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Contractible and non-contractible non-edges in 2-connected graphs

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Abstract

In this paper, we prove that for any 2-connected finite graph of order n ($n \geq 6$), the number of contractible non-edges is at most $\lfloor \frac{n(n-4)}{2} \rfloor$. All the extremal graphs with at least eight vertices are characterized. We also prove that for any 2-connected finite graph of order n that is not a cycle, the number of non-contractible non-edges is at most $\frac{(n-1)(n-4)}{2}$. This bound is attained precisely by a cycle with exactly one chord between two vertices at distance two apart.

1 Introduction

Contractible non-edges (a pair of non-adjacent vertices whose identification preserves k-connectivity) were first studied by Kriesell [6] for 3-connected finite graphs and later for triangle-free k-connected finite graphs [7]. Das et al. [2] proved that every 2-connected graph which is not a cycle and is non-complete contains a contractible non-edge. Here, we extend the above result, and study the distribution of contractible and non-contractible non-edges in 2-connected finite graphs.

All graphs considered in this paper can be finite or infinite, and are simple. Standard graph-theoretical terminology can be found in Diestel [3]. Consider any graph $G = (V(G), E(G))$. A *non-edge* is a pair of non-adjacent vertices. For any two disjoint subsets A and B of $V(G)$, denote $E_G(A, B)$ to be the set of all edges between A and B in G. Let S be a subset of $V(G)$. A *fragment* F of S is a union of at least one but not all components of $G - S$. Let H and K be any two graphs. Define $H + K$ to be the union of H and K together with all the edges joining $V(H)$ and $V(K)$. For an edge e in G, define $G\#e$ to be the graph constructed from G by adding a vertex x together with two edges joining x to $V(e)$. Denote a path of order n by P_n, a cycle of order n by C_n and a complete graph of order n by K_n. Define K_n^- to be K_n minus one edge. For a complete bipartite graph $K_{2,n}$, define $K_{2,n}^+$ to be

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the graph obtained by adding an edge between the vertices in the partition class of cardinality two. A graph G is *outerplanar* if there exists an embedding of G in the plane such that all vertices lie in the boundary of one face. A graph G is *maximally outerplanar* if adding any edge to G makes it non-outerplanar. A graph is *cubic* if every vertex has degree three. We say a graph is *almost cubic* if one vertex has degree four and all other vertices have degree three.

Consider any k-connected graph G and let H be a subgraph of G. Then H is k-*contractible* if the identification of its vertices and removal of loops and parallel edges results in a k-connected graph. Since we only deal with 2-contractible subgraphs in this paper, we will simply write *contractible* for 2-contractible. Note that for G non-isomorphic to K_3, $\{x, y\} \subseteq V(G)$ is contractible if and only if $G-x-y$ is connected. Denote the set of all non-edges by $\overline{E}(G)$, the set of all contractible non-edges by $\overline{E}_C(G)$, and the set of all non-contractible non-edges by $\overline{E}_{NC}(G)$. A graph G is *minimally* 2-*connected* if for all $e \in E(G)$, $G - e$ is not 2-connected.

The paper is organized as follows. In Section 2, we derive a tight upper bound for the number of contractible non-edges and determine all the extremal graphs except for order seven. Section 3 deals with non-contractible non-edges. We obtain an upper bound for the number of non-contractible non-edges for non-cycles together with all extremal graphs. Lastly, we list some open problems for further research in Section 4.

2 Contractible non-edges

First, we give a short proof of Das et al.'s characterization of 2-connected graphs that do not contain any contractible non-edges. They are precisely cycles or complete graphs. Interestingly, some other characterizations for a graph to be a cycle or a complete graph are also known [5]. The following lemma is elementary but very useful for finding contractible non-edges.

Lemma 2.1. *Let G be any 2-connected graph and H be a connected subgraph of G. Suppose u and v are two vertices that lie in different components of $G - H$. Then $\{u, v\}$ is a contractible non-edge.*

Proof. Note that $\{u, v\}$ is a non-edge and every u-v path intersects H. Suppose $\{u, v\}$ is non-contractible. Let C be a component of $G - u - v$ not containing H. Then $G[C \cup u \cup v]$ contains a u-v path not intersecting H which is impossible. □

Theorem 2.1 (Das et al. [2]). *Let G be any 2-connected graph. Then G contains a contractible non-edge if and only if G is not a cycle and not a complete graph.*

Proof. (⇒) Easy. (⇐) Since G is non-complete, there exists a non-edge, say $\{x, y\}$. If $\{x, y\}$ is contractible, then we are done. Otherwise, let C_1 and C_2 be any two components of $G-x-y$. Then $G[C_1 \cup x \cup y]$ contains a x-y path P_1 and $G[C_2 \cup x \cup y]$ contains a x-y path P_2. Since G is not a cycle, either (I) G is a cycle with chords or (II) there exists a vertex z outside $P_1 \cup P_2$.

(I) Without loss of generality, let ab be a chord that lies in $G[P_1]$ and x, a, b, y be the order of the vertices in P_1. Take $H := xP_1abP_1y$, u to be a vertex in $P_1 \setminus H$, and v to be a vertex in $P_2 - x - y$. By Lemma 2.1, $\{u, v\}$ is a contractible non-edge.

(II) Without loss of generality, assume $z \notin C_2$. Let a be any vertex in C_2. Take $H := P_1$. By Lemma 2.1, $\{z, a\}$ is a contractible non-edge. □

For any 2-connected finite graph of fixed order, it is obvious that adding an edge will never decrease the number of contractible edges. However, the same conclusion need not hold for contractible non-edges because adding an edge may turn a contractible non-edge into a contractible edge. Also, we see that both a cycle and a complete graph contain no contractible non-edges. Therefore, it is natural to find the maximum number of contractible non-edges and characterize all the extremal graphs.

Lemma 2.2. *Let H and K be any two 2-connected graphs. Suppose $|V(H) \cap V(K)| \geq 2$. Then $H \cup K$ is 2-connected.*

Proof. We will show that $H \cup K$ does not have a cutvertex. Let x be any vertex in $H \cup K$. Suppose $x \in H \cap K$. Then $H - x$ and $K - x$ are both connected and $(H - x) \cap (K - x) \neq \emptyset$. Therefore, $(H \cup K) - x$ is connected. Suppose $x \notin H \cap K$ and without loss of generality, assume $x \in H \setminus K$. Then $H - x$ is connected and $(H - x) \cap K \neq \emptyset$. Therefore, $(H \cup K) - x$ is connected. □

Lemma 2.3. *Let G be any 2-connected graph and H be an edge or a 2-connected subgraph of G. Suppose $G - V(H)$ is not connected. Then for any fragment F of $G - V(H)$, $G[F \cup H]$ is 2-connected.*

Proof. First, consider any component C of $G - V(H)$. Assume $G[C \cup H]$ is not 2-connected. Let x be a cutvertex of $G[C \cup H]$. Suppose $x \in C$. Consider a component D of $G[C \cup H] - x$ not containing H. Then D is a component of $G - x$, a contradiction. Hence, $x \in H$, and $H - x$ is a vertex or a connected subgraph of G. Since G is 2-connected, there exists an edge between C and $H - x$. But $G[C \cup H] - x$ is connected, a contradiction. Therefore, $G[C \cup H]$ is 2-connected. Since $|V(H)| \geq 2$, by Lemma 2.2, for any fragment F of $G - V(H)$, $G[F \cup H]$ is 2-connected. □

Lemma 2.4. *Let G be any 2-connected graph non-isomorphic to K_4 and $\{x, y\}$ be non-contractible in G. Consider any fragment C of $G - x - y$ and define $C' := G[C \cup x \cup y] \cup xy$. Then C' is 2-connected. Suppose $\{a, b\} \subseteq V(C')$ and $\{a, b\} \neq \{x, y\}$. Then $\{a, b\}$ is contractible in G if and only if $\{a, b\}$ is contractible in C'.*

Proof. Since $G \cup xy$ is 2-connected, C' is 2-connected by Lemma 2.3. Consider $\{a, b\} \subseteq V(C')$ such that $\{a, b\} \neq \{x, y\}$. Suppose $C' - a - b$ is not connected. Let D be any component of $C' - a - b$ not intersecting $\{x, y\}$. Then D is a component of $G - a - b$ not containing $\{x, y\} \setminus \{a, b\}$ and $G - a - b$ is not connected. Suppose $G - a - b$ is not connected. Let D be any component of $G - a - b$ not intersecting $\{x, y\}$. Then D is a component of $C' - a - b$ not containing $\{x, y\} \setminus \{a, b\}$ and $C' - a - b$ is not connected. □

Theorem 2.2. *Let G be any 2-connected finite graph of order n where $n \geq 6$. Then the number of contractible non-edges is at most $\lfloor \frac{n(n-4)}{2} \rfloor$.*

Proof. We proceed by induction on $|V(G)|$. For $|V(G)| = 6$, the result is true by considering all 2-connected graphs with six vertices. Suppose the statement is true for all 2-connected graphs with less than n vertices. Consider any 2-connected graph G with n vertices. Suppose G has at least $\lfloor \frac{n(n-4)}{2} \rfloor + 1$ contractible non-edges. Then G has at least $\lfloor \frac{n(n-4)}{2} \rfloor + 1$ non-edges. Therefore, G has less than $\frac{3n}{2}$ edges and thus a vertex of degree two, say x. Let y and z be the two neighbors of x. Then $\{y, z\}$ is non-contractible. Define $G' := (G - x) \cup yz$. Since $G \cup yz$ is 2-connected, G' is 2-connected by Lemma 2.3 and thus has at most $\lfloor \frac{(n-1)(n-5)}{2} \rfloor$ contractible non-edges. By considering G' and the triangle xyz, and using Lemma 2.4, G has at most $\lfloor \frac{(n-1)(n-5)}{2} \rfloor + 0 + (1)(n-3) = \lfloor \frac{n(n-4)-1}{2} \rfloor$ contractible non-edges contradicting our initial assumption. Hence, G has at most $\lfloor \frac{n(n-4)}{2} \rfloor$ contractible non-edges. □

Define $f(n)$ to be the maximum number of contractible non-edges among all 2-connected finite graphs of order n. Theorem 2.2 says that $f(n) \leq \lfloor \frac{n(n-4)}{2} \rfloor$ for $n \geq 6$. For $n \leq 6$, by considering all 2-connected finite graphs of order n, we can determine $f(n)$ and list all the extremal graphs. For $n = 7$, we determine $f(7)$ but do not give the complete list of extremal graphs since it is too large.

Proposition 2.1. *Let $f(n)$ denote the maximum number of contractible non-edges among all 2-connected finite graphs of order n. Then,*

(1) $f(3) = 0$ *and the extremal graph is* K_3.

(2) $f(4) = 1$ *and the extremal graph is* K_4^-.

(3) $f(5) = 3$ *and the extremal graphs are* $K_{2,3}$, $K_{2,3}^+$ *and* $K_1 + P_4$.

(4) $f(6) = 6$ *and the extremal graphs are* $K_{2,4}$, $K_{2,4}^+$, $K_{2,3}\#e$, $K_{2,3}^+\#e$ *where e is any edge of* $K_{2,3}$, *maximally outerplanar graphs of order six, and 3-connected cubic graphs of order six.*

(5) $f(7) = 10$ *and two classes of extremal graphs are: (i) maximally outerplanar graphs of order seven and (ii) 3-connected almost cubic graphs of order seven.*

Proposition 2.2. *Let G be any 2-connected graph of order 8. Then G has 16 contractible non-edges if and only if G is 3-connected cubic.*

Proof. (⇐) Suppose G is 3-connected cubic. Then G has 12 edges. Since any two vertices are contractible, G has 16 contractible non-edges.

(⇒) Suppose G is not 3-connected. Let $\{x, y\}$ be non-contractible in G and C be a fragment of $G - x - y$. Define $D := G - x - y - C$, $C' := G[C \cup x \cup y] \cup xy$ and $D' := G[D \cup x \cup y] \cup xy$. By applying Lemma 2.4, Theorem 2.2 and Proposition 2.1 to C' and D', we obtain upper bounds for contractible non-edges depending on the cardinality of C and D.

(1) $|C| = 1$ and $|D| = 5$. $|\overline{E}_C(G)| \leq 0 + 10 + (1)(5) = 15$.

(2) $|C| = 2$ and $|D| = 4$. $|\overline{E}_C(G)| \leq 1 + 6 + (2)(4) = 15$.

(3) $|C| = 3$ and $|D| = 3$. $|\overline{E}_C(G)| \leq 3 + 3 + (3)(3) = 15$.

Since all the above lead to a contradiction, G is 3-connected and has at least 12 edges. Therefore, G has exactly 16 non-edges and must be cubic. □

Now, we are ready to prove that $f(n) = \lfloor\frac{n(n-4)}{2}\rfloor$ for $n \geq 8$ and characterize all the extremal graphs.

Theorem 2.3. *Let G be any 2-connected finite graph of order n where $n \geq 8$. Then G has exactly $\lfloor\frac{n(n-4)}{2}\rfloor$ contractible non-edges if and only if*

(1) *for even n, G is 3-connected cubic.*

(2) *for odd n, G is 3-connected almost cubic, or $G := G'\#xy$ or $G := (G'\#xy) - xy$ where G' is a 3-connected cubic graph of order $n-1$ and $xy \in E(G')$.*

Proof. We proceed by induction on $|V(G)|$. Proposition 2.2 shows that the theorem holds for $|V(G)| = 8$. Suppose the result is true for all 2-connected graphs with less than n vertices. Let G be any 2-connected graph of order n.

($\Leftarrow$) If n is even and G is 3-connected cubic, then G has exactly $\binom{n}{2} - \frac{3n}{2} = \frac{n(n-4)}{2}$ non-edges, all of which are contractible. Suppose n is odd. If G is 3-connected almost cubic, then G has exactly $\binom{n}{2} - \frac{3n+1}{2} = \frac{n(n-4)-1}{2} = \lfloor\frac{n(n-4)}{2}\rfloor$ non-edges, all of which are contractible. Suppose $G := G'\#xy$ or $G := (G'\#xy) - xy$ where G' is a 3-connected cubic graph of order $n-1$ and $xy \in E(G')$. Let $z = V(G) \setminus V(G')$. By Lemma 2.4, contractible non-edges in G' are precisely the contractible non-edges in G that lie in G'. Let w be any vertex in $G' - x - y$. Since G' is 3-connected, $G' - w - xy$ is connected and hence $\{z, w\}$ is contractible in G. Therefore, G has exactly $\frac{(n-1)(n-5)}{2} + (1)(n-3) = \frac{n^2-4n-1}{2} = \lfloor\frac{n(n-4)}{2}\rfloor$ contractible non-edges.

($\Rightarrow$) Suppose G is 3-connected. Then all non-edges are contractible. If n is even, then G has exactly $\binom{n}{2} - \frac{n(n-4)}{2} = \frac{3n}{2}$ edges implying G is cubic. If n is odd, then G has exactly $\binom{n}{2} - \frac{n(n-4)-1}{2} = \frac{3n+1}{2}$ edges. This implies that G is almost cubic.

Now, assume G is not 3-connected. Let $\{x, y\}$ be non-contractible in G and C be a fragment of $G - x - y$. Define $D := G - x - y - C$, $C' := G[C \cup x \cup y] \cup xy$ and $D' := G[D \cup x \cup y] \cup xy$. By applying Lemma 2.4, Theorem 2.2 and Proposition 2.1 to C' and D', we obtain upper bounds for contractible non-edges depending on the cardinality of C and D. Note that $n \geq 9$.

(1) $|C| = 1$ and $|D| = n-3$. $|\overline{E}_C(G)| \leq 0 + \lfloor\frac{(n-1)(n-5)}{2}\rfloor + (1)(n-3) = \lfloor\frac{n^2-4n-1}{2}\rfloor \leq \lfloor\frac{n(n-4)}{2}\rfloor$.

(2) $|C| = 2$ and $|D| = n-4$. $|\overline{E}_C(G)| \leq 1 + \lfloor\frac{(n-2)(n-6)}{2}\rfloor + (2)(n-4) = \lfloor\frac{n^2-4n-2}{2}\rfloor < \lfloor\frac{n(n-4)}{2}\rfloor$.

(3) $|C| = 3$ and $|D| = n - 5$. $|\overline{E}_C(G)| \leq 3 + \lfloor \frac{(n-3)(n-7)}{2} \rfloor + (3)(n-5) = \lfloor \frac{n^2-4n-3}{2} \rfloor < \lfloor \frac{n(n-4)}{2} \rfloor$.

(4) $|C| = k \geq 4$ and $|D| = n - k - 2 \geq 4$. $|\overline{E}_C(G)| \leq \lfloor \frac{(k+2)(k-2)}{2} \rfloor + \lfloor \frac{(n-k)(n-k-4)}{2} \rfloor + k(n-k-2) \leq \lfloor \frac{n^2-4n-4}{2} \rfloor < \lfloor \frac{n(n-4)}{2} \rfloor$.

Therefore, only (1) is possible with equality holds. Hence, n is odd. By induction hypothesis, D' is 3-connected cubic. We have $G := D'\#xy$ or $G := (D'\#xy) - xy$ where $xy \in E(D')$. □

Corollary 2.1. *Let n be an integer at least six. For $0 \leq k \leq \lfloor \frac{n(n-4)}{2} \rfloor$, there exists a 2-connected finite graph of order n containing exactly k contractible non-edges.*

Proof. By Proposition 2.1 and Theorem 2.3, there exists a 3-connected graph G of order n containing exactly $\lfloor \frac{n(n-4)}{2} \rfloor$ non-edges all of which are contractible. By adding $\lfloor \frac{n(n-4)}{2} \rfloor - k$ edges to G, the resulting graph is still 3-connected and has exactly k non-edges all of which are contractible. □

3 Non-contractible non-edges

This section focuses on non-contractible non-edges. Obviously, for 2-connected graphs of order n, the minimum number of non-contractible non-edges is zero as demonstrated by 3-connected graphs. On the other hand, the maximum number of non-contractible non-edges is $\binom{n}{2} - n = \frac{n(n-3)}{2}$ which can only be attained by a cycle. Note that in this case, every vertex is contained in $n-3$ non-contractible non-edges. We will characterize all graphs having a vertex contained in exactly $n-3$ or $n-4$ non-contractible non-edges. Finally, we determine the maximum number of non-contractible non-edges among all 2-connected graphs of order n which are not a cycle, and find all the extremal graphs.

We start with two fundamental lemmas about contractible and non-contractible edges in 2-connected graphs whose proofs can be found in [1].

Lemma 3.1. *Let G be any 2-connected graph non-isomorphic to K_3 and e be an edge of G. Then $G - e$ or G/e is 2-connected.*

Lemma 3.2. *Let G be any 2-connected graph non-isomorphic to K_3, and e and f be two non-contractible edges of G. Then f is a non-contractible edge of $G - e$.*

By Lemma 3.1 and 3.2, for any 2-connected finite graph non-isomorphic to K_3, every vertex is incident to at least two contractible edges. Also, the lemma below follows immediately (see [1]).

Lemma 3.3. *Consider any 2-connected finite graph G non-isomorphic to K_3. Let x, y be any two vertices of G and C be a component of $G - x - y$. Then $E_G(x, C)$ contains a contractible edge.*

We now derive the following technical lemma which will facilitate the proofs of our main results in this section.

Lemma 3.4. *Let G be any 2-connected finite graph non-isomorphic to K_3 and x, a be two vertices of G. Let C be a component of $G - x - a$ and F be the set of all contractible edges in $E_G(x, C)$. Denote $S := V(C) \cap V(F)$. Let H be any connected subgraph in $G[C \cup a]$ containing $S \cup a$.*

(a) *If b is a vertex in $V(C) \setminus V(H)$, then $\{x, b\}$ is a contractible non-edge.*

(b) *Suppose $S = \{y\}$ and for every vertex z in $V(C) \setminus y$, $\{x, z\}$ is non-contractible. Let P be any a-y path in $G[C \cup a]$. Then $G[C \cup a] = P$.*

(c) *Suppose $S = \{y\}$ and for every vertex z in $V(C) \setminus y$ except one vertex w, $\{x, z\}$ is non-contractible. Then $G[C \cup a]$ has an a-y path not containing w. Let P be any such path. Then $V(C) \cup a = V(P) \cup w$, P is an induced path, and w has exactly two neighbors corresponding to consecutive vertices in P.*

Proof. (a) Suppose $\{x, b\}$ is non-contractible. Since G is 2-connected, $G - C - x$ is either a or a connected subgraph containing a. Hence, $H \cup (G - C - x)$ lies in a component of $G - x - b$. Let D be a component of $G - x - b$ that lies in $V(C) \setminus V(H)$. By Lemma 3.3, $E_G(x, D)$ contains a contractible edge contradicting the definition of S. Therefore, $\{x, b\}$ is contractible and must be a non-edge.

(b) By (a), $V(C) \setminus V(P) = \emptyset$. Suppose P has a chord uv where u is closer to a in P than v. Let t be a vertex in uPv other than u and v. By applying (a) to $aPuvPy$ and t, $\{x, t\}$ is contractible, a contradiction.

(c) Since $\{x, w\}$ is contractible, $G - x - w$ is connected and there exists an a-y path P not containing w in $G[C \cup a]$. Suppose b is a vertex in $V(C) \setminus (V(P) \cup w)$. By applying (a) to P and b, $\{x, b\}$ is contractible, a contradiction. Hence, $V(C) \cup a = V(P) \cup w$. Since the above conclusion is true for any a-y path not containing w in $G[C \cup a]$, this implies that P is an induced path. Finally, since $S = \{y\}$, $\{x, w\}$ is a non-edge. Thus, the neighbors of w lie in P. Let u and v be the two neighbors of w that are farthest apart in P with u closer to a than v. Suppose uPv contains a vertex t other than u, v. By applying (a) to $aPuwvPy$ and t, $\{x, t\}$ is contractible, a contradiction. Therefore, w has exactly two neighbors corresponding to consecutive vertices in P. □

Let us characterize all 2-connected graphs of order n having a vertex contained in exactly $n - 3$ or $n - 4$ non-contractible non-edges.

Theorem 3.1. *Let G be any 2-connected graph of order n and x be a vertex of G. Then there are exactly $n - 3$ non-contractible non-edges containing x if and only if G is a cycle.*

Proof. As the theorem is true for $n = 3$, we can assume $n \geq 4$. ($\Leftarrow$) Easy. ($\Rightarrow$) Obviously, x has degree two. Let y and z be the two neighbors of x. Then xy and

xz are contractible edges. Denote the remaining $n-3$ vertices by $x_1, x_2, \ldots, x_{n-3}$. Note that for any $1 \leq i \leq n-3$, $\{x, x_i\}$ is non-contractible. Since x has degree two, $G - x - x_i$ has exactly two components. Let Y_i be the component of $G - x - x_i$ containing y and Z_i be the component of $G - x - x_i$ containing z. By Lemma 3.4(b), $G[Y_i \cup x_i]$ is a path and $G[Z_i \cup x_i]$ is a path. Hence, G is a cycle. □

Theorem 3.2. *Let G be any 2-connected graph of order n where $n \geq 4$ and x be a vertex of G. Then there are exactly $n-4$ non-contractible non-edges containing x if and only if G is one of the following graphs:*

(1) *a cycle $C_n := x_1x_2 \ldots x_nx_1$ with exactly one chord x_1x_{n-1} and x can be any vertex in $\{x_2, x_3, \ldots, x_{n-2}\}$.*

(2) *a cycle $C_n := x_1x_2 \ldots x_nx_1$ with exactly one chord x_1x_i where $3 \leq i \leq n-1$ and $x = x_1$.*

(3) *a graph of order n consisting of three internally disjoint paths, each of length at least two, joining two vertices, one of which is x.*

(4) *a graph of order n consisting of three internally disjoint paths joining x to each of the three vertices of a K_3 not containing x.*

Proof. ($\Leftarrow$) Easy. ($\Rightarrow$) Suppose x has degree two. Let y and z be the two neighbors of x. Then xy and xz are contractible edges. Denote $V(G) \setminus \{x, y, z\}$ by $\{w\}$ if $n = 4$ and $\{w, x_1, x_2, \ldots, x_{n-4}\}$ if $n \geq 5$, where $\{x, w\}$ is contractible and $\{x, x_i\}$ is non-contractible for all $1 \leq i \leq n-4$. For $n = 4$, w is adjacent to y and z. Since $\{x, w\}$ is a contractible non-edge, y is adjacent to z. We have case (1). For $n \geq 5$, let Y be the component of $G - x - x_1$ containing y and Z be the component of $G - x - x_1$ containing z. Without loss of generality, assume w lies in Y. By Lemma 3.4 (b) and (c), we have case (1).

Suppose x has degree three. Let w, y, z be the neighbors of x, and xy and xz are contractible. For $n = 4$, since $G - x - y$ is connected, $wz \in E(G)$ and since $G - x - z$ is connected, $wy \in E(G)$. Therefore, we have case (2) if $yz \notin E(G)$ and case (4) if $yz \in E(G)$.

For $n \geq 5$, denote $V(G) \setminus \{x, y, z, w\}$ by $\{x_1, x_2, \ldots, x_{n-4}\}$. Note that $\{x, x_i\}$ is non-contractible for all $1 \leq i \leq n-4$. Suppose xw is non-contractible. By Lemma 3.4(b), we have case (2).

Now, assume xw is contractible. Suppose there exists $a \in \{x_1, x_2, \ldots, x_{n-4}\}$ such that $G - x - a$ has three components. By Lemma 3.4(b), we have case (3).

Suppose for all $a \in \{x_1, x_2, \ldots, x_{n-4}\}$, $G - x - a$ has exactly two components. Choose one such a and let C and D be the two components of $G - x - a$. Without loss of generality, assume C contains y, z and D contains w. By Lemma 3.4(b), $G[D \cup a]$ is a path. Since $G - x - y$ is connected, there exists an a-z path P in $G[C \cup a]$ not containing y. Since $G - x - z$ is connected, there exists an y-$(P - z)$ path Q in $G[C \cup a]$. Define $H := P \cup Q$, $b := P \cap Q$, $P_a := bPa$, $P_z := bPz$ and $P_y := bQy$.

Note that $b \neq y$, $b \neq z$ and b may be equal to a. Since H is a connected subgraph in $G[C \cup a]$ containing $\{y, z, a\}$, by Lemma 3.4(a), $V(C) \setminus V(H) = \emptyset$.

Now, P_a, P_z, P_y are all induced paths in G for otherwise we can construct a connected subgraph H' in $G[C \cup a]$ containing $\{y, z, a\}$ such that $V(C) \setminus V(H') \neq \emptyset$ contradicting Lemma 3.4(a). By similar arguments using Lemma 3.4(a), $E(G[P_a \cup P_z \cup P_y]) \setminus E(P_a \cup P_z \cup P_y)$ contains at most one edge uv such that b is adjacent to both u and v. If $E(G[P_a \cup P_z \cup P_y]) \setminus E(P_a \cup P_z \cup P_y) = \emptyset$, we have case (3). If $E(G[P_a \cup P_z \cup P_y]) \setminus E(P_a \cup P_z \cup P_y) \neq \emptyset$, we have case (4). □

For the graphs described in Theorem 3.2, we are interested in the number of non-contractible non-edges they contain.

Proposition 3.1. *Let G be a graph of order n consisting of three internally disjoint paths joining two vertices. Then $|\overline{E}_{NC}(G)| \leq \frac{(n-1)(n-4)}{2}$. The equality holds if and only if G is a cycle with exactly one chord between two vertices at distance two apart.*

Proof. Let P_1, P_2, P_3 be the three paths joining x and y in G. For $i = 1, 2, 3$, let $n_i = |V(P_i)|$. Note that $n_1 + n_2 + n_3 = n + 4$. Suppose $n_i \geq 3$ and $n_j \geq 3$. Consider any $x_i \in P_i - x - y$ and $x_j \in P_j - x - y$. By applying Lemma 2.1 to P_k ($k \neq i, j$) and x_i, x_j, $\{x_i, x_j\}$ is contractible. Therefore, $|\overline{E}_{NC}(G)| = \sum_{i=1}^{3}[\binom{n_i}{2} - (n_i - 1)] - 2 = \frac{1}{2}(n_1^2 + n_2^2 + n_3^2 - 3n - 10)$. Using the fact that for $a \leq b$, $a^2 + b^2 < (a-1)^2 + (b+1)^2$, and G is simple, we have $|\overline{E}_{NC}(G)| \leq \frac{1}{2}[2^2 + 3^2 + (n-1)^2 - 3n - 10] = \frac{1}{2}(n-1)(n-4)$. The equality holds if and only if $\{n_1, n_2, n_3\} = \{2, 3, n-1\}$. □

Proposition 3.2. *Let G be a graph of order n consisting of three internally disjoint paths, each of length at least one, joining a vertex to each of the three vertices of K_3. Then $|\overline{E}_{NC}(G)| \leq \frac{(n-3)(n-4)}{2}$.*

Proof. Let P_1, P_2, P_3 be the three paths joining x to x_1, x_2, x_3 respectively where x_1, x_2, x_3 are the vertices of K_3. For $i = 1, 2, 3$, let $n_i = |V(P_i)|$. Note that $n_1 + n_2 + n_3 = n + 2$. Suppose $n_i \geq 3$ and $n_j \geq 3$. Consider any $y_i \in P_i - x - x_i$ and $y_j \in P_j - x - x_j$. By applying Lemma 2.1 to $G[P_k \cup \{x_1, x_2, x_3\}]$ ($k \neq i, j$) and y_i, y_j, $\{y_i, y_j\}$ is contractible. Therefore, $|\overline{E}_{NC}(G)| = \sum_{i=1}^{3}[\binom{n_i}{2} - (n_i - 1)] = \frac{1}{2}(n_1^2 + n_2^2 + n_3^2 - 3n)$. Using the fact that for $a \leq b$, $a^2 + b^2 < (a-1)^2 + (b+1)^2$, we have $|\overline{E}_{NC}(G)| \leq \frac{1}{2}[2^2 + 2^2 + (n-2)^2 - 3n] = \frac{1}{2}(n-3)(n-4)$. □

Finally, we derive a tight upper bound for the number of non-contractible non-edges among all 2-connected graphs of order n which are not a cycle, and find all the extremal graphs.

Theorem 3.3. *Let G be any 2-connected finite graph of order n that is not a cycle. Then G has at most $\frac{(n-1)(n-4)}{2}$ non-contractible non-edges. For $n = 4$, the equality holds if and only if $G \cong K_4^-$ or K_4. For $n \geq 5$, the equality holds if and only if G is a cycle with exactly one chord between two vertices at distance two apart.*

Proof. The theorem is obviously true for $n = 4$. For the rest of the proof, assume $n \geq 5$. Suppose for all $x \in V(G)$, there are at most $n - 5$ non-contractible non-edges containing x. Then G has at most $\frac{n(n-5)}{2}$ non-contractible non-edges. Suppose there exists a vertex x in G that is contained in at least $n-4$ non-contractible non-edges. Since x has degree at least two, x is contained in at most $n-3$ non-contractible non-edges. If x is contained in exactly $n-3$ non-contractible non-edges, then G is a cycle by Theorem 3.1, a contradiction. Therefore, x is contained in exactly $n-4$ non-contractible non-edges. Then G is one of the graphs described in Theorem 3.2. By Proposition 3.1 and 3.2, the result follows. □

Here, we provide another proof of Theorem 3.3 without using the technical Lemma 3.4 because Theorem 3.1 can be proved directly.

Alternative proof of Theorem 3.3. We proceed by induction on $|V(G)|$. Obviously, the result holds for $|V(G)| = 4, 5$. Suppose the result is true for $|V(G)| = n-1$. Consider any 2-connected graph of order n. Since deleting edges while preserving 2-connectedness does not decrease the number of non-contractible non-edges. We can assume either (I) G is a cycle with one chord, or (II) G is minimally 2-connected and is not a cycle.

(I) By Proposition 3.1 (restricted to cycles with one chord),

$$|\overline{E}_{NC}(G)| \leq \frac{(n-1)(n-4)}{2}.$$

The equality holds if and only if the chord is between two vertices at distance two apart.

(II) By a well-known result for minimally 2-connected graphs [4, 8], G has a vertex of degree two, say z. Let x, y be the two neighbors of z in G. Note that by Lemma 3.1, $\{x, y\}$ is a non-contractible non-edge in G. Define $G' := (G - z) \cup xy$ which is 2-connected by Lemma 2.4. Since G is not a cycle, G' is not a cycle. By induction hypothesis, $|\overline{E}_{NC}(G')| \leq \frac{(n-2)(n-5)}{2}$. By Lemma 2.4 and Theorem 3.1, G has at most $\frac{(n-2)(n-5)}{2} + (n-4) + 1 = \frac{(n-1)(n-4)}{2}$ non-contractible non-edges. If the equality holds, then G' is a cycle C_{n-1} with one chord between two vertices of distance two apart, and z is contained in $n-4$ non-contractible non-edges in G. If xy is the chord, then by Lemma 2.1, z does not lie in any non-contractible non-edges in G, a contradiction. If xy lies in C_{n-1}, then G is a cycle with one chord, a contradiction. Therefore, G has less than $\frac{(n-1)(n-4)}{2}$ non-contractible non-edges. □

4 Open problems

We end this paper with three open problems concerning non-contractible non-edges in 2-connected finite graphs.

Problem 4.1. Characterize all 2-connected graphs of order n that do not contain any non-contractible non-edges.

Problem 4.2. Among all 2-connected graphs of order n which are neither cycles nor cycles with exactly one chord between two vertices at distance two apart, derive a tight upper bound for the number of non-contractible non-edges, and find all the extremal graphs.

Problem 4.3. Determine the largest number $g(n)$ such that for all $0 \leq k \leq g(n)$, there exists a 2-connected graph of order n containing exactly k non-contractible non-edges.

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