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Lower bounds for the Laplacian spectral radius of an oriented hypergraph

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Abstract

An oriented hypergraph is a hypergraph where each vertex-edge incidence is given a label of $+1$ or -1. This labeling allows one to naturally define adjacencies so the Laplacian matrix may be defined and studied. In this work, new lower bounds for the Laplacian spectral radius of incidence-simple oriented hypergraphs are found that improve on previous lower bounds by including edge size instead of only using vertex degree. The main result is further improved by making use of the incidence dual of the oriented hypergraph. These bounds also improve the bounds for the k-uniform cases, which also specializes to signed and unsigned graphs.

1 Introduction

An *oriented hypergraph* is a hypergraph where each vertex-edge incidence is given a label of $+1$ or -1 [13,16]. Shi also called this type of hypergraph a *signed hypergraph* and used it to model the constrained via minimization (CVM) problem or two-layer routings [17,18]. Oriented hypergraphs were independently developed to generalize oriented signed graphs [19] as well as related matroid properties [13,16]. A generalization of directed graphs, known as *directed hypergraphs*, also have this type of vertex-edge labeling (see for example [6], and the references therein). One feature that distinguishes oriented hypergraphs from these other related incidence structures

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is the notion of an *adjacency signature* that naturally allows the adjacency and Laplacian matrices to be defined and studied [1, 11, 13]. This is an alternative approach to studying matrices and hypermatrices associated to hypergraphs [2–5], [8–10], that does not require a uniformity condition on the edge sizes and allows reasonably quick spectral calculations. Rodríguez also developed a version of the adjacency and Laplacian matrices for hypergraphs without a uniformity requirement on edge sizes [15]. The definition of adjacency signature and the derived matrices could be applied to directed hypergraphs and their many applications.

The initial work done on the spectral properties of oriented hypergraphs has generalized some known properties for signed graphs and unsigned graphs [1], [11–14]. In this paper we improve upon the lower bounds for the Laplacian spectral radius of an oriented hypergraph. In [12], it was shown that the vertex-edge incidence dual can be used to calculate the nonzero Laplacian eigenvalues of an oriented hyperaph, and also can be used for developing new bounds. At the conclusion of this paper, we expand on this and show how considering the incidence dual can improve the bound performance. Several examples are also examined at the end of this paper which lead to some open questions for future research.

2 Background

2.1 Oriented Hypergraphs

Let V and E denote the set of *vertices* and *edges* respectively. An *incidence function* is a mapping $\iota : V \times E \to \mathbb{N} = \{0, 1, \ldots\}$. A vertex v and edge e are *incident* with respect to ι if $\iota(v, e) \neq 0$. An *incidence* is a triple (v, e, k), where v and e are incident and $k \in \{1, 2, 3, \ldots, \iota(v, e)\}$. The value $\iota(v, e)$ is called the *multiplicity* of the incidence.

Let $\mathcal{I}$ be the set of incidences determined by the incidence function ι. An *incidence orientation* is a function $\sigma : \mathcal{I} \to \{+1, -1\}$. An *oriented hypergraph* is a quadruple $G = (V, E, \mathcal{I}, \sigma)$. The *underlying hypergraph* is the triple $H = (V, E, \mathcal{I})$. The notation $V(G)$, $E(G)$, $\mathcal{I}(G)$ and σ_G may also be used if necessary. Let $n := |V|$ and $m := |E|$.

Two not necessarily distinct vertices v_i and v_j are said to be *adjacent with respect to edge* e if there exist incidences (v_i, e, k_1) and (v_j, e, k_2) such that $(v_i, e, k_1) \neq (v_j, e, k_2)$. Thus, an *adjacency* is a quintuple $(v_i, k_1; v_j, k_2; e)$, where v_i and v_j are adjacent with respect to edge e via incidences (v_i, e, k_1) and (v_j, e, k_2). Each adjacency $(v_i, k_1; v_j, k_2; e)$ has an associated *sign*, (or *adjacency signature*), defined as

$$\mathrm{sgn}_e(v_i, k_1; v_j, k_2) = -\sigma(v_i, e, k_1)\sigma(v_j, e, k_2). \tag{2.1}$$

An oriented hypergraph is *incidence-simple* if $\iota(v, e) \leq 1$ for all v and e. Therefore, instead of writing $(v, e, 1)$, the alternative notation (v, e) will be used in such cases. Similarly, the notation $\sigma(v, e)$ will be used for for the orientation of incidence (v, e), and $\mathrm{sgn}_e(v_i, v_j)$ for the sign of the adjacency $(v_i; v_j; e)$. After these simplifications Equation 2.1 becomes

$$\mathrm{sgn}_e(v_i, v_j) = -\sigma(v_i, e)\sigma(v_j, e). \tag{2.2}$$

Unless otherwise stated, for the remainder of this paper all hypergraphs and oriented hypergraphs are assumed to be incidence-simple. See Figure 1 for an example of an oriented hypergraph. If the reader is interested in related examples of recent research where incidence-simple is not assumed see [1, 16]. Let $V(e)$ denote the set of vertices incident with edge e.

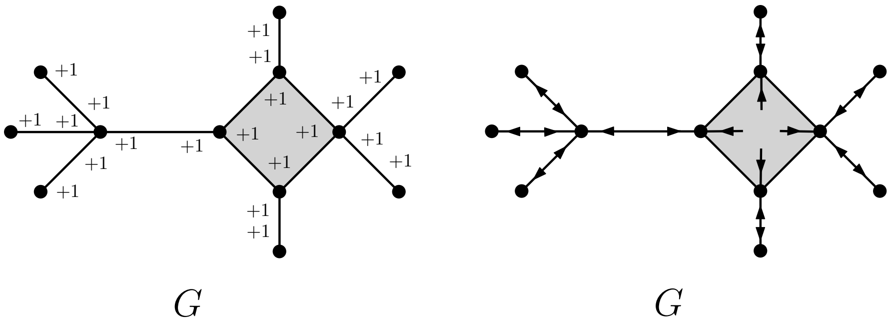


Figure 1: An oriented hypergraph G drawn using two different conventions. On the left the vertex-edge incidences are labelled with the values of the incidence orientation σ. On the right, the σ values are drawn using the arrow convention of $+1$ as an arrow going into a vertex and -1 as an arrow departing a vertex. Here the gray square is a single edge of size 4.

The *degree* of a vertex v_i, denoted by $d_i = \deg(v_i)$, is equal to the number of incidences containing v_i. The *size* of an edge e is the number of incidences containing e. A *k-edge* is an edge of size k. A *k-uniform hypergraph* is a hypergraph such that all of its edges have size k.

Let $G = (V, E, \mathcal{I}, \sigma)$ be an oriented hypergraph, and suppose $v \in V$ is some chosen vertex. The *weak vertex-deletion* of G by the vertex v, denoted $G\backslash v$, is the oriented hypergraph obtained by deleting v from the set V and restricting the incidence function ι to $V\backslash\{v\} \times E$, thus deleting any incidences containing v. That is $G\backslash v = (V\backslash\{v\}, E, \mathcal{I}_v, \sigma_v)$, where $\mathcal{I}_v$ is the set of incidences determined by the restricted incidence function $\iota|_{V\backslash\{v\}\times E}$, which also restricts the incidence orientation $\sigma_v \colon \mathcal{I}_v \to \{+1, -1\}$, but preserving orientations of undeleted incidences. Observe that edges incident to v are not deleted in $G\backslash v$, as in the vertex-deletion of a graph. That is why we call this type of deletion a weak vertex-deletion.

For an oriented hypergraph $G = (V, E, \mathcal{I}, \sigma)$ with an edge $e \in E$, the *weak edge-deletion* (or simply *edge-deletion*), denoted by $G\backslash e$ is the oriented hypergraph obtained by deleting e from the set E and restricting the incidence function ι to $V \times E\backslash\{e\}$, thus deleting any incidences containing e. That is $G\backslash e = (V, E\backslash\{e\}, \mathcal{I}_e, \sigma_e)$, where $\mathcal{I}_e$ is the set of incidences determined by the restricted incidence function $\iota|_{V\times E\backslash\{e\}}$, which also restricts the incidence orientation $\sigma_e \colon \mathcal{I}_e \to \{+1, -1\}$, but preserving orientations of undeleted incidences. The weak edge-deletion is the same as the graph version of edge-deletion.

As with hypergraphs, an oriented hypergraph has an incidence dual. The *incidence dual* of an oriented hypergraph $G = (H, \sigma)$ is the oriented hypergraph $G^* = (H^*, \sigma^*)$, where the *coincidence orientation* $\sigma^* \colon \mathcal{I}^* \to \{+1, -1\}$ is defined by $\sigma^*(e, v) = \sigma(v, e)$, and the *coadjacency signature* sgn^* is defined by

$$\text{sgn}^*_v(e_i, e_j) = -\sigma^*(e_i, v)\sigma^*(e_j, v) = -\sigma(v, e_i)\sigma(v, e_j).$$

See Figure 5 for an example of the incidence dual of G_1.

2.2 Matrices

The *adjacency matrix* $A(G) = (a_{ij}) \in \mathbb{R}^{n \times n}$ of an oriented hypergraph G is defined by

$$a_{ij} = \begin{cases} \sum_{e \in E} \text{sgn}_e(v_i, v_j) & \text{if } v_i \text{ is adjacent to } v_j, \\ 0 & \text{otherwise.} \end{cases}$$

If v_i is adjacent to v_j, then

$$a_{ij} = \sum_{e \in E} \text{sgn}_e(v_i, v_j) = \sum_{e \in E} \text{sgn}_e(v_j, v_i) = a_{ji}.$$

Therefore, $A(G)$ is symmetric.

The *incidence matrix* $\mathrm{H}(G) = (\eta_{ij})$ is the $n \times m$ matrix, with entries in $\{\pm 1, 0\}$, defined by

$$\eta_{ij} = \begin{cases} \sigma(v_i, e_j) & \text{if } (v_i, e_j) \in \mathcal{I}, \\ 0 & \text{otherwise.} \end{cases}$$

The *degree matrix* of an oriented hypergraph G is defined as $D(G) := \text{diag}(d_1, d_2, \ldots, d_n)$. The *Laplacian matrix* is defined as $L(G) := D(G) - A(G)$. Therefore, $L(G)$ is also symmetric. The Laplacian matrix of an oriented hypergraph can be written in terms of the incidence matrix. Thus, the Laplacian matrix is also positive semidefinite, and its eigenvalues are nonnegative.

Lemma 2.1 ([13], Corollary 4.4). *If G is an oriented hypergraph, then*

1. $L(G) = D(G) - A(G) = \mathrm{H}(G)\mathrm{H}(G)^T$, *and*

2. $L(G^*) = D(G^*) - A(G^*) = \mathrm{H}(G)^T\mathrm{H}(G)$.

Thus, the Laplacian matrix is also positive semidefinite, and its eigenvalues are nonnegative.

The following lemma will allow us to say that G and its incidence dual G^* have the same Laplacian spectral radius, which will be utilized to make a final bound improvement.

Lemma 2.2 ([11], Corollary 4.2). *If G is an oriented hypergraph, then $L(G)$ and $L(G^*)$ have the same nonzero eigenvalues.*

Since the eigenvalues of any symmetric matrix $S \in \mathbb{R}^{n\times n}$ are real we will assume that they are labeled and ordered according to the following convention:

$$\lambda_n(S) \leq \lambda_{n-1}(S) \leq \cdots \leq \lambda_2(S) \leq \lambda_1(S).$$

The following lemma establishes a relationship between the Laplacian eigenvalues of an oriented hypergraph G and the Laplacian eigenvalues of the weak vertex-deletion $G\backslash v$. The same can be said for the edge-deletion $G\backslash e$.

Lemma 2.3 ([11], Theorems 4.9 and 4.10)**.** *Let G be an oriented hypergraph.*

1. *If v is a vertex of G, then*

$$\lambda_{k+1}(L(G)) \leq \lambda_k(L(G\backslash v)) \leq \lambda_k(L(G)) \text{ for all } k \in \{1,\ldots,n-1\}.$$

2. *If e is an edge of G, then*

$$\lambda_{k+1}(L(G)) \leq \lambda_k(L(G\backslash e)) \leq \lambda_k(L(G)) \text{ for all } k \in \{1,\ldots,n-1\}.$$

2.3 Switching

A *vertex-switching function* is any function $\zeta\colon V \to \{-1,+1\}$. *Vertex-switching* the oriented hypergraph $G = (H,\sigma)$ means replacing σ with σ^ζ, defined by

$$\sigma^\zeta(v,e) = \zeta(v)\sigma(v,e); \tag{2.3}$$

producing the oriented hypergraph $G^\zeta = (H,\sigma^\zeta)$. Two oriented hypergraphs G_1 and G_2 are said to be *vertex-switching equivalent* (or *switching equivalent*), written $G_1 \sim G_2$, when there exists a vertex-switching function ζ such that $G_2 = G_1^\zeta$. The effect of vertex-switching on the adjacency signature is immediately determined as

$$\mathrm{sgn}_e^\zeta(v_i,v_j) = -\sigma^\zeta(v_i,e)\sigma^\zeta(v_j,e) = \zeta(v_i)\,\mathrm{sgn}_e(v_i,v_j)\zeta(v_j). \tag{2.4}$$

Switching is easily encoded with basic matrix operations. For a vertex-switching function $\zeta\colon V \to \{+1,-1\}$, we define a diagonal matrix $D(\zeta) := \mathrm{diag}\big(\zeta(v_1),\zeta(v_1),\ldots,\zeta(v_n)\big)$. The following shows how to calculate the switched oriented hypergraph's adjacency and Laplacian matrices.

Lemma 2.4 ([13], Propositions 3.1 and 4.3)**.** *Let G be an oriented hypergraph. Let ζ be a vertex-switching function on G. Then*

1. $A(G^\zeta) = D(\zeta)^T A(G) D(\zeta)$, *and*
2. $L(G^\zeta) = D(\zeta)^T L(G) D(\zeta)$.

This means that vertex-switching does not affect the Laplacian eigenvalues. This is also true for the adjacency eigenvalues, but we only need this result for the Laplacian eigenvalues in the proofs to come.

Lemma 2.5 ([11], Lemma 3.1)**.** *If G_1 and G_2 are switching equivalent oriented hypergraphs, then the eigenvalues of $L(G_1)$ and $L(G_2)$ are the same.*

3 Main Results

In this section we will produce several new lower bounds for the Laplacian spectral radius of an incidence-simple oriented hypergraph. This is accomplished by calculating the Laplacian eigenvalues of a specific oriented hypergraph T with variable parameters (see Figure 2) and using its eigenvalues to produce a lower bound for a general oriented hypergraph G. We can then specialize to the k-uniform case to obtain another improvement on previously known bounds. We finish with an even stronger lower bound by utilizing the incidence-dual G^*.

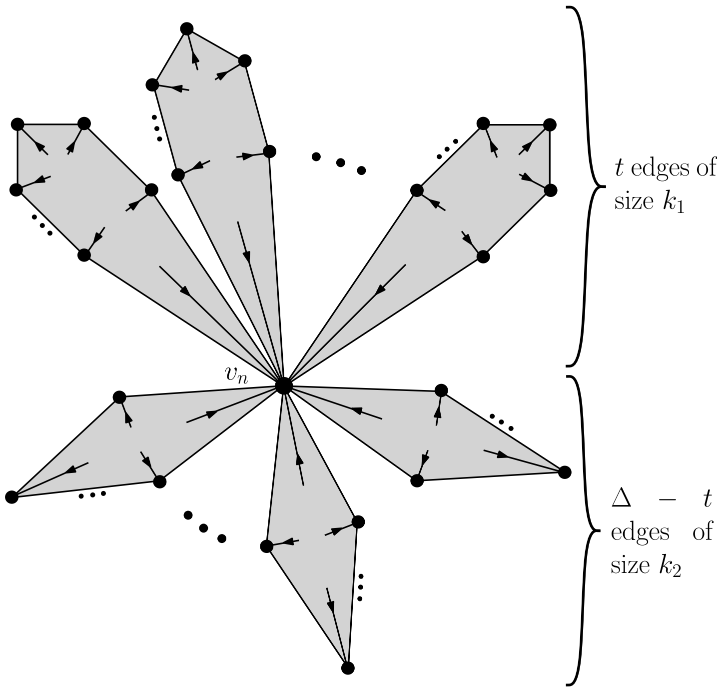


Figure 2: The oriented hypergraph T on n vertices, with t edges of size k_1 and $\Delta - t$ edges of size k_2 and v_n has degree Δ.

Let $L(T)$ be the Laplacian matrix of the oriented hypergraph in Figure 2. The matrix $L(T) - \lambda I$ can be written as

$$L(T) - \lambda I = \begin{bmatrix} J_{k_1-1} - \lambda I & \mathbf{O} & \cdots & \cdots & \cdots & \mathbf{O} & \mathbf{j} \\ \mathbf{O} & \ddots & \mathbf{O} & \cdots & \cdots & \mathbf{O} & \vdots \\ \vdots & \mathbf{O} & J_{k_1-1} - \lambda I & \mathbf{O} & \cdots & \mathbf{O} & \vdots \\ \vdots & \vdots & \ddots & J_{k_2-1} - \lambda I & \mathbf{O} & \vdots & \vdots \\ \vdots & \vdots & \cdots & \mathbf{O} & \ddots & \mathbf{O} & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \cdots & \mathbf{O} & J_{k_2-1} - \lambda I & \mathbf{j} \\ \mathbf{j}^{\mathrm{T}} & \cdots & \cdots & \cdots & \cdots & \mathbf{j}^{\mathrm{T}} & \Delta - \lambda \end{bmatrix},$$

where J_n is the $n \times n$ matrix of all 1's, and $\mathbf{j} = (1, \ldots, 1)$ is in the appropriate dimension. To calculate the characteristic polynomial of $L(T)$ one can consider the row reduction of the following blocks. Let $\mathcal{B}_k$ denote the $k \times k$ matrix defined as

$$\mathcal{B}_k = \begin{bmatrix} J_{k-1} - \lambda I & \mathbf{j} \\ \mathbf{j}^{\mathrm{T}} & 0 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 1 & \cdots & 1 & 1 \\ 1 & 1-\lambda & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & 1 \\ 1 & \cdots & 1 & 1-\lambda & 1 \\ 1 & \cdots & 1 & 1 & 0 \end{bmatrix}.$$

The following lemma will assist in the calculation of the Laplacian characteristic polynomial of the oriented hypergraph T in Figure 2.

Lemma 3.1. *The row echelon form of $\mathcal{B}_k$ is*

$$\begin{bmatrix} 1-\lambda & 1 & \cdots & \cdots & 1 \\ 0 & \frac{-\lambda(\lambda-2)}{\lambda-1} & \frac{\lambda}{\lambda-1} & \cdots & \frac{\lambda}{\lambda-1} \\ \vdots & \ddots & \ddots & & \\ 0 & \cdots & 0 & \frac{-\lambda\big(\lambda-(k-1)\big)}{\lambda-(k-2)} & \frac{\lambda}{\lambda-(k-2)} \\ 0 & \cdots & 0 & 0 & \frac{k-1}{\lambda-(k-1)} \end{bmatrix}$$

Proof. We will show by induction on j, that after reducing the first j columns of $\mathcal{B}_k$, the form of the resulting submatrix from row j to row k and column j to column k is

$$\begin{bmatrix} \frac{-\lambda(\lambda-j)}{\lambda-(j-1)} & \frac{\lambda}{\lambda-(j-1)} & \cdots & \cdots & \cdots & \frac{\lambda}{\lambda-(j-1)} \\ 0 & \frac{-\lambda(\lambda-(j+1))}{\lambda-j} & \frac{\lambda}{\lambda-j} & \cdots & \cdots & \frac{\lambda}{\lambda-j} \\ 0 & \frac{\lambda}{\lambda-j} & \frac{-\lambda(\lambda-(j+1))}{\lambda-j} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{\lambda}{\lambda-j} & \frac{\lambda}{\lambda-j} \\ 0 & \frac{\lambda}{\lambda-j} & \cdots & \frac{\lambda}{\lambda-j} & \frac{-\lambda(\lambda-(j+1))}{\lambda-j} & \frac{\lambda}{\lambda-j} \\ 0 & \frac{\lambda}{\lambda-j} & \cdots & \frac{\lambda}{\lambda-j} & \frac{\lambda}{\lambda-j} & \frac{j}{\lambda-j} \end{bmatrix}$$

Base case: The first column of $\mathcal{B}_k$ can be reduced by using the row operations

$$R_i \longrightarrow R_i + \left(\frac{-1}{1-\lambda}\right) R_1,$$

where $2 \leq i \leq k$. The resulting matrix is

$$\begin{bmatrix} 1-\lambda & 1 & \cdots & \cdots & \cdots & 1 \\ 0 & \dfrac{-\lambda(\lambda-1)}{\lambda-2} & \dfrac{\lambda}{\lambda-1} & \cdots & \cdots & \dfrac{\lambda}{\lambda-1} \\ 0 & \dfrac{\lambda}{\lambda-1} & \dfrac{-\lambda(\lambda-2)}{\lambda-1} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \dfrac{\lambda}{\lambda-1} & \dfrac{\lambda}{\lambda-1} \\ 0 & \dfrac{\lambda}{\lambda-1} & \cdots & \dfrac{\lambda}{\lambda-1} & \dfrac{-\lambda(\lambda-2)}{\lambda-1} & \dfrac{\lambda}{\lambda-1} \\ 0 & \dfrac{\lambda}{\lambda-1} & \cdots & \dfrac{\lambda}{\lambda-1} & \dfrac{\lambda}{\lambda-1} & \dfrac{1}{\lambda-1} \end{bmatrix},$$

which matches the form stated above for j=1.

Induction step:

Let ℓ be some integer such that $1 < \ell \leq k$. Suppose, after reducing the first ℓ columns of $\mathcal{B}_k$, the form of the resulting submatrix from row ℓ to row k and column ℓ to column k is as shown above (with j's replaced with ℓ's). The reduction of the $(\ell+1)^{\text{st}}$ column can be completed with the row operations

$$R_i \longrightarrow R_i + \left(\frac{-1}{\lambda-(\ell+1)}\right) R_{\ell+1},$$

where $\ell+2 \leq i \leq k$. The resulting matrix is:

$$\begin{bmatrix} \frac{-\lambda(\lambda-\ell)}{\lambda-(\ell-1)} & \frac{\lambda}{\lambda-(\ell-1)} & \cdots & \cdots & \cdots & \cdots & \frac{\lambda}{\lambda-(\ell-1)} \\ 0 & \frac{-\lambda(\lambda-(\ell+1))}{\lambda-\ell} & \frac{\lambda}{\lambda-j} & \cdots & \cdots & \cdots & \frac{\lambda}{\lambda-\ell} \\ 0 & 0 & \frac{-\lambda(\lambda-(\ell+2))}{\lambda-(\ell+1)} & \frac{\lambda}{\lambda-(\ell+1)} & \cdots & \cdots & \frac{\lambda}{\lambda-(\ell+1)} \\ \vdots & \vdots & \frac{\lambda}{\lambda-(\ell+1)} & \ddots & \ddots & \frac{\lambda}{\lambda-(\ell+1)} & \vdots \\ 0 & 0 & \vdots & \ddots & \frac{-\lambda(\lambda-(\ell+2))}{\lambda-(\ell+1)} & \frac{\lambda}{\lambda-(\ell+1)} & \vdots \\ 0 & 0 & \frac{\lambda}{\lambda-(\ell+1)} & \cdots & \frac{\lambda}{\lambda-(\ell+1)} & \frac{-\lambda(\lambda-(\ell+2))}{\lambda-(\ell+1)} & \frac{\lambda}{\lambda-(\ell+1)} \\ 0 & 0 & \frac{\lambda}{\lambda-(\ell+1)} & \cdots & \frac{\lambda}{\lambda-(\ell+1)} & \frac{\lambda}{\lambda-(\ell+1)} & \frac{\ell+1}{\lambda-(\ell+1)} \end{bmatrix}.$$

Thus, the correct form of the matrix is produced. After reducing all $k-1$ columns the result of the Lemma follows. □

Theorem 3.2. *Let T be the oriented hypergraph on n vertices, with t edges of size k_1 and $\Delta - t$ edges of size k_2 and v_n has degree Δ (see Figure 2). The Laplacian characteristic polynomial of T is $p(\lambda) =$*

$$(-1)^{\Delta}(-\lambda)^{t(k_1-2)+(\Delta-t)(k_2-2)+1}\big(\lambda-(k_1-1)\big)^{t-1}\big(\lambda-(k_2-1)\big)^{\Delta-t-1}(\lambda-\lambda_1)(\lambda-\lambda_2),$$

where λ_1 and λ_2 are the roots of the quadratic in λ

$$(k_1-k_2)t-(\lambda-k_1+1)(\lambda-\Delta-k_2+1).$$

Proof. Notice that in the previous Lemma the b_{kk} entry represents the sum of the contributions from each row reduction of the previous matrix and should be added to $\Delta - \lambda$ to get the last entry in the reduced Laplacian matrix:

$$\Delta - \lambda + \frac{(k-1)}{\lambda - (k-1)}.$$

Furthermore, in a hypergraph with t edges of size k_1 and $\Delta - t$ edges of size k_2, the total contribution from all the edges is simply:

$$t \cdot \frac{(k_1-1)}{\lambda - (k_1-1)} + (\Delta - t) \cdot \frac{(k_2-1)}{\lambda - (k_2-1)}.$$

By reducing the matrix $L(T) - \lambda I$ as in Lemma 3.1, this results in an upper triangular matrix and the determinant is given as the product along its diagonal:

$$\left(-(\lambda-1)\left(\frac{-\lambda(\lambda-2)}{\lambda-1}\right)\left(\frac{-\lambda(\lambda-3)}{\lambda-2}\right)\cdots\left(\frac{-\lambda(\lambda-(k_1-1))}{\lambda-(k_1-2)}\right)\right)^t$$
$$\cdot\left(-(\lambda-1)\left(\frac{-\lambda(\lambda-2)}{\lambda-1}\right)\left(\frac{-\lambda(\lambda-3)}{\lambda-2}\right)\cdots\left(\frac{-\lambda(\lambda-(k_2-1))}{\lambda-(k_2-2)}\right)\right)^{\Delta-t}$$
$$\cdot\left((\Delta-\lambda) + t\cdot\frac{(k_1-1)}{\lambda-(k_1-1)} + (\Delta-t)\cdot\frac{(k_2-1)}{\lambda-(k_2-1)}\right),$$

which simplifies to:

$$\left(-(-\lambda)^{k_1-2}(\lambda-(k_1-1))\right)^t \cdot \left(-(-\lambda)^{k_2-2}(\lambda-(k_2-1))\right)^{\Delta-t}$$
$$\cdot\left((\Delta-\lambda) + t\cdot\frac{(k_1-1)}{\lambda-(k_1-1)} + (\Delta-t)\cdot\frac{(k_2-1)}{\lambda-(k_2-1)}\right),$$

and further:

$$(-1)^{\Delta}(-\lambda)^{t(k_1-2)+(\Delta-t)(k_2-2)}\big(\lambda-(k_1-1)\big)^t\big(\lambda-(k_2-1)\big)^{\Delta-t}$$
$$\cdot\left(\frac{\lambda\big((k_1-k_2)t - (\lambda-k_1+1)(\lambda-\Delta-k_2+1)\big)}{(\lambda-k_2+1)(\lambda-k_1+1)}\right).$$

Hence, the characteristic polynomial of $L(T)$ is as stated. □

Now the Laplacian eigenvalues of T can be stated as follows.

Corollary 3.3. *Let T be the oriented hypergraph in Figure 2. The Laplacian eigenvalues of T are:*

eigenvalue	*multiplicty*
0	$t(k_1-2)+(\Delta-t)(k_2-2)+1$
k_1-1	$t-1$
k_2-1	$\Delta-t-1$
$\lambda_2=\dfrac{\Delta+k_1+k_2-2-\sqrt{[\Delta-(k_1-k_2)]^2+4(k_1-k_2)t}}{2}$	1
$\lambda_1=\dfrac{\Delta+k_1+k_2-2+\sqrt{[\Delta-(k_1-k_2)]^2+4(k_1-k_2)t}}{2}$	1

Now we present the main result of the paper, a new lower bound for the Laplacian spectral radius of an oriented hypergraph. This lower bound is an improvement on a lower bound obtained in [11, Theorem 4.11], which will be a corollary below. This bound also generalizes the same bound for signed graphs in [7, Theorem 3.10]. The proof method here borrows techniques from both of these papers and uses the above results to achieve this improvement. What is unique about this bound is the inclusion of new parameters beyond the maximum vertex degree.

Theorem 3.4. *Let G be an incidence-simple oriented hypergraph. Let v be a vertex of max degree Δ. Let k_1 be the size of the largest edge incident to v. Let $k_2 \leq k_1$ be the size of the smallest edge incident to v. Let t be the number of edges of size k_1 incident to v. Then*

$$\frac{\Delta+k_1+k_2-2+\sqrt{[\Delta-(k_1-k_2)]^2+4(k_1-k_2)t}}{2} \leq \lambda_1(L(G)). \tag{3.1}$$

Proof. Let v be a vertex in G with max degree $\deg(v)=\Delta$. Let G_1 be the oriented hypergraph obtained by weak edge-deletion of all edges not incident to v in G. By repeated use of Lemma 2.3 (part 2), $\lambda_1(L(G_1)) \leq \lambda_1(L(G))$. Let G_2 be the oriented hypergraph obtained by weak vertex-deletion of all isolated vertices in G_1. By repeated use of Lemma 2.3, $\lambda_1(L(G_2)) \leq \lambda_1(L(G_1))$. If $k_1=k_2$, then let G_3=G_2. Otherwise, for every edge e of G_2 with size $|e|$ where $k_2 \leq |e| < k_1$, perform weak vertex-deletion on $|e|-k_2$ vertices of e that have degree 1. After all such weak vertex-deletions, pick one of the possible resulting oriented hypergraphs G_3. By repeated use of Lemma 2.3, $\lambda_1(L(G_3)) \leq \lambda_1(L(G_2))$.

Notice that the underlying hypergraph of G_3 is the same as the underlying hypergraph of T in Figure 2 with the appropriate parameters. However, the incidence orientations do not necessarily match, so G_3 and T may not be the same oriented hypergraph. Define a vertex-switching $\zeta\colon V(G_3) \to \{+1,-1\}$ as follows. Let $\zeta(v)=+1$. For each edge e, consider all $u \in V(e)$ with $u \neq v$. Define $\zeta(u)=\sigma(u,e)\sigma(v,e)$. This will ensure that $\sigma^{\zeta}(u,e)=\sigma(v,e)$ and hence all adjacency signatures in G_3^{ζ} are -1. Therefore, even though T and G_3^{ζ} are not necessarily switching equivalent we can guarentee $L(G_3^{\zeta})=L(T)$, because their adjacency signatures are all the same in addition to their underlying hypergraphs matching. Since vertex-switching leaves the Laplacian eigenvalues unchanged by Lemma 2.5 $\lambda_1(L(G_3))=\lambda_1(L(G_3^{\zeta}))$. Furthermore, since $L(G_3^{\zeta})=L(T)$ we clearly have $\lambda_1(L(G_3^{\zeta}))=\lambda_1(L(T))$. By Corollary 3.3

$$\lambda_1(L(T))=\frac{\Delta+k_1+k_2-2+\sqrt{[\Delta-(k_1-k_2)]^2+4(k_1-k_2)t}}{2}.$$

The result follows via the string of inequalities:

$$\lambda_1(L(T)) = \lambda_1(L(G_3^\zeta)) = \lambda_1(L(G_3)) \leq \lambda_1(L(G_2)) \leq \lambda_1(L(G_1)) \leq \lambda_1(L(G)). \quad \square$$

The following bound is valid for k-uniform oriented hypergraphs, but the result as stated is more general. The following form also makes the generalized connection to signed graphs and unsigned graphs more evident.

Corollary 3.5. *Let G be an oriented hypergraph. Let v be a vertex of max degree, and let k be the size of the smallest edge incident to v. Then*

$$\Delta + k - 1 \leq \lambda_1(L(G)). \tag{3.2}$$

Proof. This is immediate from Theorem 3.4 if $t = 0$ and $k = k_2$. □

As stated before, this can be further specialized to the known bound for signed and unsigned graphs [7, Theorem 3.10]. This bound is also the known for oriented hypergraphs [11, Theorem 4.11] and was the motivation for producing the improved bound in Theorem 3.4.

Corollary 3.6. *Let G be an oriented signed graph. Then*

$$\Delta + 1 \leq \lambda_1(L(G)). \tag{3.3}$$

Proof. This is immediate from Corollary 3.5 with $k = 2$. □

As mentioned in Lemma 2.2, an oriented hypergraph G and its incidence dual G^* have the same nonzero eigenvalues. Therefore, they have the same Laplacian spectral radius, which allows us to improve Theorem 3.4 as follows.

Corollary 3.7. *Let G be an oriented hypergraph. Let v be a vertex of max degree Δ. Let k_1 be the size of the largest edge incident to v. Let $k_2 \leq k_1$ be the size of the smallest edge incident to v. Let t be the number of edges of size k_1 incident to v. Let*

$$\ell = \frac{\Delta + k_1 + k_2 - 2 + \sqrt{[\Delta - (k_1 - k_2)]^2 + 4(k_1 - k_2)t}}{2},$$

and let ℓ^* be the calculation of ℓ using the incidence dual G^* instead of G. Then

$$\max\{\ell, \ell^*\} \leq \lambda_1(L(G)). \tag{3.4}$$

Proof. By Lemma 2.2 G and G^* have the same Laplacian spectral radius, $\lambda_1(L(G)) = \lambda_1(L(G^*))$. We can calculate ℓ and ℓ^* respectively and apply Theorem 3.4 to obtain a lower bound for $\lambda_1(L(G)) = \lambda_1(L(G^*))$. Thus, the larger of the two values ℓ and ℓ^* will provide an improvement on the lower bound, and the result follows. □

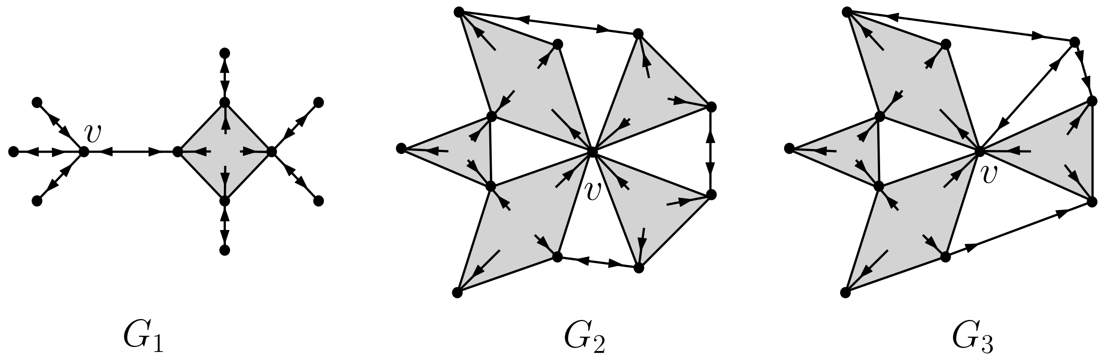


Figure 3: Three oriented hypergraphs G_1, G_2 and G_3 considered in Examples 1 and 2. The vertex v in these oriented hypergraphs is the vertex of maximum degree $\Delta = 4$.

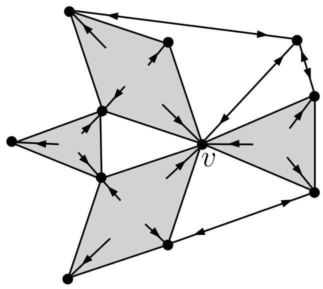
Table 4.1: In this table the performance of each bound can be compared to the Laplacian spectral radius for each respective oriented hypergraph G_1, G_2, G_3, G_4 and G_1^*.

	$\lambda_1(L(G_i))$	(3.1)	(3.2)	(3.3)	(3.4)
G_1	5.751	5	5	5	5.303
G_2	7.356	6.562	6	5	6.562
G_3	6.424	6.236	5	5	6.236
G_4	7.277	6.236	5	5	6.236
G_1^*	5.751	5.303	5	5	5.303

4 Examples, Bound Performance and the Incidence Dual

Example 1: Consider the oriented hypergraphs G_1, G_2 and G_3 in Figure 3. In each example, the vertex v is of maximum degree $\Delta = 4$. The performance of each bound presented in the paper is outlined in Table 4.1.

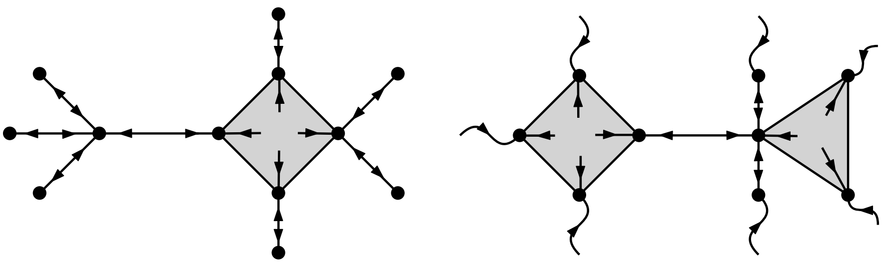
Example 2: An edge in an oriented hypergraph is *uniformly oriented* if all incidences containing that edge have the same sign. An oriented hypergraph is *uniformly oriented* if all of its edges are uniformly oriented. For example, the oriented hypergraph G_4 in Figure 4 is uniformly oriented. It has been shown in [11, Theorem 4.6] that under all possible choices of incidence signs for a given oriented hypergraph, the Laplacian spectral radius is maximized when edges are uniformly oriented (or if an oriented hypergraph is switching equivalent to an oriented hypergraph that is uniformly oriented). Notice that G_4 has the same underlying oriented hypergraph as G_3, but G_3 is not uniformly oriented (or switching equivalent to an oriented hypergraph that is uniformly oriented). This is why the spectral radius of G_3 is less than that of G_4. Hence, the new bounds perform better for G_3 as seen in Table 4.1. It would be interesting to determine which incidence orientations minimize the spectral radius for a given hypergraph. Also, other than the extreme case of having the oriented hypergraph T in Figure 2 is Inequality (3.1) strict?



G_4

Figure 4: Oriented hypergraph G_4 considered in Example 2. Note that G_4 has all incidences labeled as $+1$, and vertex v is the vertex of maximum degree $\Delta = 4$.

Example 3: Consider the oriented hypergraph G_1 and its incidence dual G_1^* in Figure 5. As mentioned in Lemma 2.2, an oriented hypergraph G and its incidence dual G^* have the same nonzero Laplacian eigenvalues. Hence, they have the same Laplacian spectral radius as we can see in Table 4.1 for G_1 and G_1^*. What is interesting in this case is that Inequality (3.1) performs better for the incidence dual G_1^*. In this particular example we can see that the improved bound from Corollary 3.7 is better than Theorem 3.4.



G_1 G_1^*

Figure 5: An oriented hypergraph G_1 and its incidence dual G_1^* considered in Example 3.

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