On the density of sumsets and product sets

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Abstract

In this paper some links between the density of a set of integers and the density of its sumset, product set and set of subset sums are presented.

1 Introduction and notation

In the field of additive combinatorics a popular topic is to compare the densities of different sets (of, say, positive integers). The well-known theorem of Kneser gives a description of the sets $A$ having lower density $\alpha$ such that the density of $A + A := \{a + b : a, b \in A\}$ is less than $2\alpha$ (see for instance [9]). The analogous question with the product set $A^2 := \{ab : a, b \in A\}$ is apparently more complicated.

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For any set $A \subset \mathbb{N}$ of natural numbers, we define the lower asymptotic density $\underline{d}A$ and the upper asymptotic density $\overline{d}A$ in the natural way:

$$\underline{d}A = \liminf_{n \to \infty} \frac{|A \cap [1, n]|}{n}, \quad \overline{d}A = \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n}.$$ 

If the two values coincide, then we denote by $dA$ the common value and call it the asymptotic density of $A$.

Throughout the paper $\mathbb{N}$ denotes the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will use the notion $A(x) = \{n \in A : n \leq x\}$ for $A \subseteq \mathbb{N}$ and $x \in \mathbb{R}$. For functions $f, g : \mathbb{N} \to \mathbb{R}_+$ we write $f = O(g)$ (or $f \ll g$), if there exists some $c > 0$ such that $f(n) \leq cg(n)$ for large enough $n$.

In Section 2 we investigate the connection between the (upper-, lower-, and asymptotic) density of a set of integers and the density of its sumset. In Section 3 we give a partial answer to a question of Erdős by giving a necessary condition for the existence of the asymptotic density of the set of subset sums of a given set of integers. Finally, in Section 4 we consider analogous problems for product sets.

## 2 Density of sumsets

For subsets $A, B$ of integers the sumset $A + B$ is defined to be the set of all sums $a + b$ with $a \in A, b \in B$. For $A \subseteq \mathbb{N}_0$ the following clearly hold:

$$\underline{d}A \leq \overline{d}A,$$

$$\underline{d}A \leq d(A + A),$$

$$\overline{d}A \leq \overline{d}(A + A).$$

We shall assume that our sets $A$ are normalized in the sense that $A$ contains 0 and $\gcd(A) = 1$.

First observe that there exists a set of integers $A$ not having an asymptotic density such that its sumset $A + A$ has a density: for instance $A = \{0\} \cup \bigcup_{n \geq 0} [2^n, 2^{n+1}]$ has lower density $1/3$, upper density $2/3$ and its sumset $A + A$ has density 1, since it contains every nonnegative integer. For this kind of sets $A$, we denote respectively

$$\underline{d}A := \alpha_A,$$

$$\overline{d}A := \beta_A,$$

$$d(A + A) := \gamma_A,$$

$$(\alpha_A, \beta_A, \gamma_A) := p_A,$$

and we have

$$\alpha_A \leq \beta_A \leq \gamma_A.$$ 

The first question arising from this is to decide whether or not for any $p = (\alpha, \beta, \gamma)$ such that $0 \leq \alpha \leq \beta \leq \gamma \leq 1$ there exists a set $A$ of integers such that $p = p_A$. This question has no positive answer in general, though the following weaker statement holds.
Proposition 2.1 Let $0 \leq \alpha \leq 1$. There exists a normalized set $A \subset \mathbb{N}$ such that $dA = \alpha$ and $d(A + A) = 1$.

Proof: Let $0 \in B$ be a thin additive basis (of order 2), that is, a basis containing 0 and satisfying $|B(x)| = o(x)$ as $x \to \infty$. For $\alpha = 0$ the choice $A = B$ is fine. For $\alpha > 0$ let $A = B \cup \{n/\alpha, \ n \geq 1\}$. Then $A$ is a normalized set satisfying $A + A = \mathbb{N}_0$ and $dA = \alpha$.

(Note that $B = \{0, 1, 2, \ldots, \lfloor 1/\alpha \rfloor\}$ is also an appropriate choice for $B$ in the case $\alpha > 0$.)

Remark 1 We shall mention that Faisant et al. [1] proved the following related result: for any $0 \leq \alpha \leq 1$ and any positive integer $k$, there exists a sequence $A$ such that $d(jA) = j\alpha/k$, $j = 1, \ldots, k$, where $jA$ denotes the $j$-fold sumset $A + A + \cdots + A$ ($j$ times). Well before that in [11, Theorem 2] the author established that for any positive real numbers $\alpha_1, \ldots, \alpha_k, \beta$ satisfying $\sum_{i=1}^{k} \alpha_i \leq \beta \leq 1$ there exist sets $A_1, \ldots, A_k$ such that $dA_i = \alpha_i$ ($1 \leq i \leq k$) and $d(A_1 + \cdots + A_k) = \beta$.

After a conjecture stated by Pichorides, the related question about the characterisation of the two-dimensional domains $\{(dB, \overline{dB}) : B \subset A\}$ has been solved (see [3] and [6]).

Note that if the density $\gamma_A$ exists, then $\alpha_A$, $\beta_A$ and $\gamma_A$ have to satisfy some strong conditions. For instance, by Kneser’s theorem, we know that if for some set $A$ we have $\gamma_A < 2\alpha_A$, then $A + A$ is, except possibly a finite number of elements, a union of arithmetic progressions in $\mathbb{N}$ with the same difference. This implies that $\gamma_A$ must be a rational number. From the same theorem of Kneser, we also deduce that if $\gamma_A < 3\alpha_A/2$, then $A + A$ is an arithmetic progression from some point onward. It means that $\gamma_A$ is a unit fraction, hence $A$ contains any sufficiently large integer, if we assume that $A$ is normalized.

Another strong connection between $\alpha_A$ and $\gamma_A$ can be deduced from Freiman’s theorem on the addition of sets (cf. [2]). Namely, every normalized set $A$ satisfies

$$\gamma_A \geq \frac{\alpha_A}{2} + \min\left(\alpha_A, \frac{1}{2}\right).$$

A related but more surprising statement is the following:

Proposition 2.2 There is a set of positive integers for which $d(A)$ does exist and $d(A + A)$ does not exist.

Proof: Let us take $U = \{0, 2, 3\}$ and $V = \{0, 1, 2\}$, then observe that

$$U + (U \cup V) = \{0, 1, 2, 3, 4, 5, 6\} \quad V + (U \cup V) = \{0, 1, 2, 3, 4, 5\}.$$ 

Let $(N_k)_{k \geq 0}$ be a sufficiently quickly increasing sequence of integers with $N_0 = 0$, $N_1 = 1$, and define $A$ by

$$A = (U \cup V) \cup \bigcup_{k \geq 1} \left( (U + 7\mathbb{Z}) \cap [7N_{2k}, 7N_{2k+1}] \cup (V + 7\mathbb{Z}) \cap [7N_{2k+1}, 7N_{2k+2}] \right).$$
Then $A$ has density $3/7$. Moreover, for any $k \geq 0$

$$[7N_{2k}, 7N_{2k+1}] \subset A + A,$$

thus $\overline{d}(A + A) = 1$, if we assume $\lim_{k \to \infty} N_{k+1}/N_k = \infty$.

We also have

$$(A + A) \cap [14N_{2k-1}, 7N_{2k}] = (\{0, 1, 2, 3, 4, 5\} + 7N) \cap [14N_{2k-1}, 7N_{2k}],$$

hence $d(A + A) = 6/7$ using again the assumption that $\lim_{k \to \infty} N_{k+1}/N_k = \infty$. \hfill \Box

For any set $A$ having a density, let

$$d_A := \alpha_A,$$

$$\overline{d}(A + A) := \gamma_A^+,$$

$$\overline{d}(A + A) := \gamma_A^-,$$

$$(\alpha_A, \gamma_A^+, \gamma_A^-) := q_A;$$

then we have

$$\alpha_A \leq \gamma_A^+ \leq \gamma_A^-.$$

A question similar to the one asked for $p_A$ can be stated as follows: given $q = (\alpha, \gamma, \gamma^+)$ such that $0 \leq \alpha \leq \gamma \leq \gamma^+ \leq 1$, does there exist a set $A$ such that $q = q_A$?

We further mention an interesting question of Ruzsa: does there exist $0 < \nu < 1$ and a constant $c = c(\nu) > 0$ such that for any set $A$ having a density,

$$d(A + A) \geq c \cdot (\overline{d}(A + A))^{1-\nu}(dA)^\nu?$$

Ruzsa proved (unpublished) that in case of an affirmative answer, we necessarily have $\nu \geq 1/2$.

### 3 Density of subset sums

Let $A = \{a_1 < a_2 < \cdots\}$ be a sequence of positive integers. Denote the set of all subset sums of $A$ by

$$P(A) := \left\{ \sum_{i=1}^{k} \varepsilon_i a_i : k \geq 0, \varepsilon_i \in \{0, 1\} \ (1 \leq i \leq k) \right\}.$$

Zannier conjectured and Ruzsa proved that the condition $a_n \leq 2a_{n-1}$ implies that the density $d(P(A))$ exists (see [8]). Ruzsa also asked the following questions:

i) Is it true that for every pair of real numbers $0 \leq \alpha \leq \beta \leq 1$, there exists a sequence of integers for which $d(P(A)) = \alpha; \overline{d}(P(A)) = \beta$? This question was answered positively in [5].
ii) Is it true that the condition \( a_n \leq a_1 + a_2 + \cdots + a_{n-1} + c \) also implies that \( d(P(A)) \) exists?

We shall prove the following statement.

**Proposition 3.1** Let \( (a_n)_{n=1}^\infty \) be a sequence of positive integers. Assume that for some function \( \theta \) satisfying \( \frac{k}{(\log k)^2} \) we have

\[
|a_n - s_{n-1}| = \theta(s_{n-1}) \quad \text{for every } n,
\]

where \( s_{n-1} := a_1 + a_2 + \cdots + a_{n-1} \).

Then \( d(P(A)) \) exists.

**Proof:** We first prove that there exists a real number \( \delta \) such that

\[
|P(A)(s_n)| = (\delta + o(1))s_n \quad \text{as } n \to \infty.
\]

Let \( n \geq 2 \) be large enough. Then

\[
P(A) \cap [1, s_n] = \left( P(A) \cap [1, s_{n-1}] \right) \cup \left( P(A) \cap (s_{n-1}, s_n - \theta(s_{n-1})) \right).
\]

Since \( a_n \geq s_{n-1} - \theta(s_{n-1}) \), we have \( P(A) \cap (s_{n-1}, s_n] \supseteq a_n + P(A) \cap (\theta(s_{n-1}), s_{n-1}] \), and thus

\[
|P(A) \cap [1, s_n]| \geq 2|P(A) \cap [1, s_{n-1}]| - 2\theta(s_{n-1}) - 1. \tag{1}
\]

On the other hand,

\[
P(A) \cap [1, s_n] \subseteq (P(A) \cap [1, s_{n-1}]) \cup (a_n + P(A) \cap [1, s_{n-1}]) \cup [s_n - \theta(s_n), s_n],
\]

since \( a_{n+1} \geq s_n - \theta(s_n) \). Therefore,

\[
|P(A) \cap [1, s_n]| \leq 2|P(A) \cap [1, s_{n-1}]| + \theta(s_n) + 1. \tag{2}
\]

Observe that \( s_n = a_n + s_{n-1} \leq 2s_{n-1} + \theta(s_{n-1}) \); hence letting

\[
\delta_n = \frac{|P(A) \cap [1, s_n]|}{s_n},
\]

we obtain from (1) and (2) that

\[
\delta_n - \delta_{n-1} = O\left( \frac{\theta(s_n)}{s_n} \right). \tag{3}
\]

Now, we show that \( s_n \gg 2^n \). Since

\[
s_n = s_{n-1} + a_n \geq 2s_{n-1} - \theta(s_{n-1}) = s_{n-1}\left( 2 - \frac{\theta(s_{n-1})}{s_{n-1}} \right), \tag{4}
\]
the condition \( \theta(k) \ll \frac{k}{(\log k)^2} \) implies that from (4) we obtain that \( s_n \gg 1.5^n \). Therefore, in fact, for large enough \( n \) we have \( s_n \geq s_{n-1} \left(2 - \frac{c}{n^2}\right) \) with some \( c > 0 \). Now, let \( 10c < K \) be a fixed integer. For \( K < n \) we have

\[
s_n \geq s_K \prod_{i=K+1}^{n} \left(2 - \frac{c}{i^2}\right) \geq s_K \left[2^n - 2^{n-k-1} \sum_{i=K+1}^{n} \frac{c}{i^2}\right] \gg 2^n,
\]

since \( \sum_{i=K+1}^{n} \frac{c}{i^2} < 1/10 \). Hence, \( s_n \gg 2^n \) indeed holds.

Therefore, using the assumption on \( \theta \) we obtain that \( \frac{\theta(s_n)}{s_n} \ll \frac{1}{n^2} \). So (3) yields that

\[
\delta_n - \delta_{n-1} = O(n^{-2}).
\]

Therefore, the sequence \( \delta_n \) has a limit which we denote by \( \delta \). Furthermore, observe that

\[
\delta_n = \delta + O(1/n). \tag{5}
\]

The next step is to consider an arbitrary sufficiently large positive integer \( x \) and decompose it as

\[
x = a_{n_1+1} + a_{n_2+1} + \cdots + a_{n_j+1} + z,
\]

where \( n_1 > n_2 > \cdots > n_j > k \) and \( 0 \leq z \) are defined in the following way. (Here \( k \) is a fixed, sufficiently large positive integer.) The index \( n_1 \) is chosen in such a way that \( a_{n_1+1} \leq x < a_{n_1+2} \). If \( x - a_{n_1+1} \geq a_{n_1} \), then \( n_2 = n_1 - 1 \), otherwise \( n_2 \) is the largest index for which \( x - a_{n_1+1} \geq a_{n_2+1} \). The indices \( n_3, n_4, \ldots \) are defined similarly. We stop at the point when the next index would be at most \( k \) and set

\[
z := x - a_{n_1+1} - a_{n_2+1} - \cdots - a_{n_j+1}.
\]

As \( z \leq \theta(s_{n_1+1}) + s_k \), we have

\[
z = o(x). \tag{6}
\]

Furthermore, let

\[
b_i = a_{n_1+1} + a_{n_2+1} + \cdots + a_{n_i+1}, \quad i = 0, 1, \ldots, j.
\]

(The empty sum is \( b_0 := 0 \), as usual.)

Let \( X_0 := (0, s_{n_1} - \theta(s_{n_1})) \) and for \( 1 \leq i \leq j - 1 \) let \( X_i := (b_i + \theta(s_{n_i}), b_i + s_{n_i+1} - \theta(s_{n_i+1})) \) and consider

\[
X := X_0 \cup X_1 \cup \cdots \cup X_{j-1} = (0, s_{n_1} - \theta(s_{n_1})) \cup (b_1 + \theta(s_{n_1}), b_1 + s_{n_2} - \theta(s_{n_2})) \cup \cdots \cup (b_{j-1} + \theta(s_{n_{j-1}}), b_{j-1} + s_{n_j} - \theta(s_{n_j})).
\]

Note that in this union each element appears at most once, since according to the definition of \( \theta \) the sets \( X_0, X_1, \ldots, X_{j-1} \) are pairwise disjoint as

\[
b_i + s_{n_i+1} - \theta(s_{n_i+1}) \leq b_{i+1} = b_i + a_{n_i+1+1}
\]
holds for every $0 \leq i \leq j - 2$.

The set of those elements of $[1, x]$ that are not covered by $X$ is:

$$[1, x] \setminus X = [s_{n_1} - \theta(s_{n_1}), b_1 + \theta(s_{n_1})] \cup [b_1 + s_{n_2} - \theta(s_{n_2}), b_2 + \theta(s_{n_2})] \cup \ldots$$

$$\cup [b_{j-2} + s_{n_{j-1}} - \theta(s_{n_{j-1}}), b_{j-1} + \theta(s_{n_{j-1}})] \cup [b_{j-1} + s_n - \theta(s_n), x].$$

Therefore,

$$|[1, x] \setminus X| \leq 3 \sum_{i=1}^{j} \theta(s_{n_i}) + z.$$

Using $\sum_{i=1}^{j} \theta(s_{n_i}) \ll \sum_{i=1}^{j} s_{n_i} \ll \frac{x}{k^2}$ and (6), we obtain that $|[1, x] \setminus X| \leq (\varepsilon_k + o(1))x$, where $\varepsilon_k \to 0$ (as $k \to \infty$). (Note that $\varepsilon_k \ll 1/k^2$.)

That is, the set $X$ covers $[1, x]$ with the exception of a “small” portion of size $O(x/k^2)$. Therefore, by letting $k \to \infty$ the density of the uncovered part tends to 0.

Let us consider $P(A) \cap X_i$. If a sum is contained in $P(A) \cap X_i$, then the sum of the elements with indices larger than $n_{i+1}$ is $b_i$. Otherwise, the sum is either at most $b_i + \theta(s_{n_i})$ or at least $b_i + s_{n_{i+1}} - \theta(s_{n_{i+1}})$.

Therefore $P(A) \cap X_i = (b_i + P(\{a_1, a_2, \ldots, a_{n_{i+1}}\})) \cap X_i$.

Hence

$$\delta_{n_{i+1}} s_{n_{i+1}} - 2\theta(s_{n_{i+1}}) - 1 \leq |P(A) \cap X_i| \leq \delta_{n_{i+1}} s_{n_{i+1}}.$$

Therefore

$$|P(A) \cap [x]| \geq \sum_{i=0}^{j-1} \left( \delta_{n_{i+1}} s_{n_{i+1}} - 2\theta(s_{n_{i+1}}) - 1 \right)$$

$$\geq \delta x - \delta z + \delta \sum_{i=0}^{j-1} (s_{n_{i+1}} - a_{n_{i+1}+1}) + \sum_{i=0}^{j-1} (\delta_{n_{i+1}} - \delta) s_{n_{i+1}} - 2 \sum_{i=0}^{j-1} (\theta(s_{n_{i+1}}) + 1)$$

(7)

and

$$|P(A) \cap [x]| \leq \sum_{i=0}^{j-1} \delta_{n_{i+1}} s_{n_{i+1}}$$

$$\leq \delta x - \delta z + \delta \sum_{i=0}^{j-1} (s_{n_{i+1}} - a_{n_{i+1}+1}) + \sum_{i=0}^{j-1} (\delta_{n_{i+1}} - \delta) s_{n_{i+1}}.$$

Now, observe that

- $|z| = o(x)$ by (6),
- $\sum_{i=0}^{j-1} |s_{n_{i+1}} - a_{n_{i+1}+1}| = o(x)$, using $|s_{n_{i+1}} - a_{n_{i+1}+1}| = \theta(s_{n_{i+1}})$ and $\sum_{i=0}^{j-1} a_{n_{i+1}+1} \leq x$. 


\[ \sum_{i=0}^{j-1} (\delta_{n_i+1} - \delta) s_{n_i+1} \ll x/k \text{ by using (5).} \]

Letting \( k \to \infty \) this term is also of size \( o(x) \).

Hence we obtain from (7) and (8) that \( |P(A) \cap [x]| = \delta x + o(x) \).

\[ \square \]

### 4 Density of product sets

For any subsets \( A, B \subseteq \mathbb{N}_0 \), we denote by \( A \cdot B \) the product set

\[ AB = A \cdot B = \{ab : a \in A, b \in B\} . \]

For brevity, for \( A = B \) we also write \( A \cdot A = A^2 \).

In this section we focus on the case \( G = (\mathbb{N}, \cdot) \), the semigroup (for multiplication) of all positive integers. The restricted case \( G = \mathbb{N} \setminus \{1\} \) is even more interesting, since \( 1 \in A \) implies \( A \subset A^2 \).

The sets of integers satisfying the small doubling hypothesis \( d(A + A) = dA \) are well described through Kneser’s theorem. The similar question for the product set does not plainly lead to a strong description. We can restrict our attention to sets \( A \) such that \( \gcd(A) = 1 \), since by setting \( B := \frac{1}{\gcd(A)}A \) we have \( dA = \frac{1}{\gcd(A)}dB \) and \( dA^2 = \frac{1}{(\gcd(A))^2}d(B^2) \).

#### Examples

1. Let \( A_{nsf} \) be the set of all non-squarefree integers. Letting \( A = \{1\} \cup A_{nsf} \) we have \( A^2 = A \) and

\[ dA = 1 - \zeta(2)^{-1}. \]

2. However, while \( \gcd(A_{nsf}) = 1 \), we have

\[ dA_{nsf}^2 < dA_{nsf} = 1 - \zeta(2)^{-1}. \]

3. Furthermore, the set \( A_{sf} \) of all squarefree integers satisfies

\[ dA_{sf} = \zeta(2)^{-1} \text{ and } dA_{sf}^2 = \zeta(3)^{-1}, \]

since \( A_{sf}^2 \) consists of all cubefree integers.

4. Given a positive integer \( k \), the set \( A_k = \{n \in \mathbb{N} : \gcd(n, k) = 1\} \) satisfies

\[ A_k^2 = A_k \text{ and } dA_k = \frac{\phi(k)}{k}, \]

where \( \phi \) is Euler’s totient function.

We have the following result:

**Proposition 4.1** For any positive \( \alpha < 1 \) there exists a set \( A \subset \mathbb{N} \) such that \( dA > \alpha \) and \( dA^2 < \alpha \).
Proof: Assume first that \( \alpha < 1/2 \).
For \( k \geq 1 \) let \( A_k = k\mathbb{N} = \{kn, \ n \geq 1\} \), then \( A_k^2 = k^2\mathbb{N} \). Therefore, \( d(A_k) = 1/k \) and \( d(A_k^2) = 1/k^2 \). If \( 1/(k+1) \leq \alpha < 1/k \), then \( A_k \) satisfies the requested condition. Since \( \bigcup_{k \geq 2} \left[ \frac{1}{k+1}, \frac{1}{k} \right) = (0, 1/2) \), an appropriate \( k \) can be chosen for every \( \alpha \in (0, 1/2) \).

Assume now that \( 1 > \alpha \geq 1/2 \).
Let \( p_1 < p_2 < \cdots \) be the increasing sequence of prime numbers and
\[
B_r := \bigcup_{i=1}^{r} p_i \mathbb{N}.
\]
The complement of the set \( B_r \) contains exactly those positive integers that are not divisible by any of \( p_1, p_2, \ldots, p_r \), thus we have
\[
d(B_r) = 1 - \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) =: \gamma_r.
\]
Similarly, the complement of the set \( B_r^2 \) contains exactly those positive integers that are not divisible by any of \( p_1, \ldots, p_r \) or can be obtained by multiplying such a number by one of \( p_1, \ldots, p_r \). Hence, we obtain that
\[
d(B_r^2) = 1 - \left( 1 + \sum_{i=1}^{r} \frac{1}{p_i} \right) \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) =: \beta_r.
\]
Note that
\[
\beta_{r+1} = 1 - \left( 1 + \sum_{i=1}^{r+1} \frac{1}{p_i} \right) \left( 1 - \frac{1}{p_{r+1}} \right) \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) < 1 - \frac{3}{2} \cdot \frac{2}{3} \cdot \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right) = \gamma_r.
\]
As \( (\beta_1, \gamma_1) = (1/4, 1/2) \), moreover \( (\beta_r)_{r=1}^{\infty} \) and \( (\gamma_r)_{r=1}^{\infty} \) are increasing sequences satisfying (8) and \( \lim_{r \to \infty} \gamma_r = 1 \), we obtain that \([1/2, 1)\) is covered by \( \bigcup_{r=1}^{\infty} (\beta_r, \gamma_r) \). That is, for every \( \alpha \in [1/2, 1) \) we have \( \alpha \in (\beta_r, \gamma_r) \) for some \( r \), and then \( A = B_r \) is an appropriate choice.

We pose two questions about the densities of \( A \) and \( A^2 \).

**Question 1** If \( 1 \in A \) and \( d(A) = 1 \), then \( d(A^2) = 1 \), too. Given two integers \( k, \ell \), the set
\[
\{n \in \mathbb{N} : \gcd(n, k) = 1\} \cup k\ell \mathbb{N}
\]
is multiplicatively stable. What are the sets \( A \) of positive integers such that \( A^2 = A \) or less restrictively
\[
1 \in A \text{ and } 1 > d(A^2) = dA > 0?
\]
**Question 2** It is clear that $dA > 0$ implies $dA^2 > 0$, since $A^2 \supset (\min A)A$.

For any $\alpha \in (0, 1)$ we denote

$$f(\alpha) := \inf_{A \subseteq \mathbb{N}; dA = \alpha} dA^2.$$

Is it true that $f(\alpha) = 0$ for any $\alpha$ or at least for $\alpha < \alpha_0$?

The next result shows that the product set of a set having density 1 and satisfying a technical condition must also have density 1.

**Proposition 4.2** Let $A$, with $1 \notin A$, be a set of positive integers with asymptotic density $dA = 1$. Furthermore, assume that $A$ contains an infinite subset of mutually coprime integers $a_1 < a_2 < \cdots$ such that

$$\sum_{i \geq 1} \frac{1}{a_i} = \infty.$$

Then the product set $A^2$ also has density $d(A^2) = 1$.

**Proof:** Let $\varepsilon > 0$ be arbitrary and choose a large enough $k$ such that

$$\sum_{i=1}^k \frac{1}{a_i} > - \log \varepsilon. \quad (9)$$

Let $x$ be a large integer. For any $i = 1, \ldots, k$, the set $A^2(x)$ contains all the products $a_i a$ with $a \leq x/a_i$. We shall use a sieve argument. Let $A'$ be a finite subset of $A$ and $X = [1, x] \cap \mathbb{N}$ for some $x > \max(A')$. For any $a \in A'$, let

$$X_a = \left\{ n \leq x : a \nmid n \text{ or } \frac{n}{a} \notin A \right\}.$$

Observe that

$$X \setminus X_a = (aA)(x).$$

Then

$$(A' \cdot A)(x) = \bigcup_{a \in A'} (X \setminus X_a).$$

By the inclusion-exclusion principle we obtain

$$|(A' \cdot A)(x)| = \sum_{k=1}^{[A']} (-1)^{k-1} \sum_{B \subseteq A' \atop |B| = j} \left| \bigcap_{b \in B} (X \setminus X_b) \right|,$$

whence

$$\left| \bigcap_{a \in A'} X_a \right| = \sum_{j=0}^{[A']} (-1)^j \sum_{B \subseteq A' \atop |B| = j} \left| \bigcap_{b \in B} (X \setminus X_b) \right|, \quad (10)$$
where the empty intersection $\bigcap_{b \in \emptyset} (X \setminus X_b)$ denotes the full set $X$.

For any finite set of integers $B$ we denote by $\text{lcm}(B)$ the least common multiple of the elements of $B$. Now, we consider

$$\bigcap_{b \in B} (X \setminus X_b) = \{ n \leq x : \text{lcm}(B) | n \text{ and } n/b \in A \text{ (}\forall b \in B\text{)} \}.$$ 

By the assumption $d_A = 1$ we immediately get

$$\left| \bigcap_{b \in B} (X \setminus X_b) \right| = \frac{x}{\text{lcm}(B)} (1 + o(1)).$$

Plugging this into (10):

$$\left| \bigcap_{a \in A'} X_a \right| = x \sum_{k=0}^{|A'|} (-1)^j \sum_{B \subseteq A'} \frac{1}{\text{lcm}(B)} + o(x).$$

Since the elements of $A' = \{ a_1, a_2, \ldots, a_k \}$ are mutually coprime,

$$x - |A' \cdot A(x)| = x \sum_{j=0}^{k} (-1)^j \sum_{1 \leq a_{i_1} < \cdots < a_{i_j} \leq k} \frac{1}{a_{i_1} a_{i_2} \cdots a_{i_j}} + o(x) = x \prod_{i=1}^{k} \left( 1 - \frac{1}{a_i} \right) + o(x).$$

(Note that for $j = 0$ the empty product is defined to be 1, as usual.) Since $1 - u \leq \exp(-u)$ we get

$$x - |A' \cdot A(x)| \leq x \exp \left( - \sum_{i=1}^{k} \frac{1}{a_i} \right) + o(x) < \varepsilon x + o(x)$$

by our assumption (9). Thus finally

$$|A^2(x)| \geq |A' \cdot A(x)| > x (1 - \varepsilon - o(1)).$$

This ends the proof. \qed

**Remark 2** Specially, the preceding result applies when $A$ contains a sequence of prime numbers $p_1 < p_2 < \cdots$ such that $\sum_{i \geq 1} 1/p_i = \infty$. For this it is enough to assume that

$$\lim \inf_{i \to \infty} \frac{i \log i}{p_i} > 0.$$

However, we do not know how to avoid the assumption on the mutually coprime integers having infinite reciprocal sum. We thus pose the following question:

**Question 3** Is it true that $d_A = 1$ implies $d(A^2) = 1$?
An example for a set $A$ such that $d(A) = 0$ and $d(A^2) = 1$.

According to the fact that the multiplicative properties of the elements play an important role, one can build a set whose elements are characterized by their number of prime factors. Let

$$A = \{ n \in \mathbb{N} : \Omega(n) \leq 0.75 \log \log n + 1 \},$$

where $\Omega(n)$ denotes the number of prime factors (with multiplicity) of $n$. An appropriate generalisation of the Hardy-Ramanujan theorem (cf. [4] and [10]) shows that the normal order of $\Omega(n)$ is $\log \log n$ and the Erdős-Kac theorem asserts that

$$d\left\{ n \in \mathbb{N} : \alpha < \frac{\Omega(n) - \log \log n}{\sqrt{\log \log n}} < \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt,$$

which implies $dA = 0$. Now we prove that $dA^2 = 1$. The principal feature in the definition of $A$ is that $A^2$ must contain almost all integers $n$ such that $\omega(n) \leq 1$.

For $n \in \mathbb{N}$ let

$$P_+(n) := \max \{ p : p \text{ is a prime divisor of } n \}.$$

Let us consider first the density of the integers $n$ such that

$$P_+(n) > n \exp(-(\log n)^{4/5}). \tag{11}$$

Let $x$ be a large number and write

$$\left| \left\{ n \leq x : P_+(n) \leq n \exp(-(\log n)^{4/5}) \right\} \right|$$

$$= \left| \left\{ n \leq x : P_+(n) \leq x \exp(-(\log x)^{4/5}) \right\} \right| + o(x).$$

By a theorem of Hildebrand (cf. [7]) on the estimation of $\Psi(x, z)$, the number of $z$-friable integers up to $x$, we conclude that the above cardinality is $x + o(x)$. Hence, we may avoid the integers $n$ satisfying (11). By the same estimation we may also avoid those integers $n$ for which $P_+(n) < \exp((\log n)^{4/5})$.

Let $n$ be an integer such that $\Omega(n) \leq 1.2 \log \log n$ and

$$\exp((\log n)^{4/5}) \leq P_+(n) \leq n \exp(-(\log n)^{4/5}).$$

Our goal is to find a decomposition $n = n_1n_2$ with $\Omega(n_i) \leq 0.75 \log \log n_i + 1$, $i = 1, 2$.

Let

$$n = p_1p_2 \cdots p_{t-1}P_+(n),$$

where $t = \Omega(n)$. We also assume that $p_1 \leq p_2 \leq \cdots \leq p_{t-1} \leq P_+(n)$. Let $m = \frac{n}{P_+(n)}$. Then

$$\exp((\log n)^{4/5}) \leq m \leq n \exp(-(\log n)^{4/5}).$$
Let 

\[ n_1 = p_1 p_2 \ldots p_{u-1} P_+(n) \quad \text{and} \quad n_2 = p_u \ldots p_{t-1}, \]

where \( u = \lfloor (t-1)/2 \rfloor \). Then \( n_2 \geq \sqrt{m} \), which yields

\[ \log \log n_2 \geq \log \log m - \log 2 \geq 0.8 \log \log n - \log 2. \]

On the other hand,

\[ \Omega(n_2) = t - u \leq \frac{t}{2} + 1 \leq 0.6 \log \log n + 1 \leq 0.75 \log \log n_2 + \frac{3 \log 2}{4}. \]

Now \( n_1 \geq P_+(n) \geq \exp((\log n)^{4/5}) \), hence

\[ \log \log n_1 \geq 0.8 \log \log n \]

and

\[ \Omega(n_1) \leq \frac{t - 1}{2} \leq 0.6 \log \log n \leq 0.75 \log \log n_1. \]

Therefore, the following statement is obtained:

**Proposition 4.3** The set

\[ A = \{ n \in \mathbb{N} : \Omega(n) \leq 0.75 \log \log n + 1 \} \]

has density 0 and its product set \( A^2 \) has density 1.

By a different approach we may extend the above result as follows.

**Theorem 4.4** For every \( \alpha \) and \( \beta \) with \( 0 \leq \alpha \leq \beta \leq 1 \), there exists a set \( A \subseteq \mathbb{N} \) such that \( d(A) = 0 \), \( d(A \cdot A) = \alpha \) and \( \overline{d}(A \cdot A) = \beta \).

**Proof:** We start with defining a set \( Q \) such that \( d(Q) = 0 \) and \( d(Q \cdot Q) = \beta \). Let us choose a subset \( P_0 \) of the primes such that \( \prod_{p \in P_0} (1 - 1/p) = \beta \). Such a subset can be chosen, since \( \sum 1/p = \infty \). Now, let \( p_k \) denote the \( k \)-th prime and let

\[ P_1 = \{ p_i : i \text{ is odd} \} \setminus P_0, \]

\[ P_2 = \{ p_i : i \text{ is even} \} \setminus P_0. \]

Furthermore, let

\[ Q_1 = \{ n : \text{all prime divisors of } n \text{ belong to } P_1 \} \]

and

\[ Q_2 = \{ n : \text{all prime divisors of } n \text{ belong to } P_2 \}. \]

Let \( Q = Q_1 \cup Q_2 \). Clearly, \( Q \cdot Q = Q_1 \cdot Q_2 \) contains exactly those numbers that do not have any prime factor in \( P_0 \), so \( d(Q \cdot Q) = \beta \). For \( i \in \{1, 2\} \) and \( x \in \mathbb{R} \) the probability that an integer does not have any prime factor being less than \( x \) from \( P_i \)
Our aim is to define a subset \( A \) of \( Q \) in such a way that \( d(A \cdot A) = \alpha \) and \( \overline{d}(A \cdot A) = \beta \). As \( A \subseteq Q \) we will have \( d(A) = 0 \) and \( \overline{d}(A \cdot A) \leq \beta \). The set \( A \) is defined recursively. We will define an increasing sequence of integers \((n_j)_{j=1}^\infty\) and sets \( A_j (j \in \mathbb{N}) \) satisfying the following conditions (and further conditions to be specified later):

1. \( A_j \subseteq A_{j-1} \).
2. \( A_j \cap [1, n_{j-1} + 1] = A_{j-1} \cap [1, n_{j-1}] \).
3. \( A_j \cap [n_j + 1, \infty] = Q \cap [n_j + 1, \infty] \).

That is, \( A_j \) is obtained from \( A_{j-1} \) by dropping out some elements of \( A_{j-1} \) in the range \([n_{j-1} + 1, n_j]\). Finally, we set \( A = \bigcup_{j=1}^\infty A_j \).

Let \( n_1 = 1 \) and \( A_1 = Q \). We define the sets \( A_j \) in such a way that the following condition holds for every \( j \) with some \( n_0 \) depending only on \( Q \):

\[
|(A_j \cdot A_j)(n)| \geq \alpha n \quad \text{for every } n \geq n_0.
\]

Since \( d(Q \cdot Q) = \beta > \alpha \), a threshold \( n_0 \) can be chosen in such a way that \((*)\) holds for \( A_1 = Q \) with this choice of \( n_0 \). Now, assume that \( n_j \) and \( A_j \) are already defined for some \( j \). We continue in the following way depending on the parity of \( j \):

Case I: \( j \) is odd.

Let \( n_j < s \) be the smallest integer such that

\[
|(A_j \setminus [n_j + 1, s]) \cdot (A_j \setminus [n_j + 1, s])(n)| < \alpha n
\]

for some \( n \geq n_0 \). We claim that such an \( s \) exists, indeed it is at most \( \lfloor n_j^2/\alpha \rfloor + 1 \).

For \( s' = \lfloor n_j^2/\alpha \rfloor + 1 \) we have

\[
|(A_j \setminus [n_j + 1, s']) \cdot (A_j \setminus [n_j + 1, s')](s')| \leq n_j^2 < \alpha s'.
\]

Hence, \( s \) is well-defined (and \( s \leq s' \)). Let \( n_{j+1} := s - 1 \) and \( A_{j+1} := A_j \setminus [n_j + 1, s - 1] \). (Specially, it can happen that \( n_{j+1} = n_j \) and \( A_{j+1} = A_j \).) Note that \( A_{j+1} \) satisfies \((*\))

Case II: \( j \) is even.

Now, let \( n_j < s \) be the smallest index for which \( |(A_j \cdot A_j)(s)| > (\beta - 1/j)s \).
We have $d(Q \cdot Q) = \beta$ and $A_j$ is obtained from $Q$ by deleting finitely many elements of it: $A_j = Q \setminus R$, where $R \subseteq [n_j]$. As $d(Q) = 0$, we have that

$$\left| ((Q \cdot Q) \setminus (Q \setminus R) \cdot (Q \setminus R)) (n) \right| \leq |R|^2 + \sum_{r \in R} |Q(n/r)| = o(n),$$

therefore, $d(A_j \cdot A_j) = \beta$. So for some $n > n_j$ we have that $(A_j \cdot A_j)(n) > (\beta - 1/j)n$, that is, $s$ is well-defined. Let $n_{j+1} := s$ and $A_{j+1} = A_j$. Clearly, $A_{j+1}$ satisfies $(*)$.

This way an increasing sequence $(n_j)_{j=1}^\infty$ and sets $A_j(j \in \mathbb{N})$ are defined; these satisfy conditions $(i)$–$(iii)$. Finally, let us set $A := \bigcap_{j=1}^\infty A_j$. Note that $A(n_j) = A_j(n_j)$.

We have already seen that $A \subseteq Q$ implies that $d(A) = 0$ and $\overline{d}(A \cdot A) \leq \beta$. At first we show that $d(A \cdot A) \geq \alpha$. Let $n \geq n_0$ be arbitrary. If $j$ is large enough, then $n_j > n$. As $A_j$ satisfies $(*)$ and $(A \cdot A)(n) = (A_j \cdot A_j)(n)$ we obtain that

$$\left| (A \cdot A)(n) \right| = \left| (A_j \cdot A_j)(n) \right| \geq \alpha n.$$ 

This holds for every $n \geq n_0$, therefore, $d(A \cdot A) \geq \alpha$.

As a next step, we show that $d(A \cdot A) = \alpha$. Let $j$ be odd. According to the definition of $n_{j+1}$ and $A_{j+1}$ there exists some $n \geq n_0$ such that

$$\left| ((A_j \setminus \{n_{j+1} + 1\}) \cdot (A_j \setminus \{n_{j+1} + 1\})) (n) \right| < \alpha n.$$

For brevity, let $s := n_{j+1} + 1$. As $A \subseteq A_j$ we get that $\left| (A \setminus \{s\}) \cdot (A \setminus \{s\}) (n) \right| < \alpha n$. Also,

$$\left| (A \cdot A) \setminus ((A \setminus \{s\}) \cdot (A \setminus \{s\}) (n)) \right| \leq 1 + |A(n/s)| \leq 1 + |Q(n/s)|,$$

since $A \subseteq Q$. Thus $\left| (A \cdot A)(n) \right| \leq \alpha n + 1 + |Q(n/s)| \leq \alpha n + 1/n + 1/s$. Clearly $s = n_{j+1} + 1 \leq n$, and as $j \to \infty$ we have $n_{j+1} \to \infty$, therefore $d(A \cdot A) = \alpha$.

Finally, we prove that $\overline{d}(A \cdot A) = \beta$. Let $j$ be even. According to the definition of $A_{j+1}$ and $n_{j+1}$, we have $\left| (A \cdot A)(n_{j+1}) \right| \geq (\beta - 1/j)n_{j+1}$. However, $(A \cdot A)(n_{j+1}) = (A_{j+1} \cdot A_{j+1})(n_{j+1})$, therefore $\overline{d}(A \cdot A) \geq \lim(\beta - 1/j) = \beta$, and thus $\overline{d}(A \cdot A) = \beta$, as was claimed.

\[\square\]

References


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