AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **73(3)** (2019), Pages 466–477

# On the complexity of the outer-connected bondage and the outer-connected reinforcement problems

M. Hashemipour M. R. Hooshmandasl

*Department of Computer Science*
*Yazd University*
*Iran*
mhashemi@stu.yazd.ac.ir hooshmandasl@yazd.ac.ir

A. Shakiba

*Department of Computer Science*
*Vali-e-Asr University of Rafsanjan*
*Iran*
ali.shakiba@vru.ac.ir

**Abstract**

Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is a *dominating set* of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. A set $\tilde{D} \subseteq V$ of a graph $G = (V, E)$ is called an *outer-connected dominating set* for $G$ if (1) $\tilde{D}$ is a dominating set for $G$, and (2) $G[V \setminus \tilde{D}]$, the induced subgraph of $G$ by $V \setminus \tilde{D}$, is connected. The minimum cardinality among all outer-connected dominating sets of $G$ is called the *outer-connected domination number* of $G$ and is denoted by $\tilde{\gamma}_c(G)$. We define the *outer-connected bondage number* of a graph $G$ as the minimum number of edges whose removal from $G$ results in a graph with an outer-connected domination number larger than the one for $G$. Also, the *outer-connected reinforcement number* of a graph $G$ is defined as the minimum number of edges whose addition to $G$ results in a graph with an outer-connected domination number which is smaller than the one for $G$. This paper shows that the decision problems for the *outer-connected bondage* and the *outer-connected reinforcement* numbers are **NP**-hard. Also, the exact values of the *outer-connected bondage number* and the *outer-connected reinforcement number* are determined for several classes of graphs.

All authors also at: The Lab. of Quantum Information Processing, Yazd University, Iran.

ISSN: 2202-3518 

# 1 Introduction

The terminology and notation on graph theory in this paper in general follows the reference [11]. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. The graph $G$ is said to be of order $|V|$ and size $|E|$. Also, we use $V(G)$ and $E(G)$ to denote the vertex set and the edge set of the graph $G$, respectively. The complement of a graph $G$, denoted by $\bar{G}$, is a graph whose vertex set is $V(G)$ and such that two vertices are adjacent if and only if they are not adjacent in $G$.

Let $v$ be a vertex in $V$. The *open neighborhood* of $v$ is denoted by $N_G(v)$ and is defined as $\{u \in V : \{u, v\} \in E(G)\}$. Similarly, the *closed neighborhood* of $v$ is denoted by $N_G[v]$ and is defined as $\{v\} \cup N_G(v)$. Whenever the graph $G$ is clear from the context, we simply write $N(v)$ and $N[v]$ to denote $N_G(v)$ and $N_G[v]$, respectively. A *leaf* in $G$ is a vertex of degree one. We denote the *path* of order $n$ by $P_n$, the *cycle* of order $n$ by $C_n$ and the *star* of order $n$ by $S_n$. A forest where each component is a star is called a *galaxy*. For a subset $S$ of vertices of $G$, we refer to $G[S]$ as the subgraph of $G$ induced by $S$. A subset $S \subseteq V$ is a *dominating set* of $G$ if every vertex not in $S$ is adjacent to a vertex in $S$. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality among all dominating sets of $G$. A dominating set $S$ is called a $\gamma - set$ of $G$ if $|S| = \gamma(G)$.

Domination is one of the most widely studied topics in graph theory; see e.g. [10, 11] and references therein. This paper studies some issues in a particular variation of domination, namely, *outer-connected domination*. The concept of the outer-connected domination number is introduced by Cyman [3] and is further studied by others [1, 17]. The outer-connected domination problem is shown to be an **NP**-complete problem for arbitrary graphs in [3]. A set $\tilde{D} \subseteq V$ of a graph $G = (V, E)$ is called an *outer-connected dominating set* for $G$ if (1) $\tilde{D}$ is a dominating set for $G$, and (2) $G[V \setminus \tilde{D}]$, the induced subgraph of $G$ by $V \setminus \tilde{D}$, is connected. The minimum cardinality among all outer-connected dominating sets of $G$ is called the *outer-connected domination number* of $G$ and is denoted by $\tilde{\gamma}_c(G)$ [3]. An outer-connected dominating set $\tilde{D}$ is called a $\tilde{\gamma} - set$ of $G$ if $|\tilde{D}| = \tilde{\gamma}_c(G)$.

In this paper, we focus on two graph alterations and their effects on the outer-connected domination number: (1) the removal of edges from a graph, and (2) the addition of edges to a graph. The bondage and the reinforcement numbers are two important parameters for measuring the vulnerability and the stability of the network domination under link failure and link addition. The *bondage number* of $G$, denoted by $b(G)$, is the minimum number of edges whose removal from $G$ results in a graph with a domination number larger than the one for $G$. The *reinforcement number* of $G$, denoted by $r(G)$, is the smallest number of edges whose addition to G results in a graph with a domination number smaller than the one for $G$. The bondage and the reinforcement numbers in graphs are very interesting research problems and were introduced by Fink et al. [4] and Kok, Mynhardt [18], respectively. Hattingh et al. [9] showed that the problem of the restrained bondage is **NP**-complete, even for bipartite graphs. Also they have determined the exact value of the bondage numbers for several classes of graphs. Moreover, the reinforcement number for digraphs has been studied by Huang, Wang and Xu [14]. Hu and Xu [13] showed that the problems of the bondage, the total bondage, the reinforcement and the total reinforcement numbers for an arbitrary graph are all **NP**-hard, in general. Recently, Xu [20] gave a review article on the bondage numbers. Moreover, Hu and Sohn [12] showed that these problems remain **NP**-hard, even for bipartite graphs. Xu, Hu and Lu [19] studied the complexity of $p$-reinforcement and paired bondage problems in general graphs. Jafari Rad [15] showed that the problems of the $p$-reinforcement, the $p$-total reinforcement, the total restrained reinforcement and the $k$-rainbow reinforcement are all **NP**-hard for bipartite graphs. In addition, he also (in [16]) showed that the problems of the paired bondage, the total restrained bondage, the independent bondage and the $k$-rainbow bondage numbers are all **NP**-hard, even if they are restricted to bipartite graphs. From the algorithmic point of view, Hartnell et al. [7] designed a linear time algorithm to compute the bondage number of a tree.

The *outer-connected bondage number* of a graph $G$, where $G$ does not have any isolated vertices, is denoted by $b_{OCD}(G)$, and is equal to the minimum number of edges whose removal from $G$ results in a graph with an outer-connected domination number larger than the one for $G$. The *outer-connected reinforcement number* of a graph $G$ which does not have any isolated vertices is denoted by $r_{OCD}(G)$ and is equal to the smallest number of edges whose addition from $G$ results in a graph with an outer-connected domination number smaller than the one for $G$.

The rest of the paper is organized as follows: In Sections 2 and 3 we describe some necessary preliminaries and determine the exact value of the bondage number for several classes of graphs. In Section 4 we show that the decision problem for the outer-connected reinforcement number in general graphs is **NP**-hard. Finally, in Section 5 we show that the outer-connected bondage number is also **NP**-hard in general graphs. In the other words, we show that there are no polynomial time algorithms to compute these values for graphs, unless $P = NP$.

## 2 Preliminaries

In this section we establish several theorems on the exact values of $b_{OCD}(G)$ and $r_{OCD}(G)$.

**Definition 2.1.** [6] The edge-connectivity, or the line connectivity, of a graph $G$ is the minimum number of edges whose deletion from a graph G disconnects G.

**Proposition 2.2.** *[6] Let $K_n$ be a complete graph of order $n$. The edge-connectivity of a graph $K_n$ is $n - 1$.*

**Theorem 2.3.** *Let $K_n$ be a complete graph of order $n$ with $n \geq 3$. Then, we have*

$$b_{OCD}(K_n) = \begin{cases} 1, & \text{if } n = 3, \\ \lceil \frac{n}{2} \rceil, & \text{otherwise.} \end{cases} \tag{1}$$

*Proof.* If $n = 3$, then $\tilde{\gamma}_c(K_3) = 1$. By removing any edges from $G$, it turns into a $P_3$. So, we have $b_{OCD}(K_3) = 1$, since $\tilde{\gamma}_c(P_3) = 2$.

Next, suppose that $n > 3$. Let the graph $G'$ be obtained by removing fewer than $\lceil \frac{n}{2} \rceil$ edges from $G$. Then, $G'$ contains at least a vertex of degree $n-1$ . Let the vertex $v$ be such a vertex. On the other hand, according to the Proposition 2.2, the induced graph $G'[V \setminus v]$ is connected. So, we have $\tilde{\gamma}_c(G') = 1$, which implies

$$b_{OCD}(G) \geq \lceil \frac{n}{2} \rceil. \tag{2}$$

Now we need to consider the following two cases:

1. $n$ is even: Let $H$ be the graph obtained by removing $\lceil \frac{n}{2} \rceil$ independent edges from $G$. Then the degree of every vertex $v \in V(H)$ is $n-2$. So we have $\tilde{\gamma}_c(H) \geq 2$.
2. $n$ is odd: Let $H$ be the graph obtained by removing $\frac{n-1}{2}$ independent edges from $G$. Then there is exactly one vertex $v \in V(H)$ such that the degree of $v$ is $n-1$. If we remove one edge incident with $v$, then we have $\tilde{\gamma}_c(H) \geq 2$.

In either case, by removing $\lceil \frac{n}{2} \rceil$ edges, we have $\tilde{\gamma}_c(G) < \tilde{\gamma}_c(H)$. So we obtain

$$b_{OCD}(G) \leq \lceil \frac{n}{2} \rceil. \tag{3}$$

Therefore, by Equations 2 and 3, we have $b_{OCD}(G) = \lceil \frac{n}{2} \rceil$. □

**Theorem 2.4.** *Let $C_n$ be a cycle graph with $n \geq 3$ vertices. Then, we have*

$$b_{OCD}(C_n) = \begin{cases} 1, & \text{if} \quad n = 3, \\ \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases} \tag{4}$$

*Proof.* If $n = 3$, then $\tilde{\gamma}_c(C_3) = 1$. By removing any edges from $C_3$, it turns into a $P_3$. So, we have $b_{OCD}(C_3) = 1$, since $\tilde{\gamma}_c(P_3) = 2$.

Cyman [3] has shown that if $n \geq 4$ then $\tilde{\gamma}_c(C_n) = \tilde{\gamma}_c(P_n) = n-2$. So, for $n \geq 4$, we have $b_{OCD}(C_3) > 1$. We remove one edge from $C_n$ to transform it to $P_n$. By removing some edges from $P_n$, a set of components $\{H_1, H_2, \cdots, H_m\}$ is obtained such that every component is a path. let $H = \bigcup_{i=1}^{m} H_i$ and $\tilde{D}(H)$ be an outer-connected dominating set for $H$. Then $\tilde{D}(H) = \bigcup_{i=1}^{m-1} H_i \cup \tilde{D}(H_m)$ (see Lemma 3.1 in [8]). Note that the ordering of $H_i$ is arbitrary and does not matter. If there exists at least one component with four vertices, then $\tilde{\gamma}_c(C_n) = \tilde{\gamma}_c(P_n) = \tilde{\gamma}_c(H) = n-2$. i.e., the $\tilde{\gamma}_c$ remains the same. Otherwise, we have $\tilde{\gamma}_c(C_n) = \tilde{\gamma}_c(P_n) < \tilde{\gamma}_c(H) = n-1$. So we need to break the path $P_n$ in such a way that no components with more than three vertices exist. Therefore we have to remove $\lceil \frac{n}{3} \rceil - 1$ edges from $P_n$, which with the deleted edge from $C_n$ number $\lceil \frac{n}{3} \rceil$. □

**Theorem 2.5.** *Let $P_n$ be a path with $n \geq 3$ vertices. Then we have*

$$b_{OCD}(P_n) = \begin{cases} 1, & \text{if} \ n = 2, \\ 2, & \text{if} \ n = 3, \\ \lceil \frac{n}{3} \rceil - 1, & \text{otherwise.} \end{cases} \tag{5}$$

*Proof.* The cases for $n = 2$ and $n = 3$ are quite clear. The proof of the case $n \geq 4$ is the same as in Theorem 2.4. □

**Theorem 2.6.** *A graph $G$ is a galaxy of order $n \geq 4$ if and only if $b_{OCD}(G) = |E(G)|$.*

*Proof.* According to Observation 2 in [3] and Lemma 3.1 in [8], it is clear that if $G$ is a galaxy then $\tilde{\gamma}_c(G) = n - 1$. Moreover, the only graph with outer-connected domination number equal to $n$ is $\bar{K_n}$. So $b_{OCD}(G) = |E(G)|$. Conversely, suppose that $b_{OCD}(G) = |E(G)|$. If a component $H$ of $G$ is not a star, then $H$ either contains a cycle or a $P_4$, which means that $\tilde{\gamma}_c(G) \leq n - 2$. Let $e = \{v_1, v_2\}$ be an edge in the cycle or the $P_4$. If $H$ is a graph obtained by removing all edges from $G$ except $e$, then we have $\tilde{\gamma}_c(H) = n - 1 > \tilde{\gamma}_c(G)$. This implies that $b_{OCD}(G) \leq |E(G)| - 1$, which is a contradiction. □

**Proposition 2.7.** *Let $G = (V, E)$ be a cycle, a path or a galaxy graph of order $n \geq 4$. Then, $r_{OCD}(G) = 1$.*

*Proof.* The proof is immediate from the definition of the outer-connected reinforcement number. □

## 3 3-Satisfiability problem

In order to show the **NP**-hardness of the *outer-connected reinforcement* and *outer-connected bondage* problems, we do a polynomial time reduction from 3-satisfiability problem, 3-SAT, which is known to be an **NP**-complete problem [5]. For concreteness, Let $\mathcal{U}$ be a set of Boolean variables. A *truth assignment* for $\mathcal{U}$ is a mapping $f : \mathcal{U} \to \{T, F\}$. If $f(u) = T$, then $u$ is said to be "*true*" with respect to $f$. In the case that $f(u) = F$, then $u$ is said to be "$false$" with respect to $f$. If $u$ is a variable in $\mathcal{U}$, then $u$ and $\bar{u}$ are literals over $\mathcal{U}$. The literal $u$ is true if and only if the variable $u$ is true with respect to $f$ and the literal $\bar{u}$ is true if and only if the variable $u$ is false with respect to $f$ .

A *clause* over $\mathcal{U}$ is a set of literals over $\mathcal{U}$ which represents the disjunction of these literals. It is said to be satisfied by a truth assignment if and only if at least one of its members is true with respect to that assignment. Similarly, a collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over $\mathcal{U}$ is satisfiable if and only if there exists some truth assignment for $\mathcal{U}$, which simultaneously satisfies all the clauses $C_i$ in $\mathcal{C}$ for $i = 1, 2, \ldots, m$. Such a truth assignment is called a satisfying truth assignment for $\mathcal{C}$. Given this notation, the 3-SAT problem is specified as follows:

3-SAT problem:

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over a finite set of variables $\mathcal{U}$ such that $|C_j| = 3$ for $j = 1, 2, \ldots, m$.

Question: Is there a truth assignment for $\mathcal{U}$ which satisfies all the clauses in $\mathcal{C}$?

**Theorem 3.1.** *(See Theorem 3.1 in [2].) The 3-SAT problem is* **NP***-complete.*

## 4 NP-hardness for the outer-connected reinforcement problem

In this section, we show that the outer-connected reinforcement problem for general graphs is an NP-hard problem. The outer-connected reinforcement problem is defined as follows:

Outer-connected reinforcement problem:

Instance: A graph $G$ with no isolated vertices and a positive integer $k$.

Question: Is $r_{OCD}(G) \leq k$?

**Theorem 4.1.** *The outer-connected reinforcement problem is NP-hard.*

*Proof.* We show the NP-hardness of the outer-connected reinforcement problem by a polynomial time reduction from the 3-SAT.

Let $I = (\mathcal{U} = \{u_1, u_2, \ldots, u_n\}\ , \mathcal{C} = \{C_1, C_2, \ldots, C_m\})$ be an arbitrary instance of the 3-SAT problem. Without loss of generality, consider that $k = 1$. We construct a graph $G$ such that this instance of 3-SAT will be satisfiable if and only if $G$ has an outer-connected reinforcement of cardinality equal to 1, i.e. $r_{OCD}(G) = 1$. Next, we describe the construction of $G$.

To each $u_i \in \mathcal{U}$, we associate a triangle $\mathcal{S}_i = \{u_i, v_i, \bar{u_i}\}$. Note that $v_i \notin \mathcal{U}$. For each clause $C_j \in \mathcal{C}$, we associate a single vertex $c_j$ and add edges $\{c_j, u_i\}(\{c_j, \bar{u_i}\})$ if the literal $u_i(\bar{u_i})$ appears in clause $C_j$, for $j = 1, 2, \cdots, m$, respectively. Finally, we add vertices $x$ and $y$ and join them to every vertex $c_j$ for $j = 1, 2, \cdots, m$ and add edges $\{y, u_i\}$ and $\{y, \bar{u_i}\}$ for $i = 1, 2, \cdots, n$.

For example, consider a 3-SAT instance ($\mathcal{U} = \{u_1, u_2, u_3, u_4, u_5\}$, $\mathcal{C} = \{C_1, C_2, C_3, C_4\}$), where $C_1 = \{\bar{u_1}, \bar{u_2}, u_3\}, C_2 = \{u_1, u_3, u_5\}, C_3 = \{\bar{u_3}, \bar{u_4}, u_5\}$ and $C_4 = \{\bar{u_1}, \bar{u_3}, u_4\}$. Figure 1 illustrates the constructed graph corresponding to this instance.

It can be easily seen that the construction can be accomplished in polynomial time, since the graph $G$ contains $3n + m + 2$ vertices and $5n + 5m + 1$ edges. All that remains to be shown is that the $I = (\mathcal{U}, \mathcal{C})$ is satisfiable if and only if $r_{OCD}(G) = 1$. Claims 4.1.1, 4.1.2 and 4.1.3, which are stated and shown next, conclude the proof. □

**Claim 4.1.1.** *For any graph $G$ constructed as is described above, we have $\tilde{\gamma}_c(G) = n + 1$.*

*Proof.* Let $\tilde{D}$ be a $\tilde{\gamma}$-set of $G$. Then $\tilde{\gamma}_c(G) = |\tilde{D}| \geq n + 1$ since it is necessary that $|\tilde{D} \cap V(\mathcal{S}_i)| \geq 1$ for $i = 1, 2, \ldots, n$ and also $|\tilde{D} \cap N[x]| \geq 1$. On the other hand, the set $\tilde{D}' = \{x, u_1, u_2, \ldots, u_n\}$ is an outer-connected dominating set for $G$, which implies that $\tilde{\gamma}_c(G) \leq |\tilde{D}'| = n + 1$. Thus, we obtain $\tilde{\gamma}_c(G) = n + 1$. □

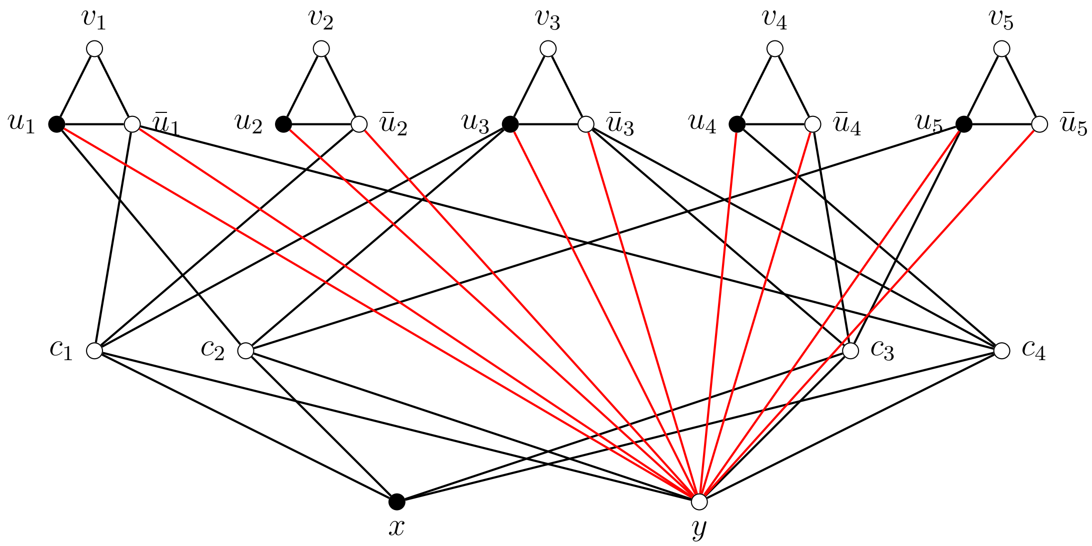


Figure 1: An instance of the outer-connected reinforcement problem. Bold points are the dominator vertices, $k = 1$ and $\tilde{\gamma}_c = 6$.

**Claim 4.1.2.** *Let $\tilde{D}_e$ denotes a $\tilde{\gamma}$-set of $G + e$ for an arbitrary edge $e \in E(\bar{G})$. If there exists an edge $e \in E(\bar{G})$ such that $\tilde{\gamma}_c(G + e) = n$, then for $i = 1, 2, \ldots, n$, we have $|\tilde{D}_e \cap V(\mathcal{S}_i)| = 1$, while $c_j \notin \tilde{D}_e$ for $j = 1, 2, \ldots, m$ and $y \notin \tilde{D}_e$.*

*Proof.* On the contrary, suppose that $|\tilde{D}_e \cap V(\mathcal{S}_\ell)| = 0$ for some $\ell = 1, 2, \ldots, n$. Since $v_\ell$ needs to be dominated by vertices in $\tilde{D}_e$ and $v_\ell, u_\ell, \bar{u}_\ell \notin \tilde{D}_e$, then one of the end-vertices of the edge $e$ should be $v_\ell$. Moreover, for every $i \neq \ell$, we have $|\tilde{D}_e \cap V(\mathcal{S}_i)| \geq 1$, since $\tilde{D}_e$ dominates all the vertices $v_i$.

There are two cases to consider:

**Case 1:** $y \notin \tilde{D}_e$**,** It is clear that the vertices $u_\ell$ and $\bar{u}_\ell$ do not simultaneously appear in the same clause in $\mathcal{C}$, so, there is no $j$ such that the vertex $c_j$ is adjacent to both of them. Since $u_\ell$ and $\bar{u}_\ell$ should be dominated by $\tilde{D}_e$, then there exists two distinct vertices $c_j, c_\ell \in \tilde{D}_e$ such that $c_j$ and $c_\ell$ dominate $u_\ell$ and $\bar{u}_\ell$, respectively.

Hence, $|\tilde{D}_e| \geq n + 1$, which is a contradiction.

**Case 2:** $y \in \tilde{D}_e$**,** In this case, $|\tilde{D}_e \cap N[x]| \geq 1$. So, $|\tilde{D}_e| \geq n + 1$, which is a contradiction.

Therefore, we have $|\tilde{D}_e \cap V(\mathcal{S}_i)| = 1$ for all $i = 1, 2, \ldots, n$, $y \notin \tilde{D}_e$ and $c_j \notin \tilde{D}_e$ for every $j$, since $|\tilde{D}_e| = n$. □

**Claim 4.1.3.** *The 3-SAT instance $I = (\mathcal{U}, \mathcal{C})$ is satisfiable if and only if $r_{OCD}(G) = 1$.*

*Proof.* Suppose that $r_{OCD} = 1$, which means that there exists an edge $e$ in $\bar{G}$ such that $\tilde{\gamma}_c(G + e) = n$. Let $\tilde{D}_e$ be a $\tilde{\gamma}$-set of $G + e$. Then, by Claim 4.1.2, for all $i =$

$1, 2, \ldots, n$, we have $|\tilde{D}_e \cap V(\mathcal{S}_i)| = 1$. To be precise, we have either $\tilde{D}_e \cap V(\mathcal{S}_i) = \{v_i\}$, $\tilde{D}_e \cap V(\mathcal{S}_i) = \{u_i\}$ or $\tilde{D}_e \cap V(\mathcal{S}_i) = \{\bar{u}_i\}$ for all $i = 1, 2, \ldots, n$.

Assume that the mapping $f : \mathcal{U} \to \{T, F\}$ is defined as

$$f(u_i) = \begin{cases} T, & \text{if} \quad u_i \in \tilde{D}_e \quad \text{or} \quad v \in \tilde{D}_e, \\ F, & \text{if} \quad \bar{u}_i \in \tilde{D}_e. \end{cases} \tag{6}$$

We want to show that the mapping $f$ is a satisfying truth assignment for $I = (\mathcal{U}, \mathcal{C})$. So, it is sufficient to show that $f$ satisfies every clause in $\mathcal{C}$. We choose an arbitrary clause $C_j \in \mathcal{C}$. Since the corresponding vertex $c_j$ to clause $C_j$ is not adjacent to any vertices in correspondence with the set $\{v_i : 1 \leq i \leq n\}$, there exists an index $i$ such that $c_j$ is dominated by $u_i \in \tilde{D}_e$ or $\bar{u}_i \in \tilde{D}_e$. Assume that $c_j$ is dominated by $u_i \in \tilde{D}_e$, then $u_i$ is adjacent to vertex $c_j$ in $G$, namely, the literal $u_i$ is in the clause $C_j$. Since $u_i \in \tilde{D}_e$, we have $f(u_i) = T$ by Equation 6. So, $f$ satisfies the clause $C_j$.
Now, suppose that the vertex $c_j$ is dominated by vertex $\bar{u}_i \in \tilde{D}_e$. So, $\bar{u}_i$ is adjacent to $c_j$ in $G$, namely, the literal $\bar{u}_i$ is in the clause $C_j$. Since $\bar{u}_i \in \tilde{D}_e$, we have $f(u_i) = F$ by Equation 6, which implies that $\bar{u}_i$ is assigned the truth value $T$ by $f$. So, the clause $C_j$ is satisfied by $f$. Since clause $C_j$ is chosen arbitrarily, all the clauses in $\mathcal{C}$ are satisfied by $f$, which implies that $I = (\mathcal{U}, \mathcal{C})$ is satisfiable.

Conversely, suppose that $f : \mathcal{U} \to \{T, F\}$ is a satisfying truth assignment for $\mathcal{C}$ and $\tilde{D}'$ is a subset of $V(G)$ that is constructed as follows.

If $f(u_i) = T$, then we put the vertex $u_i$ in $\tilde{D}'$ and if $f(u_i) = F$, we put the vertex $\bar{u}_i$ in $\tilde{D}'$. Therefore, we have $|\tilde{D}'| = n$. For $j = 1, 2, \ldots, m$, at least one of the literals in clause $C_j$ is true under the assignment of $f$, given that $f$ is a satisfying truth assignment for $I = (\mathcal{U}, \mathcal{C})$. So, by the construction of $G$, the corresponding vertex $c_j$ in $G$ is adjacent to at least one vertex in $\tilde{D}'$. Without loss of generality, let $f(u_1) = T$. Then, $\tilde{D}'$ is a dominating set for $G + \{x, u_1\}$. On the other hand, the induced graph $G[V \setminus \tilde{D}']$ is connected. Hence, $\tilde{D}'$ is an outer-connected dominating set for $G + \{x, u_1\}$ and $\tilde{\gamma}_c(G + \{x, u_1\}) \leq |\tilde{D}'| = n$. By Claim 4.1.1, we have $\tilde{\gamma}_c(G) = n + 1$. Therefore, we obtain $\tilde{\gamma}_c(G + \{x, u_1\}) \leq n < n + 1 = \tilde{\gamma}_c(G)$, which means that $r_{OCD} = 1$. □

## 5 The NP-hardness of the outer-connected bondage

In this section, we show that the outer-connected bondage problem for general graphs is an **NP**-hard problem. Consider the following decision problem.

Outer-connected bondage problem:

Instance: A graph $G$ with no isolated vertices and a positive integer k.

Question: Is $b_{OCD}(G) \leq k$?

**Theorem 5.1.** *The outer-connected bondage problem is* **NP**-*hard.*

*Proof.* Let $I = (\mathcal{U} = \{u_1, u_2, \ldots, u_n\}\ ,\ \mathcal{C} = \{C_1, C_2, \ldots, C_m\})$ be an arbitrary instance of the 3-SAT problem. For an arbitrary positive integer k, we will construct a graph $G$ such that this instance of 3-SAT will be satisfiable if and only if $G$ has an outer-connected bondage of cardinality of at most k, i.e. $b_{OCD}(G) \leq k$ . The graph $G$ is constructed as follows.

To each $u_i \in \mathcal{U}$, we associate a vertex set $\mathcal{H}_i = \{u_i, v_i, \bar{u}_i, x_i, y_i\}$ and add edges $\{x_i, u_i\}, \{y_i, \bar{u}_i\}, \{u_i, v_i\}$, and $\{\bar{u}_i, v_i\}$ for $i = 1, 2, \ldots, n$. For each clause $C_j \in \mathcal{C}$, we associate a single vertex $c_j$ and add edge $\{c_j, u_i\}(\{c_j, \bar{u}_i\})$ if the literal $u_i(\bar{u}_i)$ is present in the clause $C_j$, where $j = 1, 2, \ldots, m$. Then, we add a set of vertices $S = \{s_1, s_2, s_3, s_4\}$ and join $s_1, s_3$ and $s_4$ to vertices $c_j$ and $s_2$. Finally, we add a vertex $t$ to the graph $G$, edges $\{t, s_1\}, \{t, s_3\}, \{t, s_4\}$, $\{t, u_i\}$ and $\{t, \bar{u}_i\}$ for $i = 1, 2, \ldots, n$ and set $k = 1$.


(a) $H_i$ (b) $S$

Figure 2: The graphs $H_i$ and $S$

It can be easily seen that the construction can be accomplished in polynomial time, since the graph $G$ contains $5n + m + 5$ vertices and $6m + 6n + 6$ edges. All that remains to be shown is that $I = (\mathcal{U}, \mathcal{C})$ is satisfiable if and only if $b_{OCD}(G) \leq k$. Without loss of generality, let $k = 1$. By Claims 5.1.1 to 5.1.5, which are shown next, we have $b_{OCD}(G) = 1$ if and only if $I = (\mathcal{U}, \mathcal{C})$ is satisfiable. □

**Claim 5.1.1.** *For any graph $G$ constructed as above, we have $\tilde{\gamma}_c(G) \geq 3n + 1$.*

*Proof.* Let $\tilde{D}$ be a $\tilde{\gamma}$-set of $G$. Then, $\tilde{\gamma}_c(G) = |\tilde{D}| \geq 3n + 1$, since $|\tilde{D} \cap V(\mathcal{H}_i)| \geq 3$ for $i = 1, 2, \ldots, n$. Note that to dominate a vertex $v_i$, we need at least one vertex and the leaf vertices $x_i$ and $y_i$ to be in $\tilde{D}$. Moreover, $|\tilde{D} \cap N[s_2]| \geq 1$. □

**Claim 5.1.2.** *If $\tilde{\gamma}_c(G) = 3n+1$, then $c_j, t \notin \tilde{D}$ for $j = 1, 2, \ldots, m$, $\tilde{D} \cap V(S) = \{s_2\}$ and $|\tilde{D} \cap V(\mathcal{H}_i)| = 3$ for $i = 1, 2, \ldots, n$.*

*Proof.* Since the connection between $\mathcal{H}_i$ and $S$ is due to the vertex $t$, then $t \notin \tilde{D}$. Suppose that $\tilde{\gamma}_c(G) = 3n + 1$. Then, $|\tilde{D} \cap V(\mathcal{H}_i)| = 3$ for $i = 1, 2, \ldots, n$, while $|\tilde{D} \cap V(S)| = 1$. Consequently, $c_j \notin \tilde{D}$ for $j = 1, 2, \ldots, m$. Simultaneously, if $\tilde{D} \cap V(S) = \{s_1\}$, then, $s_3$ and $s_4$ are not dominated. Hence, $s_1 \notin \tilde{D}$ and similarly, $s_3, s_4 \notin \tilde{D}$. So, $\tilde{D} \cap V(S) = \{s_2\}$. □

**Claim 5.1.3.** *The 3-SAT instance $I = (\mathcal{U}, \mathcal{C})$ is satisfiable if and only if $\tilde{\gamma}_c(G) = 3n + 1$.*

*Proof.* Suppose that $\tilde{\gamma}_c(G) = 3n+1$ and $c_j$ is an arbitrary vertex. By Claim 5.1.2, this vertex is adjacent to either $u_i \in \tilde{D}$ or $\bar{u}_i \in \tilde{D}$, since $s_1, s_3, s_4 \notin \tilde{D}$. As $|\tilde{D} \cap V(\mathcal{H}_i)| = 3$ for $i = 1, 2, \ldots, n$, it follows that either $\tilde{D} \cap V(\mathcal{H}_i) = \{x_i, y_i, u_i\}$, $\tilde{D} \cap V(\mathcal{H}_i) = \{x_i, y_i, \bar{u}_i\}$ or $\tilde{D} \cap V(\mathcal{H}_i) = \{x_i, y_i, v_i\}$.

Let the mapping $f : \mathcal{U} \to \{T, F\}$ be defined as

$$f(u_i) = \begin{cases} T, & \text{if } \; u_i \in \tilde{D} \; \text{ or } \; v_i \in \tilde{D}, \\ F, & \text{if } \; \bar{u}_i \in \tilde{D}. \end{cases} \tag{7}$$

To prove that the values assigned by the mapping $f$ is a satisfying truth assignment for $I = (\mathcal{U}, \mathcal{C})$, it is sufficient to show that $f$ satisfies every clause in $\mathcal{C}$. Let $C_j \in \mathcal{C}$ be an arbitrarily clause. Since the corresponding vertex to the clause $C_j$ is not adjacent to any vertex in correspondence with the set $\{v_i, x_i, y_i : 1 \leq i \leq n\}$, there exists an $i$ such that $c_j$ is dominated by either $u_i \in \tilde{D}$ or $\bar{u}_i \in \tilde{D}$. Without loss of generality, assume that $c_j$ is dominated by $u_i \in \tilde{D}$. So, $u_i$ is adjacent to $c_j$ in $G$, namely the literal $u_i$ is in the clause $C_j$. Since $u_i \in \tilde{D}$, we have $f(u_i) = T$ by Equation 7. So, the values assigned by the mapping $f$ satisfies the clause $C_j$.
Now, suppose that the vertex $c_j$ is dominated by vertex $\bar{u}_i \in \tilde{D}$. So, the vertex $\bar{u}_i$ is adjacent to the vertex $c_j$ in $G$, namely the literal $\bar{u}_i$ is in the clause $C_j$. Since $\bar{u}_i \in \tilde{D}$, we have $f(u_i) = F$ by the Equation 7, which implies that $\bar{u}_i$ is assigned the truth value $T$ by $f$ and the clause $C_j$ is satisfied by $f$. Since $C_j$ was chosen arbitrarily, all the clauses in $\mathcal{C}$ are satisfied by $f$, which implies that $I = (\mathcal{U}, \mathcal{C})$ is satisfiable.

Conversely, suppose that $f : \mathcal{U} \to \{T, F\}$ is a satisfying truth assignment for $\mathcal{C}$ and $\tilde{D}'$ is a subset of $V(G)$ which is constructed as follows. If $f(u_i) = T$, then we put the vertex $u_i$ in $\tilde{D}'$ and if $f(u_i) = F$, then we put the vertex $\bar{u}_i$ in $\tilde{D}'$. Therefore, we have $|\tilde{D}'| = n$. For $j = 1, 2, \ldots, m$, at least one of the literals in $C_j$ is true under the assignment $f$, because the mapping $f$ is a satisfying truth assignment for $I = (\mathcal{U}, \mathcal{C})$. So, by the construction of $G$, the vertex in correspondence to $C_j$ in $G$ is adjacent to at least one vertex in $\tilde{D}'$. Then, $D = \tilde{D}' \cup (\bigcup_{i=1}^{n} \{x_i, y_i\}) \cup \{s_2\}$ is a dominating set for $G$. On the other hand, the induced graph $G[V \setminus \; D]$ is connected. Hence, $D$ is an outer-connected dominating set for $G$ and $\tilde{\gamma}_c(G) \leq |D| = 3n+1$. By Claim 5.1.1, we have $\tilde{\gamma}_c(G) \geq 3n+1$. Therefore, we obtain $\tilde{\gamma}_c(G) = 3n+1$. □

**Claim 5.1.4.** *For every $e \in E(G)$, we have $\tilde{\gamma}_c(G - e) \leq 3n + 2$.*

*Proof.* Suppose that $E' = \{\{s_2, s_3\}, \{s_2, s_4\}, \{s_1, c_j\}, \{u_i, v_i\}, \{y_i, \bar{u}_i\}, \{t, s_1\}, \{v_i, \bar{u}_i\}, \{t, \bar{u}_i\}\}$ and $E'' = E \setminus E'$. Let $e \in E''$ be an edge. It is clear that the set $D' = (\bigcup_{i=1}^{n} \{x_i, y_i, u_i\}) \cup \{s_1, s_2\}$ is an outer-connected dominating set for $G - e$, since every vertex in $V \setminus D'$ is adjacent to a vertex in $D'$ due to an edge in $E'$, and the induced graph $(G-e)[V \setminus D']$ is connected. This connection is established by vertices $t$ and $s_i$ for $i \neq 1, 2$. Given that $|D'| = 3n + 2$, then $\tilde{\gamma}_c(G - e) \leq 3n + 2$. We have four cases to consider:

Case 1: If either $e = \{s_2, s_3\}$, $e = \{s_1, c_j\}$ or $e = \{t, s_1\}$, then

$$D' = (\bigcup_{i=1}^{n} \{x_i, y_i, u_i\}) \cup \{s_3, s_2\}$$

is an outer-connected dominating set for $G - e$ and $\tilde{\gamma}_c(G - e) \leq |D'| = 3n + 2$.

Case 2: If $e = \{s_2, s_4\}$, then $D' = (\bigcup_{i=1}^{n} \{x_i, y_i, u_i\}) \cup \{s_4, s_2\}$ is an outer-connected dominating set for $G - e$ and $\tilde{\gamma}_c(G - e) \leq |D'| = 3n + 2$.

Case 3: If either $e = \{y_i, \bar{u}_i\}$, $e = \{\bar{u}_i, v_i\}$ or $e = \{u_i, v_i\}$, then

$$D' = (\bigcup_{i=1}^{n} \{x_i, y_i, v_i\}) \cup \{s_1, s_2\}$$

is an outer-connected dominating set for $G - e$ and $\tilde{\gamma}_c(G - e) \leq |D'| = 3n + 2$.

Case 4: If $e = \{t, \bar{u}_i\}$, then $D' = (\bigcup_{i=1}^{n} \{x_i, y_i, \bar{u}_i\}) \cup \{s_1, s_2\}$ is an outer-connected dominating set for $G - e$ and $\tilde{\gamma}_c(G - e) \leq |D'| = 3n + 2$.

□

**Claim 5.1.5.** $\tilde{\gamma}_c(G) = 3n + 1$ *if and only if* $b_{OCD}(G) = 1$.

*Proof.* First, suppose that $\tilde{\gamma}_c(G) = 3n + 1$. Let $e = \{s_1, s_2\}$ and $\tilde{\gamma}_c(G) = \tilde{\gamma}_c(G - e)$. If $\tilde{D}$ is a $\tilde{\gamma}$-set of $G - e$, then $\tilde{D}$ is a $\tilde{\gamma}$-set for $G$ of cardinality $3n + 1$. By Claim 5.1.2, we have $c_j, t \notin \tilde{D}$ for $j = 1, 2, \ldots, m$ and $\tilde{D} \cap V(S) = \{s_2\}$. So, the vertex $s_1$ is not dominated by $\tilde{D}$, which is a contradiction. Hence, $\tilde{\gamma}_c(G) < \tilde{\gamma}_c(G - e)$. So, $b_{OCD}(G) = 1$.

Next, assume that $b_{OCD}(G) = 1$. By Claim 5.1.1, it follows that $\tilde{\gamma}_c(G) \geq 3n + 1$. Suppose that $e$ is an edge such that $\tilde{\gamma}_c(G) < \tilde{\gamma}_c(G - e)$. By Claim 5.1.4, we have $3n + 1 \leq \tilde{\gamma}_c(G) < \tilde{\gamma}_c(G - e) \leq 3n + 2$ which implies that $\tilde{\gamma}_c(G) = 3n + 1$. □

## 6 Conclusion

In this paper, we have shown the **NP**-hardness of the outer-connected bondage and the outer-connected reinforcement decision problems for general graphs. Also we have obtained the exact values of the outer-connected bondage number and the outer-connected reinforcement number for several classes of graphs.

## References

[1] M. H. Akhbari, R. Hasni, O. Favaron, H. Karami and S. M. Sheikholeslami, On the outer-connected domination in graphs, *J. Comb. Optim.* 26 (2013), 1–9.

[2] S. Cook, The P versus NP problem, in: *The millennium prize problems*, 87–104, Clay Math. Inst., Cambridge, MA, 2006.

[3] J. Cyman, The outer-connected domination number of a graph, *Australas. J. Combin.* 38 (2007), 35–46.

[4] J. F. Fink, M. S. Jacobson, L. F. Kinch and J. Roberts, The bondage number of a graph, *Discrete Math.* 86(1-3) (1990), 47–57.

[5] M. R. Garey and D. S. Johnson, *Computers and intractability*, W.H. Freeman & Co, New York, (2002).

[6] F. Harary, *Graph theory*, Addison-Wesley, Reading, MA, (1969).

[7] B. L. Hartnell, L. Jørgensen, P. D. Vestergaard and C. A. Whitehead, Edge stability of the k-domination number of trees. *Bull. Inst. Comb. Appl.* 22 (1998), 31–40.

[8] M. Hashemipour, M. R. Hooshmandasl and A. Shakiba, On outer-connected domination for graph products, *arXiv preprint arXiv:1708.00188*, (2017).

[9] J. H. Hattingh and A. R. Plummer, Restrained bondage in graphs, *Discrete Math.* 308(23) (2008), 5446–5453.

[10] T. W. Haynes, S. Hedetniemi and P. Slater, (Eds.), Domination in graph, advanced topics, *Marcel Dekker, Inc.*, New York, 1998.

[11] T. W. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of domination in graphs*, CRC Press, (1998).

[12] F. T. Hu and M. Y. Sohn, The algorithmic complexity of bondage and reinforcement problems in bipartite graphs, *Theor. Comp. Sci.* 535 (2014), 46–53.

[13] F. T. Hu and J. M. Xu, On the complexity of the bondage and reinforcement problems, *J. Complex.* 28 (2) (2012), 192–201.

[14] J. Huang, J. W. Wang and J. M. Xu, Reinforcement numbers of digraphs, *Discrete Appl. Math.* 157(8) (2009), 1938–1946.

[15] N. Jafari Rad, On the complexity of reinforcement in graphs, *Discuss. Math. Graph Theory* 36 (4) (2016), 877–887.

[16] N. Jafari Rad and H. Kamarulhaili, On the complexity of some bondage problems in graphs, *Australas. J. Combin.* 68 (2) (2017), 265–275.

[17] J. Mark Keil and D. Pradhan, Computing a minimum outer-connected dominating set for the class of chordal graphs, *Inf. Process. Lett.* 113 (14) (2013), 552–561.

[18] J. Kok and C. M. Mynhardt, Reinforcement in graphs, *Congr. Numer.* 79 (1990), 225–231.

[19] Y. Lu, F. T. Hu and J. M. Xu, On the p-reinforcement and the complexity, *J. Comb. Optim.* 29 (2) (2015), 389–405.

[20] J. M. Xu, On bondage numbers of graphs: a survey with some comments, *Int. J. Comb.* (2013), Art. ID 595210, 34 pp.

(Received 22 Jan 2018; revised 23 Sep 2018)