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Firefighting on trees and Cayley graphs

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Abstract

We study Hartnell's firefighter problem on infinite trees and characterise the branching number in terms of the firefighting game. Using our results about trees, we give a partial answer to a question of Martínez-Pedroza concerning firefighting on Cayley graphs.

1 Introduction

In 1995 Hartnell [5] introduced the firefighting game which can be described as follows. Before the first round of the game, an antagonist sets some subset of the vertices of a graph G on fire. Then, in each round n, we can protect f_n vertices whereafter the fire spreads to all unprotected neighbours of burning vertices. Once a vertex is burning or protected, it remains in that state for the rest of the game. This can for example be seen as a model for the spread of a perfectly contagious disease with no cure, see [1]. The act of protecting vertices at each time step, could then be viewed as vaccinations.

There are several different goals that we might want to pursue, e.g. minimise number of rounds or number of burnt vertices, or save a certain set of vertices or a given fraction of the vertices from being burnt. The survey paper [3] gives an overview on different lines of research concerning the firefighting game.

In this paper we focus on the question of containment. We say that a fire can be contained on an infinite graph if we can prevent it from spreading to infinitely many vertices. An infinite graph G satisfies f_n-containment, if any finite initial fire can be contained by protecting f_n vertices in round n.

Containment was first studied in grids, the first results being that certain planar grids satisfy constant containment, i.e. containment for $f_n \equiv c$, see [1, 9]. Develin

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and Hartke [1] showed that higher dimensional square grids do not satisfy constant containment. However, it is easy to see that they satisfy polynomial containment, that is, f_n-containment for some polynomial f_n. In fact, Dyer, Martínez-Pedroza, and Thorne [2] showed that if the balls of radius n around some (equivalently any) vertex of G contain $O(n^d)$ vertices, then G has the f_n-containment property for some $f_n = O(n^{d-2})$.

We study the question of exponential containment. We say that a graph satisfies exponential containment of rate $\lambda \geq 1$ if it satisfies f_n-containment for some $f_n = O(\lambda^n)$. It is easy to see that for every graph G there is a threshold $\lambda_c^{\mathrm{F}} \in [1, \infty]$ such that for every $\lambda > \lambda_c^{\mathrm{F}}$ it satisfies exponential containment of rate λ whereas for $\lambda < \lambda_c^{\mathrm{F}}$ it doesn't.

We show that the critical containment rate λ_c^{F} of a tree T coincides with the branching number $\operatorname{br} T$ of this tree (see the next section for a definition). It is worth noting that the branching number also comes up as a threshold in different problems. It marks the transition from transience to recurrence of the homesick random walk on a tree and $\frac{1}{\operatorname{br} T}$ is the percolation threshold on an infinite tree, see [6].

As an application of our results about trees we make progress towards a question of Martínez-Pedroza [8]. He showed that Cayley graphs of non-amenable groups do not satisfy polynomial containment and asked whether polynomial containment always implies polynomial growth for Cayley graphs. We show that for a locally finite Cayley graph with exponential growth of rate $\alpha > 1$ we have $\lambda_c^{\mathrm{F}} = \alpha$. This implies that such a Cayley graph can never satisfy polynomial containment, only leaving open the notoriously difficult case of groups with intermediate growth.

2 Preliminaries

Throughout this paper $G = (V, E)$ denotes a graph with vertex set V and edge set E. All graphs considered will be connected and locally finite. For a set M of vertices and edges of G, denote by $G - M$ the subgraph of G obtained by removing all elements of M from G. In case M contains vertices, we remove all edges incident to these vertices as well.

Let $r \in V$ and assume that G is rooted at r. For a vertex or edge x denote by $|x|_r$ the length of a shortest path containing both r and x. For convenience we will omit the subscript and simply write $|x|$, if the root is clear from the context. Define the *ball of radius k with center r* by $B_r(k) = \{v \in V \mid |v| \leq k\}$.

The *(exponential) growth rate* of a graph is defined by $\operatorname{gr} G = \lim_{k\to\infty} (|B_r(k)|)^{\frac{1}{k}}$ if the limit exists. Note that if the growth rate exists, then it does not depend on the base point r. A graph has *exponential growth* if $\liminf_{k\to\infty} (|B_r(k)|)^{\frac{1}{k}} > 1$. In particular, if the growth rate of G exists, then the graph has exponential growth if and only if $\operatorname{gr} G > 1$.

For a tree T the *branching number* $\operatorname{br} T$ provides another measure for growth. Its logarithm was first introduced by Furstenberg [4] as the Hausdorff-dimension of the boundary of T. Lyons [6] gave the following combinatorial definition of $\operatorname{br} T$ and pointed out its close connections to random walks and percolation on trees. Let T be a tree rooted at r and call a set Π of edges a *cutset*, if $T - \Pi$ is disconnected. The branching number is defined as

$$\operatorname{br} T = \sup \left\{ \lambda \mid \inf_{\Pi} \sum_{e \in \Pi} \lambda^{-|e|} > 0 \right\},$$

where the infimum runs over all cutsets $\Pi \subseteq E$ whose removal leaves the root r in a finite component. It is worth mentioning that the branching number does not depend on the choice of the root. Note that if the degrees are unbounded, then it is possible that $\operatorname{br} T$ is infinite.

The firefighting game is defined as follows: Let G be an infinite graph and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative integers. Before the first round, a finite set X_0 of vertices of G are marked as burning. In round n, the player can pick f_n vertices which are not burning to mark as protected. Afterwards every unprotected vertex which is adjacent to a burning vertex is marked as burning. Once a vertex is marked as burning or protected, it remains in that state until the end of the game. The player wins the game, if only finitely many vertices are marked as burning throughout the game—in this case we say that the fire is *contained*. For locally finite graphs, an equivalent condition is that after finitely many rounds no new vertices are marked as burning.

An *f_n-strategy* is a a map s from $\mathbb{N}$ to the power set of V such that $|s(n)| \leq f_n$. Call an f_n-strategy *s legal for X_0*, if no vertex in $s(n)$ is burning in round n provided that the set of vertices initially on fire was X_0, and $s(i)$ is marked as protected in round i for every $i < n$. Note that any legal f_n-strategy for X_0 is also legal for any subset of X_0. An *f_n-containment strategy for X_0* is an f_n-strategy which is legal for X_0 such that marking $s(n)$ protected in round n leads to containment, provided the set of vertices initially marked as burning was X_0. A graph satisfies *f_n-containment*, if there is an f_n-containment strategy for every finite set X_0. The following result tells us that we don't need to check all sets though.

Lemma 2.1 *Let G be a locally finite graph and let r be a vertex of G. Then G satisfies f_n-containment if and only if there is a f_n-containment strategy for each $B_r(k), k \in \mathbb{N}$.*

PROOF: If k is large enough that $X_0 \subseteq B_r(k)$, then an f_n-containment strategy for $B_r(k)$ is also an f_n-containment strategy for X_0. □

On trees we can restrict the strategies even further. We say that a set V' of vertices *surrounds* a finite set of vertices X_0, if $V' \cap X_0 = \emptyset$ and no vertex of X_0 is contained in an infinite component of $G - V'$. Note that any f_n-containment strategy for X_0 protects a set V' surrounding X_0: Let X_∞ be the (finite) set of burning vertices at the end of the game and let V' be the set of vertices outside of X_∞ which have a neighbour in X_∞. Every vertex in V' is protected, otherwise the fire would have spread further. Furthermore, V' surrounds the finite set X_∞, and since $X_0 \subseteq X_\infty$ it also surrounds X_0. Conversely, it is easy to see that any legal strategy which protects a set V' surrounding X_0 is a containment strategy. The following lemma tells us that in the case of trees, a containment strategy for $B_r(k)$ is equivalent to the existence of a set V' surrounding $B_r(k)$ which does not grow too quickly.

Lemma 2.2 *A tree T satisfies f_n-containment if and only if for some vertex r and every $k \in \mathbb{N}$ there is $V' \subseteq V$ such that V' surrounds $B_r(k)$, and $V'_{\leq n} := \{x \in V' \mid |x| \leq k + n\}$ has size at most $\sum_{i \leq n} f_i$.*

PROOF: Any containment strategy for $B_r(k)$ protects such a set V'. Conversely, assume that such a set V' exists. Order the elements of V' by distance from r (with ties broken arbitrarily) and let $s(n)$ contain the f_n smallest elements not contained in $s(i)$ for $i < n$. This strategy is legal for $B_r(k)$ due to the restriction on the size of $V'_{\leq n}$ and the fact that the set X_{n-1} of burning vertices before step n is contained in $B_r(k + n - 1)$. It is a containment strategy for $B_r(k)$ since V' surrounds $B_r(k)$ and we conclude by Lemma 2.1. □

A graph G satisfies *exponential containment of rate* $\lambda \geq 1$ if there is $f_n = O(\lambda^n)$ such that G satisfies f_n-containment. Clearly, if G satisfies exponential containment of rate λ, then it also satisfies exponential containment of any rate $\lambda' > \lambda$. Hence there is a critical rate $\lambda_c^{\mathrm{F}} \in [1, \infty]$ such that for $1 \leq \lambda < \lambda_c^{\mathrm{F}}$, the graph G does not satisfy exponential containment of rate λ, whereas for $\lambda > \lambda_c^{\mathrm{F}}$ it does. Note that $\lambda_c^{\mathrm{F}} = \infty$ means that G does not satisfy exponential containment of any rate. Further note that we do not say anything about containment at the critical rate, in particular $\lambda_c^{\mathrm{F}} = 1$ only means that G satisfies exponential containment of any rate $\lambda > 1$, but not necessarily exponential containment of rate 1 (i.e. constant containment).

3 Proof of the main result

In this section we determine the critical rate λ_c^{F} for exponential containment on trees. It turns out that λ_c^{F} equals the branching number, hence our main theorem can be used to define the branching number in terms of the firefighter game.

Theorem 3.1 *If T is a locally finite tree, then $\lambda_c^{\mathrm{F}} = \operatorname{br} T$.*

PROOF: We first show that $\lambda_c^{\mathrm{F}} \leq \operatorname{br} T$, that is, T satisfies exponential containment of any rate $\lambda > \operatorname{br} T$. If $\operatorname{br} T = \infty$ there is nothing to show. Otherwise, let $\lambda > \operatorname{br} T$ and let $f_n = \lfloor \lambda^n \rfloor$. Pick and arbitrary $k \in \mathbb{N}$ and let $\epsilon = \lambda^{-k} - \lambda^{-k-1}$. Note that $\epsilon > 0$ since $\operatorname{br} T$ is by definition always at least 1 and $\lambda > \operatorname{br} T$. For $n > k$ we have

$$\epsilon \cdot \lambda^n < \lambda^{n-k} - \lambda^{n-k-1} \leq \lambda^{n-k} - 1 \leq \lfloor \lambda^{n-k} \rfloor.$$

Since $\lambda > \operatorname{br} T$, we can pick a cutset Π whose removal leaves r in a finite component such that $\sum_{e\in\Pi} \lambda^{-|e|} < \epsilon$. The set Π does not contain any edge in $B_r(k)$ because $\epsilon < \lambda^{-k}$. Hence the set V' containing the endpoint of each $e \in \Pi$ which is further away from r surrounds $B_r(k)$. Let $V'_n := \{v \in V' \colon |v| = n\}$. Then

$$\epsilon \geq \sum_{e\in\Pi} \lambda^{-|e|} = \sum_{v\in V'} \lambda^{-|v|} > \sum_{v\in V'_n} \lambda^{-|v|} = |V'_n| \cdot \lambda^{-n},$$

whence

$$|V'_n| \leq \epsilon \cdot \lambda^n < \lfloor \lambda^{n-k} \rfloor = f_n.$$

Summing up over n shows that V' satisfies the properties required by Lemma 2.2. Since k and $\lambda > \operatorname{br} T$ were arbitrary, Lemma 2.1 shows that T satisfies exponential containment of rate λ for any $\lambda > \operatorname{br} T$ and consequently $\lambda_c^{\mathrm{F}} \leq \operatorname{br} T$.

It remains to show that $\lambda_c^{\mathrm{F}} \geq \operatorname{br} T$. If $\operatorname{br} T = 1$, then there is nothing to show. Hence assume that $\operatorname{br} T > 1$, let $1 < \lambda < \operatorname{br} T$ and let $f_n = K \cdot \lfloor \lambda^n \rfloor$. Let $C = \frac{K\lambda}{\lambda - 1}$. Then

$$\sum_{i=1}^{n} K \cdot \lfloor \lambda^i \rfloor \leq C \cdot \lambda^n.$$

Choose μ such that $\lambda < \mu < \operatorname{br} T$. Note that since $\mu < \operatorname{br} T$, there is some $\epsilon > 0$ such that for every cutset Π whose removal leaves r in a finite component we have

$$\sum_{e\in\Pi} \mu^{-|e|} > \epsilon.$$

Finally let k be such that

$$C \cdot \sum_{n=k+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n < \epsilon.$$

We now claim that there is no set V' satisfying the assertions of Lemma 2.2 for k chosen as above. Assume there was one, and let $\Pi \subseteq E$ be the set containing for every $v' \in V'$ the first edge of the path from v' to r. Then Π is a cutset whose removal leaves r in a finite component.

Let $V'_n := \{v \in V' \colon |v| = n\}$. Then

$$|V'_n| \leq \sum_{i=1}^{n-k} K \cdot \lfloor \lambda^i \rfloor \leq C \cdot \lambda^n.$$

Furthermore $|V'_n| = 0$ for $n \leq k$ because V' surrounds $B_r(k)$. Putting all of the above together we get

$$\epsilon < \sum_{e\in\Pi} \mu^{-|e|} = \sum_{v\in V'} \mu^{-|v|} = \sum_{n=k+1}^{\infty} |V'_n| \cdot \mu^{-n} \leq C \cdot \sum_{n=k+1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n < \epsilon,$$

which is a contradiction. Hence T does not satisfy f_n-containment, and thus it does not satisfy exponential containment of rate λ. Since this holds for any $\lambda < \operatorname{br} T$ it follows that $\lambda_c^{\mathrm{F}} \geq \operatorname{br} T$. □

4 Cayley graphs

In this section we use the main result of the previous section as well as some known results about Cayley graphs to determine the exponential containment threshold λ_c^{F} for Cayley graphs.

For this purpose we need the following definition. Let T be a tree rooted at r. For a vertex v define T_v as the subtree induced by all vertices x such that the unique path from r to x in T uses v, and root T_v at v. The tree T is called subperiodic, if there is $k \in \mathbb{N}$ such that for every v there is v' with $|v'| \leq k$ and T_v embeds into $T_{v'}$ as a subtree in a way that maps v to v'.

It is known that for a subperiodic tree the growth rate exists and coincides with the branching number, see [6] for a proof.

Let Γ be a finitely generated group and let G be a Cayley graph of Γ with respect to the finite generating set $\{x_1, \dots, x_k\}$. The following construction due to Lyons [7] gives a subperiodic spanning tree of G with the same exponential growth rate as G: Fix an arbitrary total order on $\{x_1, \dots, x_k\}$. For every $v \in \Gamma$ there is a unique word $[v] = (x_{i_1}, \dots, x_{i_l})$ such that

- $x_{i_1} \cdots x_{i_l} = v$,
- l is the distance from v to id in G, and
- $[v]$ is lexicographically minimal among all words with the first two properties.

Let T be the graph with vertex set Γ and an edge from v to w if $[v]$ is an extension of $[w]$ by one letter (or vice versa). Then T is easily seen to be a subperiodic spanning tree, rooted at id. Furthermore balls with the same radius centred at the identity in G and T contain the same elements, and consequently the branching number of T equals the growth rate of G (and in particular, $\operatorname{gr} G$ exists). From this we can now deduce the following result.

Theorem 4.1 *For any connected locally finite Cayley graph G we have $\lambda_c^{\mathrm{F}} = \operatorname{gr} G$.*

Proof: We first show that G satisfies exponential containment of any rate $\lambda > \operatorname{gr} T$. Let $f_n = \lfloor \lambda^n \rfloor$. By Lemma 2.1 it is sufficient to give an f_n-containment strategy for $B_r(k)$. If we decide to not mark any vertices as protected, then at step n, the set of burning vertices is $B_r(k+n)$. Wait until $\lfloor \lambda^n \rfloor$ is larger than the boundary of $B_r(k+n)$, then pick all vertices in this boundary at once. This is possible since λ^n asymptotically grows quicker than $|B_r(k+n+1)|$ and hence also faster than the boundary of the ball of radius $k+n$.

For the proof of $\lambda_c^{\mathrm{F}} < \operatorname{gr} G$ note that if G satisfies exponential containment of some rate λ, then so does every subgraph of G. But the subperiodic spanning tree T of G with $\operatorname{br} T = \operatorname{gr} G$ does not satisfy exponential containment of any rate $\lambda < \operatorname{gr} G$ by Theorem 3.1. $\square$

Corollary 4.2 *A Cayley graph of a group with exponential growth never satisfies polynomial containment.*

PROOF: For any $d \in \mathbb{N}$ and $\lambda > 1$ we have $n^d = o(\lambda^n)$. □

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