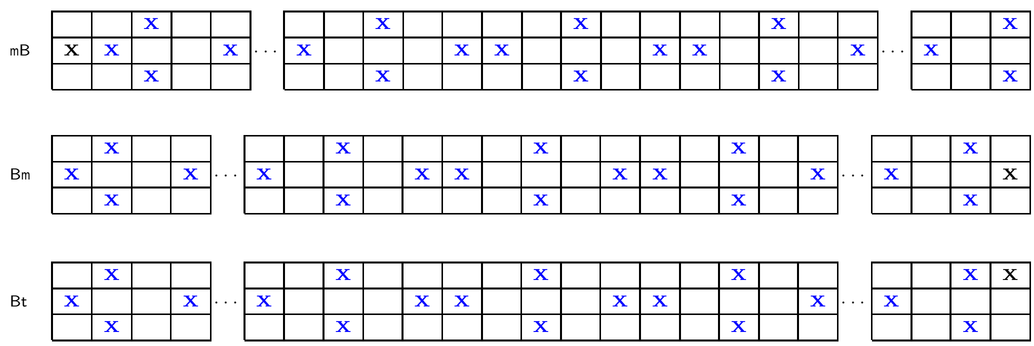

Figure 8: New configurations formed by adding one column to B or B_h.

Configuration	Response to attack
tG	tG
tO	tO
tP	$\mathtt{m}\mathrm{P}_h$
Bt	Gt
Gt	Bt
Rt	Rt
Pt	Rt

Configuration	Response to attack
xCt	Gt
RxCt	PGt
PGt	RxCt
$\mathrm{R}\overline{\mathrm{O}}\mathrm{t}$	$\mathrm{R}\overline{\mathrm{O}}\mathrm{t}$
$\mathrm{R}\mathtt{m}\mathrm{Ot}$	$\mathtt{m}\mathrm{P}$
tPG	$\mathrm{P}\mathtt{m}$
tPO	$\mathrm{P}\mathtt{m}$

Table 1: Response to attacks from Case 4.

to D'. It follows that $v \in D'_s$, the symmetry of D', and D transforms to D'_s. As $D'_s \in \mathcal{F}$, it follows that $\mathcal{F}$ is an eternal dominating family of $P_3 \square P_n$. ■

7 Main Result

We are now ready to present the main result of the paper.

Theorem 7.1 *For all $n \geq 26$,*

$$\gamma_{all}^{\infty}(P_3 \square P_n) = \left\lceil \frac{4n+7}{5} \right\rceil.$$

Proof: By Corollary 4.5, Lemma 5.1 and Lemma 6.1, the result holds when $n \equiv 2, 3 \pmod 5$. In [4], it is established that for $2 \leq k \leq 5$, $\gamma_{all}^{\infty}(P_3 \square P_k) = k$. Noting the first m columns can be guarded independently of the last k columns, $2 \leq k \leq 5$,

$$\gamma_{all}^{\infty}(P_3 \square P_{m+k}) \leq \gamma_{all}^{\infty}(P_3 \square P_m) + \gamma_{all}^{\infty}(P_3 \square P_k) = \gamma_{all}^{\infty}(P_3 \square P_n) + k. \quad (1)$$

Note that $\gamma_{all}^{\infty}(P_3 \square P_{21}) = 18$ [2]. When $n = 26$, the result now follows from (1) and Corollary 4.5. For $n > 26$, $n \equiv 0, 1, 4 \pmod 5$ there exists integers m and k so that $m \equiv 2 \pmod 4$, $2 \leq k \leq 4$ and $n = m + k$. Therefore the result follows from (1), Corollary 4.5, and Lemma 5.1. ■

With this result and previously determined values for smaller grids, the eternal domination numbers for all $3 \times n$ grid graphs are now determined, and can be summarized as:

$$\gamma_{all}^{\infty}(P_3 \square P_n) = \begin{cases} \left\lceil \frac{6n+2}{7} \right\rceil & \text{if } n \leq 11 \\ \left\lceil \frac{4n+6}{5} \right\rceil & \text{if } 11 < n \leq 22 \\ \left\lceil \frac{4n+7}{5} \right\rceil & \text{otherwise.} \end{cases}$$

As a sequence starting with the eternal domination number of a 3×1 grid graph, it is OEIS Sequence Number A289188 [10], presented as follows:

$$2, 2, 3, 4, 5, 6, 7, 8, 8, 9, 10, 11, 12, 13, 14, 14, 15, 16, 17, 18, 18, 19,$$

$$20, 21, 22, 23, 23, 24, 25, 26, 27, 27, 28, 29, 30, 31, 31, 32, 33, 34, 35, 35, \ldots$$

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References

[1] I. Beaton, S. Finbow and J. A. MacDonald, Eternal Domination Numbers of $4 \times n$ Grid Graphs, *J. Combin. Math. Combin. Comput.* **85** (2013), 33–48.

[2] S. Finbow, M. E. Messinger and M. F. van Bommel, Eternal Domination on $3 \times n$ Grid Graphs, *Australas. J. Combin.* **61** (2) (2015), 156–174.

[3] W. Goddard, S. M. Hedetniemi and S. T. Hedetniemi, Eternal Security in Graphs, *J. Combin. Math. Combin. Comput* **52** (2005), 169–180.

[4] J. L. Goldwasser, W. F. Klostermeyer and C. M. Mynhardt, Eternal Protection in Grid Graphs, *Util. Math.* **91** (2013), 47–64.

[5] D. Gonçalves, A. Pinlou, M. Rao and S. Thomassé, The Domination Number of Grids, *SIAM J. Discrete Math.* **25** (3) (2011), 1443–1453.

[6] M. S. Jacobson and L. F. Kinch, On the Domination Number of Products of Graphs: I, *Ars Combin.* **18** (1983), 33–44.

[7] W. F. Klostermeyer and G. MacGillivray, Eternal Dominating Sets in Graphs, *J. Combin. Math. Combin. Comput.* **68** (2009), 97–111.

[8] W. F. Klostermeyer and C. M. Mynhardt, Protecting a Graph with Mobile Guards, *Appl. Anal. Discrete Math.* **10** (2016), 1–29.

[9] M. E. Messinger and A. Z. Delaney, Closing the Gap: Eternal Domination on $3 \times n$ Grids, *Contrib. Discrete Math.* **12** (1) (2017), 47–61.

[10] N. J. A. Sloane, ed., Sequence A289188, *The On-Line Encyclopedia of Integer Sequences*, https://oeis.org (2017).

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