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On Sylvester colorings of cubic graphs*

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Abstract

If G and H are two cubic graphs, then an H-coloring of G is a proper edge-coloring f with edges of H, such that for each vertex x of G, there is a vertex y of H with $f(\partial_G(x)) = \partial_H(y)$. If G admits an H-coloring, then we will write $H \prec G$. The Petersen coloring conjecture of Jaeger states that for any bridgeless cubic graph G, one has: $P \prec G$. The second author has recently introduced the Sylvester coloring conjecture, which states that for any cubic graph G, one has: $S \prec G$. Here S is the Sylvester graph on 10 vertices. In this paper we prove the analogue of the Sylvester coloring conjecture for cubic pseudo-graphs. Moreover, we show that if G is any connected simple cubic graph G with $G \prec P$, then $G = P$. This implies that the Petersen graph does not admit an S_{16}-coloring, where S_{16} is the smallest connected simple cubic graph without a perfect matching. S_{16} has 16 vertices. Finally, we obtain two results towards the Sylvester coloring conjecture. The first result states that any cubic graph G has a coloring with edges of the Sylvester graph S such that at least $\frac{4}{5}$ of the vertices of G meet the conditions of the Sylvester coloring conjecture. The second result states that any claw-free cubic graph admits an S-coloring. This result is an application of our result on cubic pseudo-graphs.

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1 Introduction

In this paper we consider finite pseudo-graphs, which may contain loops and parallel edges. The edges of pseudo-graphs are not directed, and as usual, a loop contributes 2 to the degree of a vertex. A graph is a pseudo-graph which may contain parallel edges but no loops. A graph is simple if it does not contain any parallel edges.

Within this paper, we assume that graphs, pseudo-graphs and simple graphs are considered up to isomorphism. This implies that the equality $G = G'$ means that G and G' are isomorphic.

For a graph G, let $V(G)$ and $E(G)$ be the set of vertices and edges of G, respectively. Moreover, let $\partial_G(x)$ be the set of the edges of G which are incident to the vertex x of G. A matching of G is a set of edges of G such that any two of them do not share a vertex. A matching of G is perfect if it contains $\frac{|V(G)|}{2}$ edges. For a positive integer k, a k-factor of G is a spanning k-regular subgraph of G. Observe that the edge-set of a 1-factor of G is a perfect matching of G. Moreover, if G is cubic and F is a 1-factor of G, then the set $E(G)\backslash E(F)$ is an edge-set of a 2-factor of G. This 2-factor is said to be complementary to F. Conversely, if $\bar{F}$ is a 2-factor of a cubic graph G, then the set $E(G)\backslash E(\bar{F})$ is an edge-set of a 1-factor of G or is a perfect matching of G. This 1-factor is said to be complementary to $\bar{F}$.

If P is a path of a graph G, then the length of P is the number of edges of G lying on P. For a connected graph G and two of its vertices u and v, the distance between u and v is the length of the shortest path connecting these vertices. The distance between edges e and f of G, denoted by $\rho_G(e, f)$, is the shortest distance among end-vertices of e and f. Clearly, adjacent edges are at distance zero.

A subgraph H of G is even if every vertex of H has even degree in H. A block of G is a maximal 2-connected subgraph of G. An end-block is a block of G containing at most one vertex that is a cut-vertex of G. If G is a cubic graph containing cut-vertices, then any end-block B of G is adjacent to a unique bridge e. We will refer to e as a bridge corresponding to B. Moreover, if $e = (u, v)$ and $u \in V(B)$, $v \notin V(B)$, then v is called the root of B.

If K is a triangle in a cubic pseudo-graph G, then let G/K be the cubic pseudo-graph obtained from G by contracting K. If G is a cubic graph and K does not contain a parallel edge, then we will say that K is contractible in G. Observe that if K is not contractible, then two vertices of K are joined with two parallel edges, and the third vertex is incident to a bridge (see the end-blocks of the graph from Figure 2). If K is a contractible triangle and e is an edge of K, then let f be the edge of G that is incident to a vertex of K and is not adjacent to e. The edges e and f will be called opposite edges.

If T is a set, H is a subgraph of a graph G, and $f : E(G) \to T$, then a mapping $g : E(H) \to T$, such that $g(e) = f(e)$ for any $e \in E(H)$, is called the restriction of f to H.

Let G and H be two cubic graphs, and let $f : E(G) \to E(H)$. Define:

$$V(f) = \{x \in V(G) : \exists y \in V(H)\ f(\partial_G(x)) = \partial_H(y)\}.$$

An H-coloring of G is a mapping $f : E(G) \to E(H)$ such that $V(f) = V(G)$. If G admits an H-coloring, then we will write $H \prec G$. It can be easily seen that if $H \prec G$ and $K \prec H$, then $K \prec G$. In other words, $\prec$ is a transitive relation defined on the set of cubic graphs.

If $H \prec G$ and f is an H-coloring of G, then for any adjacent edges e, e' of G, the edges $f(e)$, $f(e')$ of H are adjacent. Moreover, if the graph H contains no triangle, then the converse is also true; that is, if a mapping $f : E(G) \to E(H)$ has a property that for any two adjacent edges e and e' of G, the edges $f(e)$ and $f(e')$ of H are adjacent, then f is an H-coloring of G.

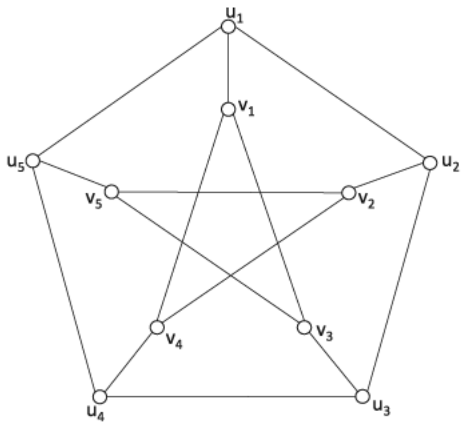

Figure 1: The Petersen graph

Let P be the well-known Petersen graph (Figure 1) and let S be the graph from Figure 2. The graph S is called the Sylvester graph [11]. We would like to point out that usually the name "Sylvester graph" is used for a particular strongly regular graph on 36 vertices, and this graph should not be confused with S, which has 10 vertices.

The Petersen coloring conjecture of Jaeger states:

Conjecture 1.1 *(Jaeger, 1988 [6]) For each bridgeless cubic graph G, one has $P \prec G$.*

The conjecture is difficult to prove, since it can be seen that it implies the following two classical conjectures [6]:

Conjecture 1.2 *(Berge-Fulkerson, 1972 [4, 12]) Any bridgeless cubic graph G contains six (not necessarily distinct) perfect matchings $F_1, \ldots, F_6$ such that any edge of G belongs to exactly two of them.*

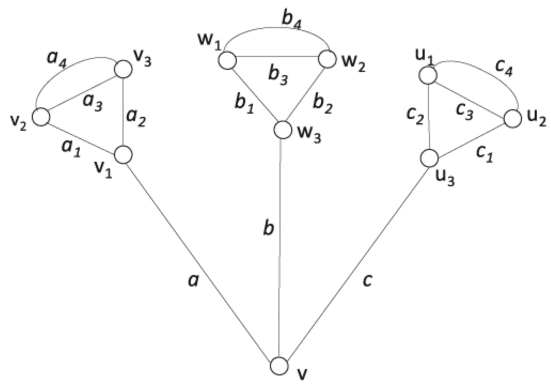

Figure 2: The Sylvester graph

Conjecture 1.3 *((5, 2)-cycle-cover conjecture, [1, 10]) Any bridgeless graph G (not necessarily cubic) contains five even subgraphs such that any edge of G belongs to exactly two of them.*

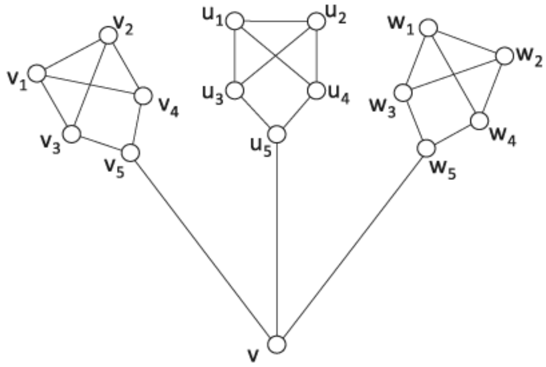

Figure 3: The graph S_{16}

Related to the Jaeger conjecture, the following conjecture was introduced in [8]:

Conjecture 1.4 *(Mkrtchyan, 2012 [8]) For each cubic graph G, one has $S \prec G$.*

In direct analogy with Conjecture 1.1, we call Conjecture 1.4 the Sylvester coloring conjecture.

In this paper, we consider the analogues of this conjecture for simple cubic graphs and cubic pseudo-graphs. Let S_{16} be the simple graph from Figure 3, and let S_4 be the pseudo-graph from Figure 4.

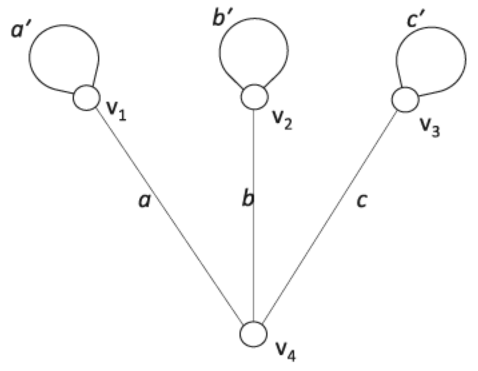

Figure 4: The pseudo-graph S_4

In this paper we show that not all simple cubic graphs admit an S_{16}-coloring. On the positive side, we prove that all cubic pseudo-graphs have an S_4-coloring. We complete the paper by proving two results towards Conjecture 1.4. The first one states that for any cubic graph G there is a mapping $f : E(G) \to E(S)$, such that $|V(f)| \geq \frac{4}{5} \cdot |V(G)|$. The second one states that any claw-free cubic graph admits an S-coloring. The latter result is derived as a consequence of the S_4-colorability of cubic pseudo-graphs.

Terms and concepts that we do not define in this paper can be found in [5, 15].

2 Some Auxiliary Statements

In this section we present some auxiliary statements that will be used in Section 3.

Theorem 2.1 *(Petersen, 1891 [7]) Let G be a cubic graph containing at most two bridges. Then G has a 1-factor.*

Lemma 2.1 *Let G be a bridgeless graph with $d(v) \in \{2, 3\}$ for any $v \in V(G)$. Assume that all vertices of G are of degree 3 except one. Then G has a 2-factor.*

Proof: Take two copies G_1 and G_2 of G, and consider a graph H obtained from them by joining degree 2 vertices by an edge e. Observe that H is a cubic graph containing only one bridge, which is the edge e. By Theorem 2.1, H contains a 1-factor F. Since e is a bridge of H, we have $e \in F$. Consider the complementary 2-factor $\bar{F}$ of F. Clearly, the edges of the set $E(\bar{F}) \cap E(G_1)$ form a 2-factor of G_1. $\square$

Proposition 2.1 *Let G be a simple cubic graph without a perfect matching and $|V(G)| \leq 16$. Then $G = S_{16}$.*

Lemma 2.2 *Suppose that G and H are two cubic graphs with $H \prec G$, and let f be an H-coloring of G. Then:*

(a) if M is any matching of H, then $f^{-1}(M)$ is a matching of G;

(b) $\chi'(G) \leq \chi'(H)$, where $\chi'(G)$ is the chromatic index of G;

(c) if M is a perfect matching of H, then $f^{-1}(M)$ is a perfect matching of G;

(d) for every even subgraph C of H, $f^{-1}(C)$ is an even subgraph of G;

(e) for every bridge e of G, the edge $f(e)$ is a bridge of H.

Proof: Statements (a), (b) and (c) are proved in Lemma 2.3 of [8]. For the proof of (d), let C be an even subgraph of H. We shall show that any vertex v of G has even degree in $f^{-1}(C)$. Since H is cubic, C is a disjoint union of cycles. Assume that in f the three edges incident to v are colored with three edges incident to a vertex w of H. Now if w is isolated in C, then clearly v is isolated in $f^{-1}(C)$. On the other hand, if w has degree 2 in C then v is of degree two in $f^{-1}(C)$. Thus v always has even degree in $f^{-1}(C)$, or $f^{-1}(C)$ is an even subgraph of G.

Finally, for the proof of (e) let e be a bridge of G and let $(X, V(G)\backslash X)$ be a partition of $V(G)$ such that $\partial_G(X) = \{e\}$. Here $\partial_G(X)$ denotes the set of edges of G which connect a vertex of X to a vertex of $V(G)\backslash X$. Assume that the edge $f(e)$ is not a bridge in H. Then there is a cycle C in H containing the edge $f(e)$. By (d), $f^{-1}(C)$ is an even subgraph of G that has non-empty intersection with $\partial_G(X)$. Since the intersection of an even subgraph with $\partial_G(X)$ always contains an even number of edges, $\partial_G(X)$ contains at least two edges, which contradicts our assumption. □

Proposition 2.2 *Let G be a connected non-3-edge-colorable simple cubic graph such that $|V(G)| \leq 10$. Then $G = P$ or $G = S'$ (see Figure 5).*

We will also need some results that were obtained in [3, 13, 14]. Let G be a graph of maximum degree at most 3, and assume that c is a proper coloring of some edges of G with colors 1, 2 and 3. The edges of G that have not received colors in c are called uncolored edges. Now assume that c is chosen so that the number of uncolored edges is smallest. It is known that, for such a choice of c, uncolored edges must form a matching [3, 13, 14].

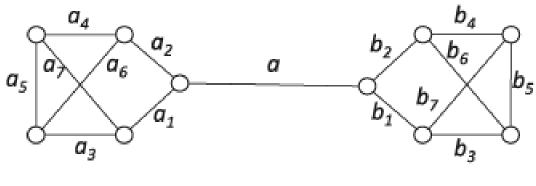

Figure 5: The graph S' on 10 vertices

Take an arbitrary uncolored edge $e = (u, v)$. Since c is chosen so that the number of uncolored edges is smallest, we see that e is incident to edges that have colors 1, 2, and 3. Since G is of maximum degree at most 3, it follows that there are colors $\alpha, \beta \in \{1, 2, 3\}$ such that no edge incident to u and v is colored with β and α, respectively. Consider a maximal $\alpha - \beta$-alternating path P_e starting from u. As shown in [3, 13, 14], this path must terminate in v, and hence it is of even length. This means that P_e together with the edge e forms an odd cycle C_e. The cycle C_e is called the cycle corresponding to the uncolored edge e. It is known that

Lemma 2.3 *([3, 13, 14]) If G is a graph of maximum degree at most 3, and e, e' ($e \neq e'$) are two uncolored edges of G, then $V(C_e) \cap V(C_{e'}) = \emptyset$.*

Finally, we will need some results on claw-free cubic graphs. Recall that a graph G is claw-free if it does not contain 4 vertices such that the subgraph of G induced on these vertices is isomorphic to $K_{1,3}$. In [2], arbitrary claw-free graphs are characterized. In [9], Oum has characterized simple, claw-free bridgeless cubic graphs. Following the approach of Oum, below we will characterize claw-free (not necessarily bridgeless) cubic graphs.

We need some definitions. A 2-cycle is a cycle of length 2 (2 parallel edges). In a claw-free cubic graph G any vertex belongs to 1, 2, or 3 triangles or 1 or 3 2-cycles. If a vertex v belongs to 3 triangles of G, then the component of G containing v is isomorphic to K_4 (Figure 6). An induced subgraph of G that is isomorphic to $K_4 - e$ is called a diamond [9]. It can be easily checked that in a claw-free cubic graph no two diamonds intersect. If v belongs to three 2-cycles of G, then the component of G containing v is isomorphic to K_2^3 (Figure 6).

Figure 6: The graphs K_4 and K_2^3.

A string of diamonds or 2-cycles of G is a maximal sequence $F_1, ..., F_k$ of diamonds or 2-cycles, in which F_i has a vertex adjacent to a vertex of F_{i+1}, $1 \leq i \leq k - 1$ (Figure 7). A string of diamonds or 2-cycles has exactly 2 vertices of degree 2, which are called the head and the tail of the string. A string J of a claw-free cubic graph G is trivial, if J is comprised of 1 2-cycle, and G contains a vertex that is adjacent to both the head and tail of J. Replacing an edge $e = (u, v)$ with a string of diamonds or 2-cycles with the head x and the tail y is to remove e and add edges (u, x) and (v, y).

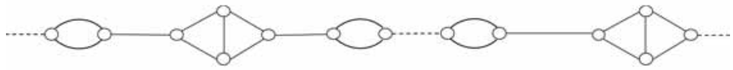

Figure 7: A string of diamonds or 2-cycles.

If G is a connected claw-free cubic graph such that each vertex lies in a diamond or a 2-cycle, then G is called a ring of diamonds or 2-cycles. It can easily be checked that each vertex of a ring of diamonds or 2-cycles lies in exactly one diamond or a 2-cycle. As in [9] we require that a ring of diamonds or 2-cycles contains at least two diamonds or 2-cycles.

We are now ready to present the characterization of claw-free cubic graphs.

Proposition 2.3 *G is a connected claw-free cubic graph, if and only if*

(1) G is isomorphic to K_4 or K_2^3, or

(2) G is a ring of diamonds or 2-cycles, or

(3) there is a cubic pseudo-graph H, such that G can be obtained from H by replacing some edges of H with strings of diamonds or 2-cycles, and by replacing any vertex of H with a triangle.

Proof: We omit the proof of the proposition, since it can be done in the same way as the proof of Proposition 1 of [9]. □

3 The Main Results

In this section we obtain the main results of the paper. Our first theorem shows that the statement analogous to the Sylvester coloring conjecture holds for cubic pseudo-graphs. Actually, we will show that all cubic pseudo-graphs G admit an S_4-coloring. We use the labels of edges of S_4 from Figure 4.

In the theorem, we need the concept of a list. A list is a collection of elements, where the order of elements is not important and elements may appear more than once. Two lists are equal if they consist of the same elements, with the frequency of appearance of each element in the lists being the same. In contrast with sets, which we denote by $\{\dots\}$, lists will be denoted by $\langle\dots\rangle$. According to our definitions, we have for example $\langle 1,1,2\rangle = \langle 1,2,1\rangle$ and $\langle 1,2,2\rangle \neq \langle 2,1,1\rangle$.

Theorem 3.1 *Let G be a cubic pseudo-graph, and let $\partial(S_4)$ be the following set of lists:*

$$\partial(S_4) = \{\langle a,b,c\rangle, \langle a,a',a'\rangle, \langle b,b',b'\rangle, \langle c,c',c'\rangle\}.$$

Then there is a mapping $f : E(G) \to E(S_4)$ such that *(a) if a vertex v of G is incident to a loop e' and a bridge e, then the list $\langle f(e), f(e'), f(e')\rangle$ is one of the three lists of $\partial(S_4)\backslash\{\langle a,b,c\rangle\}$,*

(b) if a vertex v is incident to 3 edges e, e' and e'', then the list $\langle f(e), f(e'), f(e'')\rangle$ is one of the four lists of $\partial(S_4)$.

Proof: It is clear that we can prove the theorem only for connected cubic pseudographs G. First of all, we prove the theorem when G is a connected graph. We proceed by induction on the number of bridges of G.

If there are at most two bridges in G, then due to Theorem 2.1, the graph G has a 1-factor. Color the edges of the 1-factor by a, and the edges of the complementary 2-factor by a'. One can verify that the described coloring meets condition (b) of the theorem.

Now assume that the statement is true for graphs with at most k bridges; we prove it for those with $k+1$ bridges ($k+1 \geq 3$). We will consider the following cases:

Case 1: For any two end-blocks B and B' of G, the bridges e and e', corresponding to them, are adjacent.

In this case, G consists of three end-blocks B_1, B_2, B_3, such that the bridges e_1, e_2, e_3 corresponding to them are incident to the same cut-vertex v. We obtain an S_4-coloring of G as follows: let $\bar{F}_1$, $\bar{F}_2$, $\bar{F}_3$ be 2-factors in B_1, B_2, B_3, respectively (see Lemma 2.1). Color the edges of $\bar{F}_1$ with a' and the edges of $(E(B_1)\backslash\bar{F}_1)\cup\{e_1\}$ with a, the edges of $\bar{F}_2$ with b' and the edges of $(E(B_2)\backslash\bar{F}_2)\cup\{e_2\}$ with b, the edges of $\bar{F}_3$ with c' and the edges of $(E(B_3)\backslash\bar{F}_3)\cup\{e_3\}$ with c. One can check that the described coloring meets condition (b) of the theorem.

Case 2: There are two end-blocks B and B' of G, such that the bridges e and e' corresponding to them, are not adjacent.
Assume that $e=(u,u')$, $e'=(v,v')$ and $u'\in V(B)$, $v'\in V(B')$. Consider a cubic graph H obtained from G as follows:

$$H = [G\backslash(V(B)\cup V(B'))]\cup\{(u,v)\}.$$

If initially G had an edge (u,v), then in H we will just get two parallel edges between u and v.
Observe that H is a cubic graph containing at most k bridges. By induction hypothesis, H admits an S_4-coloring g satisfying condition (b) of the theorem. Now we obtain an S_4-coloring for the graph G using the coloring g of H. For that purpose we consider 2 cases.
Subcase 2.1: $g((u,v))\in\{a,b,c\}$.
Without loss of generality, we can assume that $g((u,v))=a$. Other cases can be come up in a similar way. By Lemma 2.1, B and B' contain 2-factors $\bar{F}$ and $\bar{F}'$, respectively. Consider the restriction of g to G. We extend it to an S_4-coloring of G as follows: color the edges of $\bar{F}\cup\bar{F}'$ with a' and the edges of $(E(B)\backslash\bar{F})\cup(E(B')\backslash\bar{F}')\cup\{e,e'\}$ with a. Observe that the described coloring meets the condition (b) of the theorem.

Subcase 2.2: $g((u,v)) \in \{a', b', c'\}$.

Without loss of generality, we can assume that $g((u,v)) = a'$. Other cases can be come up in a similar way. Observe that the edges of H colored with a' (the edges of $f^{-1}(a')$) form vertex disjoint cycles in H. Consider the cycle C_{uv} of G containing the edge (u,v), and let $P_{uv} = C_{uv} - (u,v)$ (Figure 8). By Lemma 2.1, B and B' contain 2-factors $\bar{F}$ and $\bar{F}'$, respectively.

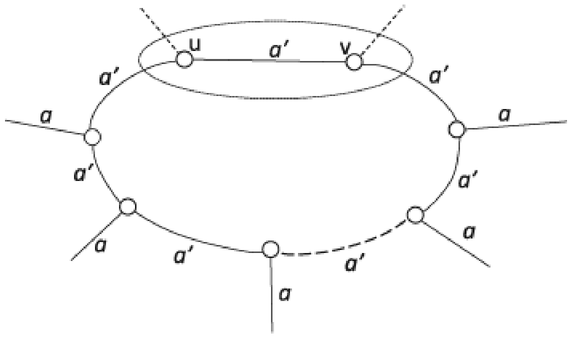

Figure 8: The cycle C_{uv} and the path P_{uv} in the graph H

Define edges d and d' of S_4 as follows: if P_{uv} is of odd length, then $d = b$, $d' = b'$, and $d = c$, $d' = c'$, otherwise. Consider the restriction of g to G. We extend it to an S_4-coloring of G as follows: color the edges of $\bar{F}$ with b' and the edges of $(E(B)\backslash\bar{F}) \cup \{e\}$ with b, re-color the edges of P_{uv} by coloring them with colors b and c alternatively beginning from c, color the edges of $\bar{F}'$ with d' and the edges of $(E(B')\backslash\bar{F}') \cup \{e'\}$ with d (Figure 9). One can check that the described coloring meets condition (b) of the theorem.

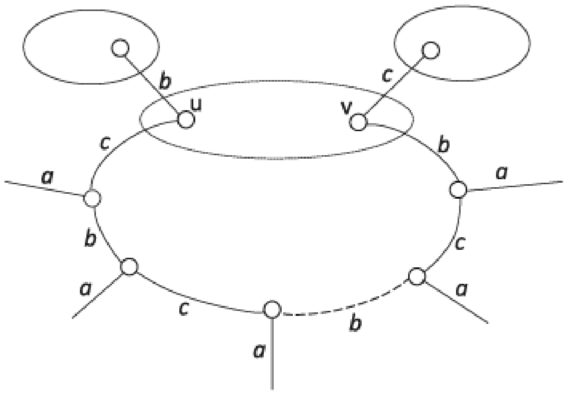

Figure 9: The path P_{uv} in the graph G

Finally, we consider the case when G contains loops. Let H be a graph obtained from G by replacing all vertices of G incident to loops by triangles (Figure 10).

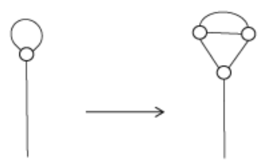

Figure 10: Modification of loops of G

Now H has an S_4-coloring f satisfying condition (b) of the theorem. We claim that any bridge of H has a color from $\{a, b, c\}$.

For the sake of contradiction, assume that there is a bridge e of H such that $f(e) = a'$ (the cases $f(e) = b'$ and $f(e) = c'$ can arise in a similar way). Observe that the edges of H colored with a' (the edges of $f^{-1}(a')$) form vertex disjoint cycles in H. Consider the cycle C containing e. Since e is a bridge, C intersects this cut in one edge, which is a contradiction to the fact that cycles intersect edge-cuts in an even number of edges.

Thus all bridges of H have colors from $\{a, b, c\}$. In particular, any bridge e of H which is adjacent to a loop of G has a color $f(e) = d$, where $d \in \{a, b, c\}$. Color the loop e' of G that is adjacent to e with d', where $d' = a'$, $d' = b'$ or $d' = c'$, if $d = a$, $d = b$ or $d = c$, respectively. Observe that the described coloring meets conditions (a) and (b) of the theorem. □

Conjecture 1.4 states that all cubic graphs admit an S-coloring. On the other hand, in the previous theorem we have shown that all cubic pseudo-graphs have an S_4-coloring. One may wonder whether there is a statement analogous to these in the class of simple cubic graphs? More precisely, is there a connected simple cubic graph H such that all simple cubic graphs admit an H-coloring? A natural candidate for H is the graph S_{16}. Next we prove a theorem that justifies our choice of S_{16}. On an intuitive level it states that the only way of coloring the graph S_{16} with some connected simple cubic graph H is to take $H = S_{16}$. This result is analogous to the following theorem proved in [8].

Theorem 3.2 *(Mkrtchyan, [8]) Let G be a connected cubic graph with $G \prec S$. Then $G = S$.*

This is the precise formulation of our second result.

Theorem 3.3 *Let G be a connected simple cubic graph with $G \prec S_{16}$. Then $G = S_{16}$.*

Proof: As $G \prec S_{16}$ and S_{16} has no a perfect matching, then due to (c) of Lemma 2.2, the graph G also has no a perfect matching.

Let f be a G-coloring of S_{16}. If $e \in E(G)$, then we will say that e is used (with respect to f), if $f^{-1}(e) \neq \emptyset$. First of all, let us show that if an edge e of G is used, then any edge adjacent to e is also used.

So let $e = (u, v)$ be a used edge of G. For the sake of contradiction, assume that v is incident to an edge $z \in E(G)$ that is not used. Assume that $\partial_G(u) = \{a, b, e\}$ (Figure 11).

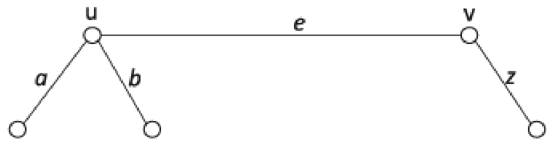

Figure 11: The edge e in the graph G

We will use the labels of edges of S_{16} from Figure 3. The following cases are possible:

Case 1: The edge e colors an edge that is not a bridge in S_{16}. So it is used in end-blocks of S_{16}. Due to the symmetry of S_{16}, there will be the following subcases:

Subcase 1.1: $f((v_1, v_2)) = e$.
Since z is not a used edge, we can assume that $f((v_1, v_3)) = a$ and $f((v_1, v_4)) = b$. Since adjacent edges receive different colors and z is not a used edge, we have $f((v_2, v_3)) = b$ and $f((v_2, v_4)) = a$. This implies that

$$f((v_3, v_5)) = f((v_4, v_5)) = e,$$

which contradicts the fact that adjacent edges receive different colors.

Subcase 1.2: $f((v_1, v_3)) = e$.
Since z is not a used edge, we can assume that $f((v_1, v_2)) = a$ and $f((v_1, v_4)) = b$. As adjacent edges receive different colors and z is not a used edge, we have that $f((v_2, v_3)) = b$ and $f((v_3, v_5)) = a$. But then $f((v_2, v_4)) = e$, which implies that $f((v_4, v_5)) = a$. This contradicts the fact that adjacent edges receive different colors.

Subcase 1.3: $f((v_3, v_5)) = e$.
Since z is not a used edge, we can assume that $f((v_1, v_3)) = a$ and $f((v_2, v_3)) = b$. Consider the edge (v_1, v_2). Observe that its color is either e, or else there is an edge h of G such that a, b and h form a triangle in G and h is not incident to u. Since we have ruled out Subcase 1.1, we can assume that the color of (v_1, v_2) is not e, and hence there is the above-mentioned edge h. Let x and y be the edges of G that are adjacent to b and h, and a and h, respectively, that are not incident to u. Observe that $f((v_1, v_4)) = y$ and $f((v_2, v_4)) = x$. On the other hand, observe that since z is not a used edge and $f((v_3, v_5)) = e$, we have $f((v_4, v_5)) \in \{a, b\}$. This is a contradiction since there is no vertex w of G such that $\partial_G(w) = \{a, x, y\}$ or $\partial_G(w) = \{b, x, y\}$.

Case 2: The edge e colors an edge that is a bridge in S_{16}.

The consideration of Case 1 implies that, without loss of generality, we can assume that the edge e is not used in end-blocks of S_{16}. Assume that $f((v, v_5)) = e$. Since z is not a used edge, we can assume that $f((v_3, v_5)) = a$ and $f((v_4, v_5)) = b$.

Assume that $a = (u, u_a)$ and $b = (u, u_b)$. We claim that u_a and u_b are not joined with an edge in G. Assume the opposite. Let $h = (u_a, u_b) \in E(G)$. Let x and y be the edges of G incident to u_a and u_b, respectively, that are different from a and h, and b and h, respectively. Then, we can assume that

$$f((v_1, v_3)) = x, f((v_2, v_3)) = h,$$

and

$$f((v_2, v_4)) = y, f((v_1, v_4)) = h.$$

This implies that

$$a = f((v_1, v_2)) = b.$$

Hence a and b are parallel edges of G, which contradicts the simpleness of G. Thus, u_a and u_b are not joined with an edge in G. Let x and y be the edges of G incident to u_a, that are different from a. Similarly, let α and β be the edges of G incident to u_b, that are different from b. We can assume that

$$f((v_1, v_3)) = x, f((v_2, v_3)) = y,$$

and

$$f((v_2, v_4)) = \beta, f((v_1, v_4)) = \alpha.$$

This implies that x and α are sharing a vertex $u_{x,\alpha}$ of G, y and β are sharing a vertex $u_{y,\beta}$ of G, and $u_{x,\alpha}$, $u_{y,\beta}$ are joined with an edge g of G, such that $f((v_1, v_2)) = g$. Observe that the edges a and b are lying on a cycle of G. Now, since z is not a used edge, we have that the other two bridges of G ($\neq (v, v_5)$) are colored with a and b. This contradicts (e) of Lemma 2.2, since a and b are not bridges of G.

The consideration of the above two cases implies that any used edge of G is adjacent to a used edge. Since G is connected, we have all edges of G used. Since $|E(S_{16})| = 24$, we have $|E(G)| \leq 24$, or $|V(G)| \leq 16$. Proposition 2.1 implies that $G = S_{16}$. $\square$

In [8], the following result is obtained:

Theorem 3.4 *(Mkrtchyan, [8]) Let G be a connected bridgeless cubic graph with $G \prec P$; then $G = P$.*

Below, we prove the analogue of this result for simple cubic graphs that may contain bridges. Our proof strategy is similar to that of given in [8].

Theorem 3.5 *If G is a connected simple cubic graph with $G \prec P$, then $G = P$.*

Proof: By (b) of Lemma 2.2, G is non-3-edge-colorable. Let f be a G-coloring of P. As in the proof of the previous theorem, we say that an edge $e \in E(G)$ is used (with respect to f) if $f^{-1}(e) \neq \emptyset$. First of all let us show that if an edge of G is used, then all edges adjacent to it are used.

Suppose that $e = (u, v)$ is a used edge, and for the sake of contradiction, assume that the edge z incident to v is not used. Assume that $\partial_G(u) = \{a, b, e\}$. We use the labels of vertices of P given on Figure 1.

Since e is a used edge, due to symmetry of P, we can assume that $f((u_3, u_4)) = e$, $f((u_4, u_5)) = a$ and $f((u_4, v_4)) = b$. As z is not a used edge, due to the symmetry of P we can assume that $f((u_3, v_3)) = b$, $f((u_2, u_3)) = a$.

Define

$$a_1 = f((u_1, u_5)), \text{ and } a_2 = f((u_1, u_2)).$$

Observe that since f is a G-coloring of P, we have a_1 and a_2 adjacent edges of G. Moreover, each of them is adjacent to a.

Similarly, define the edges

$$b_1 = f((v_1, v_4)), \text{ and } b_2 = f((v_1, v_3)).$$

Again, we have b_1 and b_2 adjacent edges of G. Moreover, each of them is adjacent to b.

We will consider three cases:

Case 1: The edges a_1, a_2 and a do not form a triangle in G.
Observe that in this case $f((u_1, v_1)) = a$. This implies that the edges a, b_1, b_2 must be incident to the same vertex w. Moreover, b_1 and b_2 differ from b. Hence $w \neq u$. This is possible only when b_1 and b_2 are two parallel edges connecting the other ($\neq u$) end-vertices of a and b. This is a contradiction, since G is a simple graph.

Case 2: The edges b_1, b_2 and b do not form a triangle in G.
This case is similar to Case 1.

Case 3: The edges a_1, a_2 and a form a triangle in G. Similarly, b_1, b_2 and b form a triangle.
Let a_3 be the edge of G that is adjacent to a_1, a_2 and is not adjacent to a. Similarly, let b_3 be the edge of G that is adjacent to b_1, b_2 and is not adjacent to b. Note that the edges a_3 and b_3 exist, since G is simple.
Observe that

$$a_3 = f((u_1, v_1)) = b_3,$$

and hence $a_3 = b_3$. Depending upon whether a and b belong to the same triangle or not, we will consider the following sub-cases:

Subcase 3.1: a and b belong to different triangles.
Observe that in this case the edge e must belong to both of them, hence we have the situation depicted on Figure 12. In this case $a_3 \neq b_3$, which is a contradiction.

Subcase 3.2: a and b belong to the same triangle (See Figure 13).
In this case b_3 should be adjacent to a, and a_3 should be adjacent to b. In this case $a_3 \neq b_3$, which is a contradiction.

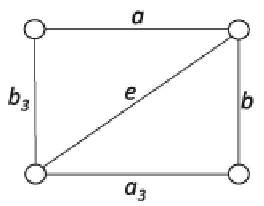

Figure 12: a and b belong to different triangles.

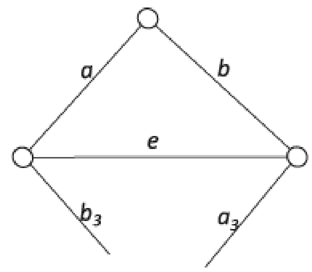

Figure 13: a and b belong to the same triangle.

Consideration of the above three cases implies that any used edge of G is adjacent to a used edge. Since G is connected, all edges of G are used. Since $|E(P)| = 15$, we have $|E(G)| \leq 15$, or $|V(G)| \leq 10$. Proposition 2.2 implies that $G = P$ or $G = S'$.

In order to complete the proof of the theorem, it suffices to show that P does not admit an S'-coloring such that all edges of S' are used. We will use the labels of edges of S' given in Figure 5.

For the sake of contradiction, assume that P admits an S'-coloring f such that all edges of S' are used. Due to symmetry of P, we can assume that $f((u_3, u_4)) = a$. Consider the connected components A and B of $S' - a$. Since P is 2-edge-connected, there is a vertex w of P such that w is incident to at least one edge that has color from A, and at least one edge that has color from B. Observe that this vertex violates the definition of an S'-coloring. This is a contradiction, and hence P does not admit an S'-coloring either. □

The theorem proved above implies:

Corollary 3.1 *P does not admit an S_{16}-coloring.*

This corollary and Theorem 3.3 suggest that a statement analogous to the Sylvester coloring conjecture is impossible in the class of simple cubic graphs.

Our next result states that any cubic graph admits a coloring with edges of S, such that 80% of vertices meet the constraints of the Sylvester coloring conjecture.

Theorem 3.6 *Let G be a cubic graph. Then there is a mapping $f : E(G) \to S$ such that*

$$|V(f)| \geq \frac{4}{5}|V(G)|,$$

and for any $v \in V(G) \backslash V(f)$ there are two edges $e, e' \in \partial_G(v)$, such that $f(e) = f(e')$.

Proof: We prove the theorem by induction on the number of vertices. If $|V(G)| = 2$, we can take an arbitrary vertex w of S, and color the three edges of G with edges incident to w. It is trivial to see that this coloring satisfies the condition of the theorem. Now, assume that the statement of the theorem holds for all cubic graphs with $|V(G)| < n$, and consider an arbitrary cubic graph G containing $n \geq 4$ vertices. Clearly we can assume that G is connected.

We will consider two cases.

Case 1: G contains a contractible triangle T.
Consider the cubic graph $H = G/T$, and let v_T be the vertex of H obtained by contracting T. Since H contains $n-2$ vertices, we have that there is a mapping $g : E(H) \to S$, such that

$$|V(g)| \geq \frac{4}{5} \cdot |V(H)|,$$

and for any $v \in V(H) \backslash V(g)$ there are two edges $e, e' \in \partial_H(v)$ such that $g(e) = g(e')$. We will consider two subcases.

Subcase 1.1: $v_T \in V(g)$.

There is a vertex $s \in V(S)$, such that $g(\partial_H(v)) = \partial_S(s)$. Let $\partial_S(s) = \{\alpha, \beta, \gamma\}$. Consider a mapping $f : E(G) \to S$, obtained from g as follows: color the edges of T with a color from $\{\alpha, \beta, \gamma\}$, such that its end-vertices are not incident to an edge with that color. Observe that

$$|V(f)| = |V(g)| + 2, \text{ and } |V(G)| = |V(H)| + 2,$$

and hence

$$\frac{|V(f)|}{|V(G)|} = \frac{|V(g)| + 2}{|V(H)| + 2} \geq \frac{|V(g)|}{|V(H)|} \geq \frac{4}{5},$$

and therefore

$$|V(f)| \geq \frac{4}{5}|V(G)|.$$

Subcase 1.2: $v_T \notin V(g)$.

There are two edges $e, e' \in \partial_H(v_T)$, such that $g(e) = g(e')$. Let $x = g(e)$, and let y and z be two edges of S that are incident to the same end-vertex of x in S.
Consider a mapping $f : E(G) \to S$, obtained from g as follows: color the edges of T that are opposite to the edges with color x by y, and color the remaining third edge of T with z. Observe that

$$|V(f)| = |V(g)| + 2, \text{ and } |V(G)| = |V(H)| + 2,$$

hence

$$\frac{|V(f)|}{|V(G)|} = \frac{|V(g)| + 2}{|V(H)| + 2} \geq \frac{|V(g)|}{|V(H)|} \geq \frac{4}{5},$$

and therefore

$$|V(f)| \geq \frac{4}{5} \cdot |V(G)|.$$

Moreover, for each vertex $w \notin V(f)$, there are two edges $h, h' \in \partial_G(w)$ such that $f(h) = f(h')$.

Case 2: No triangles of G are contractible.

Let $\mathcal{T}$ be the set of all triangles of G. Observe that $\mathcal{T}$ can be empty. If each vertex of G lies on a triangle $\mathcal{T}$, then, since G is connected, we have that G is the unique cubic graph with six vertices and one bridge. It is a matter of direct verification that this graph admits an S-coloring, and hence the statement of the theorem is true in this case. Thus we can assume that there is a vertex of G that does not lie on a triangle of $\mathcal{T}$. Consider a graph G' obtained from G by removing all vertices of G that lie on a triangle of $\mathcal{T}$. Observe that G' is a non-empty, connected, triangle-free graph of maximum degree at most 3.

We will use the labels of edges of S given in Figure 2. Consider a coloring of edges of G' with colors a, b and c, such that the number of uncolored edges is smallest. Let e be an uncolored edge. Color e with a color d from $\{a, b, c\}$, such that there is only one edge adjacent to e, such that it has also color d. Observe that all edges of G' are colored.

Now we are going to extend this coloring to that of G. Choose a triangle T from $\mathcal{T}$. As T is not contractible, it follows that the subgraph of G induced by the vertices of T form an end-block B of G. Moreover, B is isomorphic to end-blocks of S. Let v the root of B. Choose a color $d \in \{a, b, c\}$ such that d is missing on the vertex v in the coloring of G'. Color the bridge joining a vertex of T to v by d, and color the edges of B by corresponding edges of the end-block of S, which contains a vertex incident to d. Let f be the resulting coloring.

Observe that all edges of G are colored in f. Moreover, vertices of $V(G) \backslash V(f)$ lie in G'. Since G' is triangle-free, we have that the cycles corresponding to uncolored edges are of length at least 5. Since they are vertex-disjoint (Lemma 2.3), we have that their number is at most $\frac{|V(G')|}{5}$. Observe that each uncolored edge e is incident to a vertex v such that $v \in V(G) \backslash V(f)$. Moreover, $|V(G) \backslash V(f)|$ coincides with the number of uncolored edges, which implies that

$$|V(G) \backslash V(f)| \leq \frac{|V(G')|}{5} \leq \frac{|V(G)|}{5},$$

or

$$|V(f)| \geq \frac{4}{5} \cdot |V(G)|.$$

Finally, for each vertex $w \notin V(f)$ there are two edges $h, h' \in \partial_G(w)$ such that $f(h) = f(h')$.

□

Corollary 3.2 *Let G be a cubic graph. Then there is a mapping $f : E(G) \to S$ such that*

$$|V(f)| \geq \frac{4}{5}|V(G)|.$$

In the end of the paper, we verify Conjecture 1.4 in the class of claw-free cubic graphs. Our main ingredients are the characterization of claw-free cubic graphs (Proposition 2.3) and Theorem 3.1 about S_4-colorability of arbitrary cubic pseudographs.

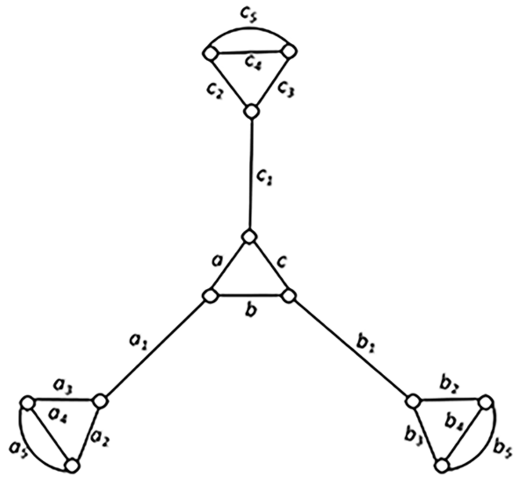

Figure 14: The graph S_{12}

Let S_{12} be the cubic graph from Figure 14. We prove a somewhat stronger statement.

Theorem 3.7 *Let G be a claw-free cubic graph. Then $S_{12} \prec G$.*

Proof: Our proof is by induction on $|V(G)|$. Clearly, the statement of the theorem is true when $|V(G)| = 2$ (K_2^3 is 3-edge-colorable). Assume that it remains true for all claw-free cubic graphs with $|V(G)| < n$, and consider a claw-free cubic graph with $|V(G)| = n$. Without loss of generality, we can assume that G is connected.

We apply Proposition 2.3. If G meets conditions (1) or (2) of the proposition, then G is 3-edge-colorable, and hence this case is similar to the base case of induction. Thus we can assume that G meets condition (3) of Proposition 2.3.

Let us show that we can assume, that in G all strings of diamonds and 2-cycles of G are trivial. On the opposite assumption, consider a non-trivial string J of diamonds and 2-cycles of G. Let a and b be the head and tail of J, respectively. Moreover, let c and d be the neighbors of a and b, respectively, that lie outside J. If $c \neq d$, then consider a cubic graph G' defined as follows:

$$G' = (G - V(J)) + (c, d).$$

Observe that G' is a claw-free cubic graph with $|V(G')| < n$, hence by the induction hypothesis, it admits an S_{12}-coloring g. Let $g((c, d)) = \alpha$, where α is an edge of S_{12}. Moreover, let β and γ be 2 edges of S_{12} leaving the same end-vertex of α in S_{12}.

Color the edges (a, c) and (b, d) with α. Since rings of diamonds and 2-cycles are 3-edge-colorable, we can color the edges of J with α, β and γ, so that each vertex of J is incident to edges with colors α, β and γ. It can be easily checked that this new coloring is an S_{12}-coloring of G.

If $c = d$, then since G is claw-free, we have that a and b are joined by 2 parallel edges, hence J is a trivial string contradicting our assumption.

Thus all strings of diamonds or 2-cycles of G are trivial. This and (3) of Proposition 2.3 imply that there is a cubic pseudo-graph H, such that G can be obtained from H by replacing any vertex of H with a triangle. By Theorem 3.1, H admits an S_4-coloring such that its loops are colored by loops of S_4 (see (a) of Theorem 3.1). Now, observe that S_{12} can be obtained from S_4 by replacing any vertex of S_4 by a triangle.

Extend the S_4-coloring of H to an S_{12}-coloring of G by coloring the edges of new triangles of G by the edges of the corresponding new triangles of S_{12}. One can easily see that there is always a way of doing this, which results in an S_{12}-coloring of G. □

Taking into account that $S \prec S_{12}$, and $\prec$ is transitive, we have the following corollary of Theorem 3.7:

Corollary 3.3 *Let G be a claw-free cubic graph. Then $S \prec G$.*

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