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Maximal partial transversals in a class of latin squares

Anthony B. Evans

Department of Mathematics and Statistics
Wright State University
Dayton, Ohio 45435
U.S.A.

Abstract

We construct a class of latin squares of order $2n+1$ constructed from the Cayley tables of cyclic groups of orders n and $n+1$. We show that for n even each of these latin squares has maximal partial transversals of all possible lengths, and for n odd each of these latin squares has maximal partial transversals of all possible lengths except one. In the case $n > 1$ odd, by switching an intercalate we obtain latin squares of order $2n+1$ that have maximal partial transversals of all possible lengths. Hence, for n odd, $n \neq 3$, there exists a latin square of order n that has maximal partial transversals of all possible lengths.

1 Introduction

A *latin square* L of order n is a square matrix of order n with entries from a symbol set S of order n, each element of S appearing exactly once in each row and column of L. As an example, the *Cayley table* of $\mathbb{Z}_n$ is the latin square $M_n = (m_{ij})$, $i, j = 0, \ldots, n-1$, where $m_{ij} = i+j \mod n$. Any latin square obtained from a latin square L by permuting rows, columns, and symbols or replacing the symbol set by another set of the same order is an *isotope* of L and is said to be *isotopic* to L. A *partial transversal* T of a latin square L is a set of cells of L, at most one in each row and at most one in each column, each symbol of L appearing at most once in a cell of T: $|T|$ is the *length* of T. A *maximal partial transversal* of L is a partial transversal that cannot be extended to a partial transversal of greater length. Maximal partial transversals are also called *non-extendable* in the literature: see [8]. For L a latin square of order n, a *near transversal* of L is a partial transversal of length $n-1$, and a *transversal* of L is a partial transversal of length n. If T is a near transversal of L, then the unique cell of L that is not in the same row or column as a cell of T will be called the *missing cell* of T, and the symbol of L that is not in T will be called the *missing symbol* of T. The following is a well-known and easily established bound on the lengths of maximal partial transversals.

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Theorem 1.1 *If m is the length of a maximal partial transversal of a latin square of order n, then*

$$\left\lceil \frac{n}{2} \right\rceil \leq m \leq n.$$

Throughout this paper, when we say that a latin square has maximal partial transversals of all possible lengths, we will mean that the latin square has maximal partial transversals of all lengths allowed by Theorem 1.1. There is much work in the literature on transversals and partial transversals: see the surveys by Wanless [7, 8] or the book by Dénes and Keedwell [3]. However, little has been done on the possible lengths of maximal partial transversals. It is known that, for $n > 4$, there exists a latin square of order n that has a maximal partial transversal of length $\lceil n/2 \rceil$: a construction that proves this was described by Wanless in [7, 8]. It was recently proved by Best, Marbach, Stones, and Wanless that, for any $n \geq 5$ and any m satisfying $\lceil n/2 \rceil \leq m \leq n$, there exists a latin square of order n with a maximal partial transversal of length m: see Theorem 12 in [1].

We will study maximal partial transversals of bicyclic latin squares. A *bicyclic* latin square of order $2n+1$ is a latin square

$$L = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right),$$

in which

- A is a latin square of order n, on the symbol set $\mathbb{Z}_n$, that is isotopic to M_n;
- B is M_{n+1} with the last row removed;
- C is M_{n+1} with the last column removed; and
- D is obtained from a circulant latin square E of order $n+1$, on the symbol set $\mathbb{Z}_n \cup \{\infty\}$, with the ∞s on the main diagonal, by replacing the main diagonal by the last row of B/last column of C.

Note that E is isotopic to M_{n+1}. If the first row of D is $(n, d_1, \ldots, d_n)$, we will denote L by $L_{A,d_1,\ldots,d_n}$. To distinguish the elements of $\mathbb{Z}_n$ from those of $\mathbb{Z}_{n+1}$, we will write $\mathbb{Z}_n$ multiplicatively as $\mathbb{Z}_n = \langle g \rangle = \{g^0, g^1, \ldots, g^{n-1}\}$ and $\mathbb{Z}_{n+1}$ additively as $\mathbb{Z}_{n+1} = \{0, 1, \ldots, n\}$, addition modulo $n+1$. Note that, if $d_i = g^i$, then, for $i \neq j$, the ijth entry of D is

$$d_{ij} = \begin{cases} g^{j-i} & \text{if } i < j, \\ g^{j-i+1} & \text{if } i > j. \end{cases}$$

The bicyclic latin square, L_{M_2,g^1,g^2}, is shown in Figure 1.

As an immediate consequence of Theorem 1.1 we obtain the following bounds on lengths of maximal partial transversals of bicyclic latin squares.

Theorem 1.2 *If T is a maximal partial transversal of length k of a bicyclic latin square of order $2n+1$, then $n+1 \leq k \leq 2n+1$.*

$$\left(\begin{array}{cc|ccc} g^0 & g^1 & 0 & 1 & 2 \\ g^1 & g^0 & 1 & 2 & 0 \\ \hline 0 & 1 & 2 & g^1 & g^0 \\ 1 & 2 & g^0 & 0 & g^1 \\ 2 & 0 & g^1 & g^0 & 1 \end{array}\right)$$

Figure 1: The bicyclic latin square L_{M_2,g^1,g^2}

In Section 2 we will give some needed results on transversals, near transversals, and latin subsquares of M_n. In Section 3 we will construct maximal partial transversals of bicyclic latin squares of order $2n+1$, n even. We will prove that, if n is even, then every bicyclic latin square of order $2n+1$ has maximal partial transversals of lengths $n+1$ and $2n$. We will also prove that, if n is even, then $L_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of all possible lengths, i.e., $n+1,\ldots,2n+1$. In Section 4 we will construct maximal partial transversals of bicyclic latin squares of order $2n+1$, n odd. We will prove that, if n is odd, then every bicyclic latin square of order $2n+1$ has maximal partial transversals of lengths $n+2$ and $2n+1$. We will also prove that, if n is odd, $n \neq 3$, then $L_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of all possible lengths but one, $n+1$. For the exceptional case, $n=3$, we will show that L_{M_3,g^1,g^2,g^3} has maximal partial transversals of all possible lengths. In Section 5 we will modify $L_{M_n,g^1,\ldots,g^n}$, n odd, to obtain a class of latin squares with maximal partial transversals of all possible lengths, thus establishing the existence of latin squares of all odd orders except three that have maximal partial transversals of all possible lengths.

2 Transversals and near transversals of M_n

In our constructions of maximal partial transversals of bicyclic latin squares, the existence of transversals and near transversals of M_n will play a role. Euler [4] showed that M_n has transversals if n is odd and no transversals if n is even. Using Euler's approach the missing symbol of a near transversal of M_n is easily determined.

Lemma 2.1 *Let T be a near transversal of M_n with missing symbol a, and let b be the entry in the missing cell of T. Then, modulo n,*

$$b = \begin{cases} a & \text{if } n \text{ is odd;} \\ a + n/2 & \text{if } n \text{ is even.} \end{cases}$$

Proof: First note that, by simple computation in $\mathbb{Z}_n$,

$$S = \sum_{i \in \mathbb{Z}_n} i = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

Let (s,t) be the missing cell of T: hence $b = s + t$. Let (i_k, j_k), $k = 1, \ldots, n-1$, be the cells of T. Then

$$S = \sum_{k=1}^{n-1}(i_k + j_k) + a = 2S + a - b.$$

Therefore

$$b = a + S = \begin{cases} a & \text{if } n \text{ is odd;} \\ a + n/2 & \text{if } n \text{ is even;} \end{cases}$$

as claimed. □

Paige [6], by characterizing finite abelian groups that admit complete mappings, generalized Euler's result, implicitly showing that the Cayley table of a finite abelian group has a transversal if and only if it does not contain a unique involution. Paige's proof implicitly yielded a near transversal for the Cayley table of a finite abelian group that contains a unique involution, in particular for M_n when n is even.

Theorem 2.1 *If n is even, then M_n has no transversal, and any cell of M_n can be the missing cell of a near transversal of M_n.*

If n is odd, then M_n has a transversal, and every near transversal of M_n can be extended to a transversal of M_n.

Proof: If n is even and T is a transversal of M_n, then for the near transversal of M_n, obtained by removing a cell of T, the missing symbol will be the entry in the missing cell, contradicting Lemma 2.1. Hence, if n is even, then M_n has no transversal. For n even, let T consist of the $n-1$ cells $(0,0), (1,1), \ldots, (n/2 - 1, n/2 - 1)$ and $(n/2, n/2 + 1), (n/2 + 1, n/2 + 2), \ldots, (n-2, n-1)$. The entries in cells $(0,0), (1,1), \ldots, (n/2 - 1, n/2 - 1)$ are $0, 2, \ldots, n-2$, which are even and distinct, and the entries in cells $(n/2, n/2+1), (n/2+1, n/2+2), \ldots, (n-2, n-1)$ are $1, 3, \ldots, n-3$, which are odd and distinct. It follows that T is a near transversal with missing cell $(n-1, n/2)$. If, for $s, t \in \mathbb{Z}_n$, $T+(s,t)$ consists of the cells $(i+s, j+t)$, (i,j) a cell of T, where row and column indices are added modulo n, then $T + (s,t)$ is a near transversal of M_n with missing cell $(n-1+s, n/2+t)$, where row and column indices are added modulo n. Hence, if n is even, then any cell of M_n can be the missing cell of a near transversal of M_n.

If n is odd, then the entries on the main diagonal of M_n are $0, 2, \ldots, 2(n-1)$. As $\gcd(2,n) = 1$, these entries are distinct and so the main diagonal of M_n is a transversal of M_n. Further, if n is odd and T is a near transversal of M_n, then, by Lemma 2.1, the entry in the missing cell of T is the missing symbol of T, and so T can be extended to a transversal of M_n. □

A *latin subsquare* of a latin square L is a square submatrix of L that is a latin square. If L is a latin square of order n and N is a latin subsquare, of L, of order m, then N is a *trivial latin subsquare* if $m = 1$ or n, and N is a *proper latin subsquare* of L otherwise. While the order of a latin subsquare of a latin square L need not divide the order of L in general, this is the case when L is isotopic to M_n.

Theorem 2.2 *If N is a latin subsquare of order m of an isotope of M_n, then m divides n.*

Proof: This is given as a corollary to Theorem 1.6.4 in [3]. □

An important role in the study of latin squares is played by the intercalates: an *intercalate* is a latin subsquare of order two. To *switch* an intercalate $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ is to replace it by $\begin{pmatrix} b & a \\ a & b \end{pmatrix}$. Switching an intercalate in a latin square yields a latin square with possibly different properties. As an example, while M_n has no transversals if n is even by Theorem 2.1, this is not the case for a latin square obtained from M_n by switching an intercalate.

Theorem 2.3 *If n is even and L is obtained from M_n by switching an intercalate, then L has a transversal.*

Proof: Let us first note that if (i, j) is a cell of M_n, n even, then the cells (i, j), $(i+n/2, j)$, $(i, j+n/2)$, and $(i+n/2, j+n/2)$, addition modulo n, form an intercalate of M_n, the unique intercalate containing the cell (i, j).

Let T be the near transversal of M_n constructed in Theorem 2.1. The missing cell of T is $(n-1, n/2)$. No cell of the intercalate containing the cell $(n-1, n/2)$ is a cell of T and, if L is the latin square obtained from M_n by switching the intercalate containing the cell $(n-1, n/2)$. The entry in the $(n-1, n/2)$th cell of L is $n-1$, the missing symbol of T. Hence L has a transversal. This argument can be repeated with $T+(s,t)$ instead of T and $(n-1+s, n/2+t)$ in place of $(n-1, n/2)$, as in the proof of Theorem 2.1. □

An immediate corollary:

Corollary 2.1 *If n is even and L is isotopic to a latin square obtained from M_n by switching an intercalate, then L has a transversal.*

There are several tests to determine whether a latin square is isotopic to the Cayley table of a group. The oldest such test is the Quadrangle Criterion.

Theorem 2.4 (The Quadrangle Criterion) *A latin square $L = (l_{ij})$, of order n, is isotopic to the Cayley table of a group of order n if and only if, for all $i_1, i_2, j_1, j_2, s_1, s_2, t_1, t_2 \in \{0, 1, \ldots, n-1\}$, $i_1 \neq i_2$, $j_1 \neq j_2$, $s_1 \neq s_2$, $t_1 \neq t_2$, if $l_{i_1 j_1} = l_{s_1 t_1} = a$, $l_{i_1 j_2} = l_{s_1 t_2} = b$, $l_{i_2 j_1} = l_{s_2 t_1} = c$, and $l_{i_2 j_2} = d$, then $l_{s_2 t_2} = d$.*

Proof: See Theorem 1.2.1 in [3]. □

Figure 2 elucidates the Quadrangle Criterion. If the Criterion holds then the "?" represents the symbol d.

From the Quadrangle Criterion, we can show that certain subarrays in a latin square L, where L is isotopic to M_n, can be extended to proper latin subsquares of L.

$$L = \begin{pmatrix} a & \cdots & b & & & \\ \vdots & \ddots & \vdots & & & \\ c & \cdots & d & & & \\ & & & a & \cdots & b \\ & & & \vdots & \ddots & \vdots \\ & & & c & \cdots & ? \end{pmatrix}$$

Figure 2: A depiction of the Quadrangle Criterion

Corollary 2.2 *Let L be isotopic to M_n and let $m \geq 3$. If N is an $m \times (m-1)$ or $(m-1) \times m$ subarray of L that contains exactly m symbols of L, then N can be extended to a latin subsquare, of L, of order m.*

Proof: Suppose that N is an $m \times (m-1)$ subarray of L, and let S be the set of symbols in N. As $|S| = m$ and each row of N contains $m-1$ symbols, in each row of N one symbol of S, a "missing" entry, does not occur. It is an exercise to show that these missing entries are distinct: see Exercise 1.5 in [5]. Further, using isotopisms, we may assume that N occupies rows $0, \ldots, m-1$ and columns $0, \ldots, m-2$ of L.

Set $s_1 = t_1 = 0$ and let a denote the s_1t_1th entry of N. For some $t_2 \geq m-1$, the s_1t_2th entry of L is a symbol of N, b say. Pick s_2, $1 \leq s_2 \leq m-1$, and let c denote the s_2t_1th entry of N. For some i_1, $1 \leq i_1 \leq m-1$, there exists j_1, j_2, $0 \leq j_1, j_2 \leq m-2$, for which the i_1j_1th entry of N is a and the i_1j_2th entry is b: the row with missing entry c for instance. For some i_2, the i_2j_1th entry of N is c. Let d denote the i_2j_2th entry of N. Then, by the Quadrangle Criterion, the s_2t_2th entry of L is d, a symbol of N. It follows that the $0,t_2$th through $(m-1),t_2$th entries of L are symbols of N, and, hence, that N can be extended to a latin subsquare of L of order m.

The case N an $(m-1) \times m$ subarray of L is similar. □

3 Maximal partial transversals when n is even

In this section we will construct maximal partial transversals of bicyclic latin squares of order $2n+1$, n even, and will prove that $L_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of all possible lengths when n is even. For n even, every bicyclic latin square of order $2n+1$ has a maximal partial transversal of the minimum possible length, $n+1$.

Lemma 3.1 *If n is even, then any bicyclic latin square of order $2n+1$ has a maximal partial transversal of length $n+1$.*

Proof: Let n be even and let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a bicyclic latin square of order $2n+1$. Let the cells $\{(0, j_0), \ldots, (n, j_n)\}$ be a transversal of M_{n+1}, and form a partial transversal T of L consisting of the n cells $\{(0, j_0), \ldots, (n-1, j_{n-1})\}$ of B and the cell (j_n, j_n) of D. The symbols of T are precisely the elements of $\mathbb{Z}_{n+1}$ and every cell containing an element of $\mathbb{Z}_n$ is in the same row and/or column as a cell of T. Hence T is a maximal partial transversal of L of length $n+1$. □

An alternative proof of Lemma 3.1 is implied in the constructions described in [7, 8]. As D is constructed from a circulant latin square of order $n+1$, and any such square is isotopic to $\mathbb{Z}_{n+1}$, we can construct a partial transversal T of a bicyclic latin square of order $2n+1$ consisting of $n+1$ cells of D, n of these cells containing elements of $\mathbb{Z}_n$. It is easy to see that T is maximal. This construction fails when n is odd.

When n is even, every bicyclic latin square of order $2n+1$ has a near transversal that is maximal.

Lemma 3.2 *If n is even, then any bicyclic latin square of order $2n+1$ has a maximal partial transversal of length $2n$.*

Proof: Let n be even and let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a bicyclic latin square of order $2n+1$. Let T consist of the $n-1$ cells of a near transversal N of A and the $n+1$ cells on the main diagonal of D. As every element of $\mathbb{Z}_{n+1}$ is represented on the main diagonal of D, T contains every element of $\mathbb{Z}_{n+1}$. The cells of N contain $n-1$ distinct elements of $\mathbb{Z}_n$, and the only cell of L, containing an element of $\mathbb{Z}_n$, that is not in the same row or column as a cell of T is the missing cell of N, which, by Lemma 2.1, cannot contain the missing symbol of N. Hence, T is a maximal partial transversal of L of length $2n$. □

As an example to illustrate Lemmas 3.1 and 3.2, for the latin square L in Figure 1, maximal partial transversals of L of lengths 3 and 4 are shown in Figure 3: the entries of the maximal partial transversal are shown as **i** or $\mathbf{g^i}$.

$$\left(\begin{array}{cc|ccc} g^0 & g^1 & \mathbf{0} & 1 & 2 \\ g^1 & g^0 & 1 & \mathbf{2} & 0 \\ \hline 0 & 1 & 2 & g^1 & g^0 \\ 1 & 2 & g^0 & 0 & g^1 \\ 2 & 0 & g^1 & g^0 & \mathbf{1} \end{array}\right) \qquad \left(\begin{array}{cc|ccc} \mathbf{g^0} & g^1 & 0 & 1 & 2 \\ g^1 & g^0 & 1 & 2 & 0 \\ \hline 0 & 1 & \mathbf{2} & g^1 & g^0 \\ 1 & 2 & g^0 & \mathbf{0} & g^1 \\ 2 & 0 & g^1 & g^0 & \mathbf{1} \end{array}\right)$$

Figure 3: Two maximal partial transversals of L_{M_2,g^1,g^2}

The main result of this section is that, if n is even, then $L_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of all possible lengths.

Theorem 3.1 *If n is even, then $L_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of lengths $n+1,\ldots,2n+1$.*

Proof: As the result is trivially true if $n=0$, we shall assume that $n \geq 2$. Suppose that $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = L_{M_n,g^1,\ldots,g^n}$ and that n is even. By Lemma 3.1, L has a maximal partial transversal of length $n+1$ and by Lemma 3.2, L has a maximal partial transversal of length $2n$.

Three constructions will yield maximal partial transversals of L of the remaining lengths. The first construction will be of maximal partial transversals of L of length $n+2m+1$ when $1 \leq m \leq n/2$ and $n/2$ is even and $1 \leq m < n/2$ when $n/2$ is odd. The second construction will be of maximal partial transversals of L of length $n+2m$ when $1 \leq m < n/2$. The last construction will be of a transversal of L.

First construction: For n even, we will construct maximal partial transversals of L of length $n+2m+1$; for $1 \leq m \leq n/2$, when $n/2$ is even; and for $1 \leq m < n/2$, when $n/2$ is odd.

Let T consist of the $(0,0),(1,1),\ldots,(m-1,m-1)$th cells of B, the $(m,m),(m+1,m+1),\ldots,(n-1,n-1)$th cells of C, and the (n,n)th cell of D. T is a partial transversal of L of length $n+1$, and the cells of T contain each element of $\mathbb{Z}_{n+1}$ exactly once.

T can only be extended to a longer partial transversal by incorporating cells from the submatrix $\bar{A} = (a_{ij})$ of A, $i = m,\ldots,n-1$, $j = 0,\ldots,m-1$; and/or the submatrix $\bar{D} = (d_{ij})$ of D, $i = 0,\ldots,m-1$, $j = m,\ldots,n-1$. We can extend T by choosing at most m cells from $\bar{A}$ and at most m cells from $\bar{D}$.

Let us extend T to a partial transversal T' of L by choosing the $(m,0),(m+1,1),\ldots,(2m-1,m-1)$th cells from $\bar{A}$. The entries in these cells are m distinct elements of $\mathbb{Z}_n$. There are two cases to consider, m even and m odd.

Case 1: If m is even, then the entries in the chosen cells of $\bar{A}$ are all even powers of g. Let us extend T' to a partial transversal T'' of L by choosing the $(m-1,m),(m-2,m+1),\ldots,(0,2m-1)$th cells of $\bar{D}$: the entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all odd powers of g.

The partial transversal T'' that we have constructed is a maximal partial transversal of length $n+2m+1$.

Case 2: If m is odd, then the entries in the chosen cells of $\bar{A}$ are all odd powers of g. If m is odd and $m \neq n/2$, let us extend T' to a partial transversal T'' of L by choosing the $(m-1,m+1),(m-2,m+2),\ldots,(0,2m)$th cells of D: the entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all even powers of g.

The partial transversal T'' that we have constructed is a maximal partial transversal of length $n+2m+1$.

In each case the partial transversal T'' that we have constructed is a maximal partial transversal of length $n+2m+1$, where $1 \leq m \leq n/2$ if $n/2$ is even and $1 \leq m < n/2$ if $n/2$ is odd.

Second construction: For n even and $1 \leq m < n/2$, we will construct maximal partial transversal of length $n + 2m$ of L.

Suppose that $1 \leq m < n/2$. Let T consist of the $(0,0), (1,1), \ldots, (m-1, m-1)$th cells of B, the $(m+1, m-1), (m+2, m), \ldots, (n, n-2)$th cells of C, and the $(0, n-1)$th cell of C. These cells contain each element of $\mathbb{Z}_{n+1}$ exactly once.

T can only be extended to a longer partial transversal by incorporating cells from the submatrix $\bar{A} = (a_{ij})$ of A, $i = m, \ldots, n-1$, $j = 0, \ldots, m-2$; and/or the submatrix $\bar{D} = (d_{ij})$ of D, $i = 1, \ldots, m$, $j = m, \ldots, n$. We can extend T by choosing at most $m-1$ cells from $\bar{A}$ and at most m cells from $\bar{D}$.

Let us extend T to a partial transversal T' of L by choosing the $(m, 0), (m+1, 1), \ldots, (2m-2, m-2)$th cells from $\bar{A}$. The entries in these cells are distinct elements of $\mathbb{Z}_n$. There are two cases to consider, m even and m odd.

Case 1: If m is even, then the entries in the chosen cells of $\bar{A}$ are all even powers of g. Let us extend T' to a partial transversal T'' of L by choosing the $(m, m+1), (m-1, m+2), \ldots, (1, 2m)$th cells of $\bar{D}$: the entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all odd powers of g.

Case 2: If m is odd, then the entries in the chosen cells of $\bar{A}$ are all odd powers of g. Let us extend T' to a partial transversal T'' of L by choosing the $(m, m+2), (m-1, m+3), \ldots, (1, 2m+1)$th cells from $\bar{D}$: the entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all even powers of g.

In each case the partial transversal T'' that we have constructed is a maximal partial transversal of length $n+2m$. This construction yields a maximal partial transversal of length $n+2m$, where $1 \leq m < n/2$.

Third construction: The first construction yielded a transversal of L when $n/2$ is even. It remains to show that L has a transversal when $n/2$ is odd.

Let T be a near transversal of A with missing cell $(n/2-2, n/2+1)$: the existence of such a near transversal is guaranteed by Theorem 2.1. As the entry of this missing cell is g^{n-1}, by Lemma 2.1, the missing symbol of T is $g^{(n/2)-1}$.

Let us extend T to T' by adding the cell $(n/2-2, 0)$ of B, the cell $(n/2-1, n/2+1)$ of C, and the cells $(1,1), \ldots, (n/2-2, n/2-2)$ and $(n/2, n/2), \ldots, (n, n)$ of D. The entries of these added cells are precisely the elements of $\mathbb{Z}_{n+1}$ and so T' is a partial transversal of L of length $2n$. The only cell of L that is not in a row or column of T' is the $(0, n/2-1)$th cell of D. The entry of this cell is $g^{(n/2)-1}$, the missing symbol of T. It follows that adding the $(0, n/2-1)$th cell of D to T' yields a transversal of L. □

4 Maximal partial transversals when n is odd

In this section we will construct maximal partial transversals of bicyclic latin squares of order $2n{+}1$, n odd, and will prove that $L_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of all possible lengths except one when n is odd, $n \neq 3$, and of all possible lengths if $n = 3$. For n odd, every bicyclic latin square of order $2n+1$ has a maximal partial transversal of length, $n+2$.

Lemma 4.1 *If n is odd, then any bicyclic latin square of order $2n{+}1$ has a maximal partial transversal of length $n+2$.*

Proof: Let n be odd and let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a bicyclic latin square of order $2n{+}1$.

Let the cells $\{(0, j_0), \ldots, (n-1, j_{n-1})\}$ form a near transversal of M_{n+1} with missing cell (n, j_n) and missing symbol $n-1$: the existence of such a near transversal follows from Lemma 2.1 and Theorem 2.1. Let T consist of the cells $\{(0, j_0), \ldots, (n{-}1, j_{n-1})\}$ of B, and the cell $(0, n-1)$ of C. The symbols of T are precisely the $n+1$ elements of $\mathbb{Z}_{n+1}$. Let us next extend T to a partial transversal T' of L by adding the $(1, j_n)$th cell of D if $j_n \neq 1$, or the $(2, j_n)$th cell of D if $j_n = 1$.

As every cell of L with an entry from $\mathbb{Z}_n$ is in the same row and/or column as a cell of T', T' is a maximal partial transversal of L of length $n+2$. □

When n is odd, every bicyclic latin square of order $2n+1$ has a transversal, i.e., a maximal partial transversal of length $2n+1$.

Lemma 4.2 *If n is odd, then any bicyclic latin square of order $2n{+}1$ has a maximal partial transversal of length $2n+1$.*

Proof: Let n be odd and let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a bicyclic latin square of order $2n{+}1$.

A transversal of A combined with the main diagonal of D is a maximal partial transversal of length $2n+1$, i.e., a transversal. □

Theorem 4.1 *If n is odd, $n \neq 3$, then $L_{M_n,g^1,\ldots,g^n}$ has no maximal partial transversal of length $n+1$.*

Proof: Let n be odd, let $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a bicyclic latin square of order $2n+1$, and suppose that T is a maximal partial transversal of L of length $n+1$. Let α be the number of rows among the first n rows of L that contain cells of T, and let β be the number of columns among the first n columns of L that contain cells of T. Then $n+1-\alpha$ is the number of rows among the last $n+1$ rows of L that contain cells of T, and $n+1-\beta$ is the number of columns among the last $n+1$ columns of L that contain cells of T. There are several cases to consider. First, let us note, as any latin square of order 3 is isotopic to M_3 and no near transversal of M_3 is maximal by Theorem 2.1, the result is true for $n = 1$, and so we will assume that $n \geq 5$.

Case 1. $\alpha = \beta = 0$.

All the cells of T are in D. For T to be maximal, the cells of T must contain all n elements of $\mathbb{Z}_n$ and one element of $\mathbb{Z}_{n+1}$. But this would imply that M_{n+1} has a transversal, violating Theorem 2.1.

Case 2. $\alpha = \beta = n$.

There are two possibilities; either T consists of a transversal of A and one cell of D; or T consists of $n-1$ cells of A and one cell each from B and C.

Subcase 2a. Suppose that T consists of a transversal of A and one cell of D. As every element of $\mathbb{Z}_n$ is a symbol of T in A, the cell of T in D must be on the main diagonal of D. But then T is not maximal as it can be extended to a longer partial transversal by adding another cell from the main diagonal of D.

Subcase 2b. Suppose that T consists of $n-1$ cells of A and one cell each from B and C. As $n \geq 5$, there must exist a cell on the main diagonal of D, not in the same row as the cell of T in C or the same column as the cell of T in B, whose entry is not a symbol of T. Hence, T cannot be maximal.

Case 3. $\{\alpha, \beta\} = \{0, n\}$.

Without loss of generality $\alpha = 0$ and $\beta = n$. T consists of n cells from C and one from D. The cell of D cannot contain an element of $\mathbb{Z}_{n+1}$ as that would imply the existence of a transversal of M_{n+1}, violating Theorem 2.1. Thus the cell of T from D contains an element of $\mathbb{Z}_n$. One element of $\mathbb{Z}_{n+1}$ is not contained in T and this is an entry in at least one cell of B that is not in the same column as any cell of T. Hence, T is not maximal.

Case 4. Exactly one of α, β is 0 and the other is neither 0 nor n.

Without loss of generality $0 = \alpha < \beta < n$. T contains β cells of C and $n+1-\beta$ cells of D. As $n - \beta > 0$ of the columns of A contain no cells of T and T is maximal, each element of $\mathbb{Z}_n$ must be a symbol of T. This can only occur if $\beta = 1$; C contains one cell of T, whose entry is $c \in \mathbb{Z}_{n+1}$; and D contains n cells of T, none on the main diagonal. T can then be extended to a longer partial transversal by adding a cell of B that is not in the same column as any cell of T, and that does not have entry c.

Case 5. Exactly one of α, β is n and the other is neither 0 nor n.

Without loss of generality $0 < \alpha < \beta = n$. T has n cells in the first n columns of L and one "lonely" cell in either B or D. In either case let M be the submatrix of L consisting of those cells that are not in the same row or column as a cell of T. Now M contains n cells from at least one row of B and n cells from at least one row of D. It follows that the entries of M consist of at least n elements of $\mathbb{Z}_{n+1}$ and at least $n-1$ elements of $\mathbb{Z}_n$. Hence T can be extended to a longer partial transversal by appending a cell of M.

Case 6. $\alpha, \beta \neq 0, n$.

Let $\bar{A}$ be the submatrix of A consisting of those cells of A that are not in the same row or column as a cell of T. Let us define $\bar{B}$, $\bar{C}$, $\bar{D}$ similarly. It follows that $\bar{A}$ is an $(n-\alpha) \times (n-\beta)$ array, $\bar{B}$ is an $(n-\alpha) \times \beta$ array, $\bar{C}$ is an $\alpha \times (n-\beta)$ array, and $\bar{D}$ is an $\alpha \times \beta$ array: see Figure 4. As in Case 5, let M be the submatrix of L consisting of those cells that are not in the same row or column as a cell of T. Thus $M = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}$.

$$\begin{array}{rc} & \begin{array}{cc} n-\beta & \quad\beta \end{array} \\ \begin{array}{c} n-\alpha \\[2em] \alpha \end{array} & \left(\begin{array}{c|c} (\bar{A}) & (\bar{B}) \\[1em] \hline \\ (\bar{C}) & (\bar{D}) \end{array}\right) \end{array}$$

Figure 4: Cells of L not in the same row and/or column as cells of T

As T is maximal, each entry of $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ must be a symbol of T. Now, the number of elements of $\mathbb{Z}_n$ in M is at least $\max\{n-\alpha, n-\beta, \alpha-1, \beta-1\}$, and the number of elements of $\mathbb{Z}_{n+1}$ in M is at least $\max\{\alpha, \beta, n-\alpha, n-\beta\}$. Hence

$$\max\{n-\alpha, n-\beta, \alpha-1, \beta-1\} + \max\{\alpha, \beta, n-\alpha, n-\beta\} \leq n+1.$$

It follows that $(n-\alpha)+(n-\alpha) \leq n+1$ and $\alpha + (\alpha-1) \leq n+1$: similarly for β. Hence

$$\frac{n-1}{2} \leq \alpha, \beta \leq \frac{n+1}{2}.$$

Without loss of generality $\alpha \leq \beta$. There are three subcases to consider.

Subcase 6a. $\alpha = \beta = (n-1)/2$.

M contains exactly $(n+1)/2$ elements of $\mathbb{Z}_n$ and exactly $(n+1)/2$ elements of $\mathbb{Z}_{n+1}$. It follows that $\bar{A}$ is a latin subsquare of A. Hence, by Theorem 2.2, $(n+1)/2$ divides n, an impossibility.

Subcase 6b. $\alpha = (n-1)/2$, and $\beta = (n+1)/2$.

M contains exactly $(n+1)/2$ elements of $\mathbb{Z}_n$ and exactly $(n+1)/2$ elements of $\mathbb{Z}_{n+1}$. Hence, A has an $(n+1)/2 \times (n-1)/2$ array, $\bar{A}$, containing exactly $(n+1)/2$ symbols. Thus, A contains a latin subsquare of order $(n+1)/2$ by Corollary 2.2. Hence, $(n+1)/2$ divides n by Theorem 2.2: an impossibility.

Subcase 6c. $\alpha = \beta = (n+1)/2$.

$\bar{A}$ contains either $(n-1)/2$ or $(n+1)/2$ elements of $\mathbb{Z}_n$. If $\bar{A}$ contains $(n-1)/2$ elements of $\mathbb{Z}_n$, then $\bar{A}$ is a latin subsquare of order $(n-1)/2$ of A. Then, by Theorem 2.2, $(n-1)/2$ divides n, an impossibility. It follows that $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ contain exactly $(n+1)/2$ elements of $\mathbb{Z}_n$ and exactly $(n+1)/2$ elements of $\mathbb{Z}_{n+1}$.

By Corollary 2.2, both $\bar{B}$ and $\bar{C}$ can be "extended" to latin subsquares of M_{n+1} of order $(n+1)/2$. Thus, if N is the set of entries of $\bar{B}$ and $\bar{C}$, then N is a coset of the subgroup of $\mathbb{Z}_{n+1}$ of index two. Hence, if $j_1, \ldots, j_{(n-1)/2}$ are the columns of $\bar{C}$, then, for $1 \leq s < t \leq (n-1)/2$, $j_t - j_s$ is even. Similarly for the rows $i_1, \ldots, i_{(n-1)/2}$ of $\bar{B}$.

For $n > 5$, we will show that $\bar{A}$ contains more than $(n+1)/2$ elements in $\mathbb{Z}_n$. Let $\bar{A}_{st}$ denote the submatrix of A with rows $i_{1+s}, \ldots, i_{(n-1)/2+s}$ and columns $j_{1+t}, \ldots, j_{(n-1)/2+t}$, the subscripts being added modulo n. As $\bar{A}_{st}$ contains the same number of distinct entries as $\bar{A}$, we need only show that the submatrix $\bar{A}_0$ of A with rows $0, 2, \ldots, n-3$ and columns $0, 2, \ldots, n-3$ contains more than $(n+1)/2$ distinct entries if $n > 5$. But then, the entries of $\bar{A}_0$ include $0, 2, \ldots, n-3$ and $n-1$ and 1. Hence, $\bar{A}_0$ contains more than $(n+1)/2$ distinct entries: a contradiction.

If $n = 5$, let T contain x cells from A. Then, T contains $3-x$ cells from B, $3-x$ cells from C, and x cells from D. Let y be the number of cells in T that are in D and have entries that are elements of $\mathbb{Z}_6$. Counting the number of symbols from $\mathbb{Z}_5$ in T, we see that $2x - y = 3$. It follows that $x = 2$ and $y = 1$, or $x = y = 3$.

If $x = 2$ and $y = 1$, then $\bar{B}$ is a 2×3 submatrix of B whose entries must consist of precisely three elements of $\mathbb{Z}_6$. By Corollary 2.2, $\bar{B}$ extends to a latin subsquare N of order three of M_6. Let H be the set consisting of the three elements of $\mathbb{Z}_6$ that are elements of $\bar{B}$: these are also the three elements of $\mathbb{Z}_6$ that are also symbols of T. As $y = 1$, all rows of N are included in B. Thus no cell on the main diagonal of D, whose entry is not an element of H, can be in the same column as a cell of T. By a similar argument no cell on the main diagonal of D, whose entry is not an element of H, can be in the same row as a cell of T. It follows that T can be extended to a longer partial transversal of L by appending any cell on the main diagonal of D, whose entry is not an element of H. Thus, T is not a maximal partial transversal of L.

Hence the only possibility is $x = y = 3$. It follows that T can be extended to a longer partial transversal of L by including another cell from the main diagonal of D. $\square$

A maximal partial transversal of L_{M_3,g^1,g^2,g^3} of length 4 is shown in Figure 5: the entries of the maximal partial transversal are shown as $\mathbf{i}$ or $\mathbf{g}^{\mathbf{i}}$.

The main result of this section is that, if n is odd, $n \neq 3$, then $L_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of all possible lengths except one. For the exception, $n = 3$, L_{M_3,g^1,g^2,g^3} has maximal partial transversals of all possible lengths.

Theorem 4.2 *If n is odd, $n \neq 3$, then $L_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of each of the lengths $n+2, \ldots, 2n+1$ and no other. If $n = 3$, then L_{M_3,g^1,g^2,g^3} has maximal partial transversals of each of the lengths 4, 5, 6, and 7.*

$$\left(\begin{array}{ccc|cccc}
g^0 & g^1 & g^2 & \mathbf{0} & 1 & 2 & 3 \\
g^1 & g^2 & g^0 & 1 & 2 & 3 & 0 \\
g^2 & g^0 & \mathbf{g^1} & 2 & 3 & 0 & 1 \\
\hline
0 & 1 & 2 & 3 & g^1 & \mathbf{g^2} & g^0 \\
1 & 2 & 3 & g^0 & 0 & g^1 & g^2 \\
\mathbf{2} & 3 & 0 & g^2 & g^0 & 1 & g^1 \\
3 & 0 & 1 & g^1 & g^2 & g^0 & 2
\end{array}\right)$$

Figure 5: A maximal partial transversal of L_{M_3,g^1,g^2,g^3} of length 4.

Proof: Suppose that $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = L_{M_n,g^1,\ldots,g^n}$ and n is odd. By Lemma 4.1, L has a maximal partial transversal of length $n+2$, by Lemma 4.2, L has a maximal partial transversal of length $2n+1$, and by Theorem 4.1, if $n \neq 3$, then L has no maximal partial transversal of length $n+1$. A maximal partial transversal of L_{M_3,g^1,g^2,g^3} of length 4 is shown in Figure 5.

Three constructions will yield maximal partial transversals of L of the remaining lengths. The first construction will be of maximal partial transversals of L of length $2n-2m-1$ when $0 \leq m < (n-3)/2$. The second construction will be of maximal partial transversals of L of length $2n-2(m+1)$ when $0 \leq m \leq (n-5)/2$. The last construction will be of a maximal partial transversal of length $2n$.

First construction: For n odd and $0 \leq m < (n-3)/2$, we will construct maximal partial transversals of L of length $2n-2m-1$.

Suppose that $0 \leq m < (n-3)/2$. Let T consist of the $(m+1, m+1), (m+2, m+2), \ldots, ((n-1)/2, (n-1)/2)$th cells of B, the $(0,0), (1,1), \ldots, (m,m)$th cells of C, and the $((n+1)/2, (n-1)/2), ((n+3)/2, (n+1)/2), \ldots, (n, n-1)$th cell of C. These cells contain each element of $\mathbb{Z}_{n+1}$ exactly once.

T can only be extended to a longer partial transversal by incorporating cells from the submatrix $\bar{A} = (a_{ij})$ of A, $i \in \{0, 1, \ldots, m\} \cup \{(n+1)/2, (n+3)/2, \ldots, n-1\}$, $j = m+1, \ldots, (n-3)/2$; and/or the submatrix $\bar{D} = (d_{ij})$ of D, $i = m+1, \ldots, (n-1)/2$, $j \in \{0, \ldots, m\} \cup \{(n+1)/2, \ldots, n\}$. We can extend T by choosing at most $(n-3)/2 - m$ cells from $\bar{A}$ and at most $(n-1)/2 - m$ cells from $\bar{D}$.

There are two cases to consider.

Case 1. $(n+3)/2 + m$ even.

Set $2M = (n+3)/2 + m$. Let us extend T to a partial transversal T' of L by choosing the cells $((n+1)/2 + j, m+1+j)$, $j = 0, \ldots, (n-1)/2 - M$, and the cells $((n+1)/2 + j + 1, m+1+j)$, $j = (n+1)/2 - M, \ldots, (n-5)/2 - m$, from $\bar{A}$: the entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all even powers of g.

Let us extend T' to a partial transversal T'' of L by choosing the cells $((n-1)/2 - i, (n+1)/2+i)$, $i = 0, \ldots, (n-3)/2 - m$, from $\bar{D}$: the entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all odd powers of g.

The partial transversal T'' that we have constructed is a maximal partial transversal of length $2n-2m-1$. This construction yields a maximal partial transversal of length $2n-2m-1$, where $1 \leq m < n/2$.

Case 2. $(n+3)/2+m$ odd.

Set $2M = (n+1)/2+m$. Let us extend T to a partial transversal T' of L by choosing the cells $((n+1)/2+j, m+1+j)$, $j = 0, \ldots, (n-3)/2-M$, and the cells $((n+1)/2+j+1, m+1+j)$, $j = (n-1)/2-M, \ldots, (n-5)/2-m$, from $\bar{A}$: the entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all odd powers of g.

Let us extend T' to a partial transversal T'' of L by choosing the cells $((n-1)/2 - i, (n+1)/2+1+i)$, $i = 0, \ldots, (n-3)/2-m$, from $\bar{D}$: the entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all even powers of g.

The partial transversal T'' that we have constructed is a maximal partial transversal of length $2n-2m-1$. This construction yields a maximal partial transversal of length $2n-2m-1$, where $1 \leq m < n/2$.

Second construction: For n odd and $0 \leq m \leq (n-5)/2$, we will construct maximal partial transversals of L of length $2n-2(m+1)$.

Suppose that $0 \leq m \leq (n-5)/2$. Let T consist of the $(0,0), (1,1), \ldots, (m,m)$th cells of B, the $((n-1)/2, (n-3)/2), ((n+1)/2, (n-1)/2), \ldots, (n-1, n-2)$th cells of B, the $(m+1, m+1), (m+2, m+2), \ldots, ((n-3)/2, (n-3)/2)$th cells of C, and the (n,n)th cell of D. These cells contain each element of $\mathbb{Z}_{n+1}$ exactly once.

T can only be extended to a longer partial transversal by incorporating cells from the submatrix $\bar{A} = (a_{ij})$ of A, $i = m+1, \ldots, (n-3)/2$, $j =\in \{0, \ldots, m\} \cup \{(n-1)/2, \ldots, n-1\}$; and/or the submatrix $\bar{D} = (d_{ij})$ of D, $i =\in \{0, \ldots, m\} \cup \{(n-1)/2, \ldots, n-1\}$, $j \in \{m+1, \ldots, (n-5)/2\} \cup \{n-1\}$. We can extend T by choosing at most $(n-3)/2-m$ cells from $\bar{A}$ and at most $(n-3)/2-m$ cells from $\bar{D}$.

There are two cases to consider.

Case 1. $m = (n-5)/2$.

Let us extend T to a partial transversal T' of L by choosing the $((n-3)/2, 0)$th cell from $\bar{A}$ and the $(0, n-1)$th cell of $\bar{D}$.

The partial transversal T' that we have constructed is a maximal partial transversal of length $n+3$.

Case 2. $m < (n-5)/2$.

If $(n-1)/2+m$ is odd, let us extend T to a partial transversal T' of L by choosing the $(m+j, (n-1)/2+j)$th cells, $j = 1, \ldots, ((n-1)/2-m-1)/2$, and the $(m+$

$j,(n+1)/2+j)$th cells, $j=((n-1)/2-m+1)/2,\ldots,(n-3)/2-m$, from $\bar{A}$. The entries in these cells are distinct elements of $\mathbb{Z}_n$ that are odd powers of g.

If $(n-1)/2+m$ is even, let us extend T to a partial transversal T' of L by choosing the $(m+j,(n+1)/2+j)$th cells, $j=1,\ldots,((n-1)/2-m-2)/2$, and the $((m+j,(n+3)/2+j))$th cells, $j=((n-1)/2-m)/2,\ldots,(n-3)/2-m$, from $\bar{A}$. The entries in these cells are distinct elements of $\mathbb{Z}_n$ that are odd powers of g.

If m is even, let us extend T' to a partial transversal T'' of L by choosing the $(0,n-1)$th cell, and the $(n-i-1,m+i)$th cells, $i=1,\ldots,(n-5)/2-m$, of $\bar{D}$. The entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all even powers of g.

If m is odd, let us extend T' to a partial transversal T'' of L by choosing the $(0,n-1)$th cell, and the $(n-i,m+i)$th cells, $i=1,\ldots,(n-5)/2-m$, of $\bar{D}$. The entries in these cells are all distinct elements of $\mathbb{Z}_n$, and are all even powers of g.

In each case, the partial transversal T'' that we have constructed is a maximal partial transversal of length $2n-2(m+1)$. This construction yields a maximal partial transversal of length $2n-2(m+1)$ when $1\le m<n/2$.

Third construction: For n odd, we will show that L has a maximal partial transversal of length $2n$.

Let T be a near transversal of A with missing cell $(0,n-1)$: the existence of such a near transversal is guaranteed by Theorem 2.1. As the entry of this missing cell is g^{n-1}, by Lemma 2.1, the missing symbol of T is g^{n-1}.

Let us extend T to T' by adding the cell $(0,n-1)$ of B, the cell $(n,n-1)$ of C, and the cells $(0,0),\ldots,(n-2,n-2)$ of D. The entries of these added cells are precisely the elements of $\mathbb{Z}_{n+1}$ and so T' is a partial transversal of L of length $2n$. The only cell of L that is not in a row or column of T' is the $(n-1,n)$th cell of D. The entry of this cell is g, which is not the missing symbol of T. It follows that T' is a maximal partial transversal of L of length $2n$. □

5 A related class of latin squares

In this section we will construct a class of latin squares of order $2n+1$, n odd, $n\ge 5$, that have maximal partial transversals of all possible lengths. These latin squares will be obtained from bicyclic latin squares by switching an intercalate. If n is odd and $L=\begin{pmatrix} A & B\\ C & D\end{pmatrix}$ is a bicyclic latin square of order $2n+1$, then a bicyclic latin square $L'=\begin{pmatrix} A & B\\ C & D'\end{pmatrix}$ obtained from L by switching an intercalate in D will be called a *switched square*. Note that no intercalate in D can contain a cell on the main diagonal of D. One class of switched squares will be of particular interest to us: if $L'=\begin{pmatrix} A & B\\ C & D'\end{pmatrix}$ is the latin square obtained from $L_{M_n,g^1,\ldots,g^n}$ by switching the intercalate in D, with cells $(0,1)$, $(0,(n+3)/2)$, $((n+1)/2,1)$, and

$((n+1)/2, (n+3)/2)$, we will denote L' by $L'_{M_n,g^1,\ldots,g^n}$. We proved in Theorem 4.1 that $L_{M_n,g^1,\ldots,g^n}$ has no maximal partial transversal of length $n+1$ if n is odd, $n \neq 3$. This is not the case for switched squares.

Lemma 5.1 *If n is odd, $n \geq 5$, and $L' = \begin{pmatrix} A & B \\ C & D' \end{pmatrix}$ is a switched square of order $2n+1$, then L' has a maximal partial transversal of length $n+1$.*

Proof: Recall that, if $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a bicyclic latin square of order $2n+1$, n odd, then D is obtained from a circulant latin square E, isotopic to M_{n+1}, by replacing the elements on the main diagonal. By Theorem 2.3, if any intercalate of E is switched, then the resulting latin square has a transversal. This yields a "transversal" of D' that is a maximal partial transversal of L of length $n+1$. □

Any switched square of order $2n+1$, n odd, $n \neq 1$, has a maximal partial transversal of length $n+2$.

Lemma 5.2 *If n is odd, $n \neq 1$, and $L' = \begin{pmatrix} A & B \\ C & D' \end{pmatrix}$ is a switched square of order $2n+1$, then L' has a maximal partial transversal of length $n+2$.*

Proof: As in the proof of Lemma 5.1, D' has a "transversal" T. Form a partial transversal T' of L' by removing the cell of T on the main diagonal of D'; adding a cell of B in the same column as this removed cell; and adding a cell of C, containing a different entry than the added cell in B, in the same row as this removed cell. It is easy to see that T' is a maximal partial transversal of L' of length $n+2$. □

Any switched square of order $2n+1$, n odd, $n \geq 5$, has a transversal.

Lemma 5.3 *If n is odd, $n \geq 5$, and L' is a switched square of order $2n+1$, then L' has a transversal.*

Proof: The proof is the same as for Lemma 4.2. □

Our class of switched squares have maximal partial transversals of all possible lengths.

Theorem 5.1 *If n is odd, $n \geq 5$, then $L'_{M_n,g^1,\ldots,g^n}$ has maximal partial transversals of all possible lengths.*

Proof: Suppose that $L' = \begin{pmatrix} A & B \\ C & D' \end{pmatrix} = L'_{M_n,g^1,\ldots,g^n}$ and n is odd, $n \geq 5$. By Lemma 5.1, L' has a maximal partial transversal of length $n+1$; by Lemma 5.2, L' has a maximal partial transversal of length $n+2$; and by Lemma 5.3, L' has a maximal partial transversal of length $2n+1$.

Let $L = \begin{pmatrix} A & B \\ C & D' \end{pmatrix} = L_{M_n,g^1,\ldots,g^n}$ and let I be the intercalate of D that is switched to yield L'. Then the cells of I are the cells $(0,1)$, $(0,(n+3)/2)$, $((n+1)/2,1)$, and $((n+1)/2,(n+3)/2)$ of D; and the symbols of I are g and $g^{(n+3)/2}$. We will show that most of the maximal partial transversals of L of the remaining lengths, constructed in Theorem 4.2, are also maximal partial transversals of L'.

The proof of Theorem 4.2 uses three constructions. Let T'' be a maximal partial transversal of L obtained from the first construction. As T'' contains the cells $(0,0)$ and $((n+1)/2,(n-1)/2)$ of C, no cell of I can be a cell of T'' and each cell of I is in the same row as a cell of T''. Hence, T'' is also a maximal partial transversal of L'.

As part of the second construction of maximal partial transversals of L in Theorem 4.2, a maximal partial transversal of length eight was obtained for $L_{M_5,g^1,g^2,g^3,g^4,g^5}$. A maximal partial transversal of length eight for $L'_{M_5,g^1,g^2,g^3,g^4,g^5}$ is shown in Figure 6: the entries of the maximal partial transversal are shown as $\mathbf{i}$ or $\mathbf{g^i}$. For $n \geq 7$, let T'' be a maximal partial transversal of L obtained from the second construction in the proof of Theorem 4.2. Thus, T'' is of length $2n-2(m+1)$, where $0 \leq m \leq (n-5)/2$. No cell of I is also a cell of T''. If $m \geq 1$, then the cells $((n+5)/2,(n+3)/2)$ and $(1,1)$ of B are also cells of T'' and so each cell of I is in the same column as a cell of T''. Hence, T'' is also a maximal partial transversal of L'. If $m = 0$, then the cell $((n+5)/2,(n+3)/2)$ of B and the cell $(n-2,1)$ of D are also cells of T'' and so each cell of I is in the same column as a cell of T''. Hence, T'' is also a maximal partial transversal of L'.

Let T' be a maximal partial transversal of L obtained from the third construction in the proof of Theorem 4.2. No cell of I is also a cell of T'. As the cells $(0,0)$ and $(1,1)$ of D are cells of T', the only cell of I that might not be in the same row or column as a cell of T' is $((n+1)/2,(n+3)/2)$. In L', the entry of this cell is $g^{(n+3)/2}$, which, if $n \neq 5$, is also the entry of a cell of T' in A. Hence, if $n \neq 5$, T' is also a maximal partial transversal of L'. A maximal partial transversal of L' of length 10 is shown in Figure 7: the entries of the maximal partial transversal are shown as $\mathbf{i}$ or $\mathbf{g^i}$.

It follows that L' has maximal partial transversals of all possible lengths. □

We are now in a position to determine all odd orders for which there exists a latin square that has maximal partial transversals of all possible lengths.

Corollary 5.1 *If n is odd, $n \neq 3$, then there exists a latin square of order n that has maximal partial transversals of all possible lengths.*

Proof: This follows from Theorems 3.1 and 5.1. □

6 Final remarks

In this paper, we have studied the spectra of lengths of maximal partial transversals of bicyclic latin squares and switched squares. More generally, we might ask the following question.

$$\left(\begin{array}{ccccc|cccccc}
g^0 & g^1 & g^2 & g^3 & g^4 & \mathbf{0} & 1 & 2 & 3 & 4 & 5 \\
g^1 & g^2 & \mathbf{g^3} & g^4 & g^0 & 1 & 2 & 3 & 4 & 5 & 0 \\
g^2 & g^3 & g^4 & g^0 & g^1 & 2 & \mathbf{3} & 4 & 5 & 0 & 1 \\
g^3 & g^4 & g^0 & g^1 & g^2 & 3 & 4 & \mathbf{5} & 0 & 1 & 2 \\
g^4 & g^0 & g^1 & g^2 & g^3 & 4 & 5 & 0 & \mathbf{1} & 2 & 3 \\
\hline
0 & 1 & 2 & 3 & 4 & 5 & g^4 & g^2 & g^3 & g^1 & g^0 \\
1 & \mathbf{2} & 3 & 4 & 5 & g^0 & 0 & g^1 & g^2 & g^3 & g^4 \\
2 & 3 & 4 & 5 & 0 & g^4 & g^0 & 1 & g^1 & \mathbf{g^2} & g^3 \\
3 & 4 & 5 & 0 & 1 & g^3 & g^1 & g^0 & 2 & g^4 & g^2 \\
4 & 5 & 0 & 1 & 2 & g^2 & g^3 & g^4 & g^0 & 3 & g^1 \\
5 & 0 & 1 & 2 & 3 & g^1 & g^2 & g^3 & g^4 & g^0 & \mathbf{4}
\end{array}\right)$$

Figure 6: A maximal partial transversal of $L'_{M_5,g^1,g^2,g^3,g^4,g^5}$ of length 8.

$$\left(\begin{array}{ccccc|cccccc}
g^0 & g^1 & g^2 & g^3 & g^4 & 0 & \mathbf{1} & 2 & 3 & 4 & 5 \\
\mathbf{g^1} & g^2 & g^3 & g^4 & g^0 & 1 & 2 & 3 & 4 & 5 & 0 \\
g^2 & \mathbf{g^3} & g^4 & g^0 & g^1 & 2 & 3 & 4 & 5 & 0 & 1 \\
g^3 & g^4 & \mathbf{g^0} & g^1 & g^2 & 3 & 4 & 5 & 0 & 1 & 2 \\
g^4 & g^0 & g^1 & \mathbf{g^2} & g^3 & 4 & 5 & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 2 & 3 & 4 & \mathbf{5} & g^4 & g^2 & g^3 & g^1 & g^0 \\
1 & 2 & 3 & 4 & 5 & g^0 & 0 & g^1 & g^2 & g^3 & g^4 \\
2 & 3 & 4 & 5 & \mathbf{0} & g^4 & g^0 & 1 & g^1 & g^2 & g^3 \\
3 & 4 & 5 & 0 & 1 & g^3 & g^1 & g^0 & \mathbf{2} & g^4 & g^2 \\
4 & 5 & 0 & 1 & 2 & g^2 & g^3 & g^4 & g^0 & \mathbf{3} & g^1 \\
5 & 0 & 1 & 2 & 3 & g^1 & g^2 & g^3 & g^4 & g^0 & \mathbf{4}
\end{array}\right)$$

Figure 7: A maximal partial transversal of $L'_{M_5,g^1,g^2,g^3,g^4,g^5}$ of length 10.

Question. Given a latin square L, what is the spectrum of lengths of maximal partial transversals of L?

We may also ask the following:

Question. Given a positive integer n, what are the possible spectra for lengths of maximal partial transversals of latin squares of order n?

An alternative way to describe latin squares is as a set of ordered triples. For convenience, let us assume that the symbol set of a latin square L is the same as the indexing set for its rows and columns. A latin square L of order n can then be regarded as a set of n^2 ordered triples in which each symbol of L appears exactly once in each position: think of the first position in each triple as representing a row, the second a column, and the third an entry. A *conjugate* of L is any latin square obtained from the set of ordered triples representing L by permuting the positions. Two latin squares L and L' of the same order are *main class isotopic* if L is isotopic to a conjugate of L'. Main class isotopy is an equivalence relation and the equivalence classes are called *main classes*. The spectrum of lengths of maximal partial transversals of a latin square is a main class invariant.

For small n, representatives of the main classes of latin squares of order n can be found in Section III.1.3 of [2]. For $n \leq 3$, the representatives of the main classes are M_1, M_2, and M_3. The spectrum of lengths of maximal partial transversals of M_1 is trivially $\{1\}$, while the spectra of lengths of maximal partial transversals of M_2 and M_3 follow from Theorem 2.1. For $n = 4$, the representatives of the main classes are M_4 and $M_{2\times 2}$, the Cayley table of the elementary abelian group of order 4. By Theorem 2.1, M_4 has no transversal, but does possess a near transversal that is maximal, and by results in [6], $M_{2\times 2}$ has no near transversal that is maximal, but does possess a transversal. The reader can verify that neither of these latin squares has a maximal partial transversal of length two. For $n = 5$, representatives of the main classes are M_5 and the bicyclic latin square L_{M_2,g,g^2}. In Theorem 3.1, we showed that L_{M_2,g,g^2} possesses maximal partial transversals of all possible lengths; and in Theorem 2.1, we showed that M_5 has a transversal, but has no near transversal that is maximal. A maximal partial transversal of M_5 of length 3 is shown in Figure 8: the entries in this maximal partial transversal are shown in bold.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 0 \\ 2 & 3 & 4 & \mathbf{0} & 1 \\ 3 & 4 & 0 & 1 & \mathbf{2} \\ 4 & 0 & \mathbf{1} & 2 & 3 \end{pmatrix}$$

Figure 8: A maximal partial transversal of M_5 of length 3.

In Theorems 3.1, 4.2, and 5.1, we gave a partial answer to the following question.

Question. For which n does there exist a latin square of order n with maximal partial transversals of all possible lengths?

We answered this question when n is odd. It remains to answer this question when n is even. Closely related to this question: an implication of Theorem 13 in [1] is that almost all latin squares do not have maximal partial transversals of all possible lengths.

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